COUNTABLE-CODIMENSIONAL SUBSPACES OF LOCALLY CONVEX SPACES

by J. H. WEBB

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A *barrel* in a locally convex Hausdorff space $E[\tau]$ is a closed absolutely convex absorbent set. A *σ-barrel* is a barrel which is expressible as a countable intersection of closed absolutely convex neighbourhoods. A space is said to be *barrelled* (*countably barrelled*) if every barrel (σ-barrel) is a neighbourhood, and *quasi-barrelled* (*countably quasi-barrelled*) if every bornivorous barrel (σ-barrel) is a neighbourhood. The study of countably barrelled and countably quasi-barrelled spaces was initiated by Husain (**2**).

It has recently been shown that a subspace of countable codimension of a barrelled space is barrelled ((**4**), (**6**)), and that a subspace of finite codimension of a quasi-barrelled space is quasi-barrelled (**5**). It is the object of this paper to show how these results may be extended to countably barrelled and countably quasi-barrelled spaces. It is known that these properties are not preserved under passage to arbitrary closed subspaces (**3**). Theorem 6 shows that a subspace of countable codimension of a countably barrelled space is countably barrelled.

Let $\{E_n\}$ be an expanding sequence of subspaces of E whose union is E. Then $E' \subseteq \bigcap_1^\infty E_n'$, and the reverse inclusion holds as well (**1**) if either

(i) $E'[\sigma(E', E)]$ is sequentially complete

or (ii) $E'[\beta(E', E)]$ is sequentially complete, and every bounded subset of E is contained in some E_n.

Theorem 1. *Let $E[\tau]$ be a locally convex space with $\tau = \mu(E, E')$ (the Mackey topology). Suppose $E = \bigcup_1^\infty E_n$, where $\{E_n\}$ is an expanding sequence of subspaces of E. If $E' = \bigcap_1^\infty E_n'$, then E is the strict inductive limit of the sequence $\{E_n\}$.*

Proof. Let $F[\chi]$ be any locally convex space, and T: $E \rightarrow F$ a linear mapping whose restriction T_n to E_n is continuous. We show that T is then continuous, which proves the result.

Let $f \in F'$. Then the composite mapping $f \circ T_n$: $E_n \rightarrow K$ (scalars) is continuous, i.e. $f \circ T_n \in E_n'$ for each n. Hence $f \circ T \in E'$, so T is $\sigma(E, E') - \sigma(F, F')$ continuous, hence $\mu(E, E') - \mu(F, F')$ continuous, hence $\tau - \chi$ continuous.

The following result has been proved by M. de Wilde and C. Houet (**9**).

Theorem 2. *Let* $E[\tau]$ *be a locally convex space, with* $E = \bigcup_1^\infty E_n$*, where* $\{E_n\}$ *is an expanding sequence of subspaces. Then* E *is the inductive limit of* $\{E_n\}$ *in either of the following cases*:

(i) *E is countably barrelled*;

(ii) *E is countably quasi-barrelled, and every bounded subset of E is contained in some* E_n.

Note that Theorem 1 is not a generalization of Theorem 2, for although it is true that the dual of a countably barrelled (countably quasi-barrelled) space is weakly (strongly) sequentially complete, there exist countably barrelled spaces $E[\tau]$ with $\tau \neq \mu(E, E')$. In fact, Theorem 1 is false if the condition $\tau = \mu(E, E')$ is dropped. (Consider $E = \phi$ with the topology $\sigma(\phi, \omega)$ and

$$E_n = \{x \in \phi \colon x_i = 0 \ \forall i > n\}.)$$

However, both Theorems 1 and 2 generalize Valdivia's result ((**6**), Corollary 1.5).

Theorem 3. *Let E be a countably barrelled* (*countably quasi-barrelled*) *space, and F a closed subspace of E of countable codimension* (*of countable codimension, and such that for every bounded subset B of E, F is of finite co-dimension in* span $\{F \cup B\}$). *Then F is countably barrelled* (*countably quasi-barrelled*).

Proof. Let $\{x_n\}$ be a sequence in E forming a base for a complementary subspace G of F. Put $E_1 = F$, $E_n = \text{span}\{E_{n-1}, x_{n-1}\}$ $(n > 1)$. Then $E = \bigcup_1^\infty E_n$ and by Theorem 2, E is the strict inductive limit of the sequence $\{E_n\}$. Since F is closed, each E_n is closed.

Consider the projection map $\pi \colon E \to F$, parallel to G. The restriction $\pi_n \colon E_n \to F$ is continuous, since F is closed and of finite codimension in E_n. Since E is the inductive limit of the sequence $\{E_n\}$, π is continuous.

It follows that F has a closed complement in E, and that F is isomorphic with a quotient of E by a closed subspace. Since the property of being countably barrelled (countably quasi-barrelled) is preserved when passing to quotients ((**2**) Corollary 14), F is countably barrelled (countably quasi-barrelled).

Corollary. *A closed subspace of finite codimension of a countably quasi-barrelled space is countably quasi-barrelled.*

A simple adaptation of Theorem 3 enables us to prove a corresponding result for quasi-barrelled spaces:

Theorem 4. *Let E be a quasi-barrelled space, and F a closed subspace of E of countable codimension, such that for every bounded subset B of E, F is of finite codimension in* span $\{F \cup B\}$. *Then F is quasi-barrelled.*

To illustrate the relevance of the condition on the bounded sets imposed in Theorems 3 and 4, we give the following examples.

Example 1. Let E be a countably barrelled space and F a closed subspace of countable codimension. Let B be a bounded subset of E. Let $\{F_n\}$ be constructed as in Theorem 3. Since E is a strict inductive limit of closed subspaces, it follows that B is contained in some F_n. So F is of finite codimension in span $\{F \cup B\}$.

Example 2. Let E and F be subspaces of the sequence space l^1, defined as follows:

$$E = \{x \in l^1 : x_{2n+1} = 0 \text{ for all but finitely many } n\}$$

$$F = \{x \in l^1 : x_{2n+1} = 0 \text{ for all } n\}.$$

Give E the topology of coordinatewise convergence. Then F is a closed subspace of countable codimension in E. If $B = \{x \in l^1 : |x_n| \leqq 1 \text{ for all } n\}$ then B is a bounded subset of E, but F is not of finite codimension in span $\{F \cup B\}$.

The main problem is to remove the hypothesis that F is closed from Theorem 3. The countably barrelled case may be dealt with by means of a very useful result, due to Saxon and Levin (**4**). Our proof is a simplified version of the original.

Theorem 5. *Let E be a locally convex space such that $E'[\sigma(E', E)]$ is sequentially complete. If A is a closed absolutely convex subset of E such that* span A *is of countable codimension in E, then* span A *is closed.*

Proof. Let $E = \text{span } A \oplus \text{span } \{x_n\}$ where $\{x_n\}$ is a linearly independent sequence. We construct $g_k \in E'$ such that $g_k(x_i) = \delta_{ki}$ and $g_k(a) = 0$ for each $a \in A$. The construction is as follows:

Let $B_r = \bar{\Gamma}\{A, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_r\}$ $(r > k)$. Then B_r is absolutely convex and closed, and $x_k \notin rB_r$. By the Hahn-Banach Theorem, there exists $f_r \in E'$ such that $f_r(x_k) = 1$ and $|f_r(x)| \leqq \frac{1}{r}$ for each $x \in B_r$. The sequence $\{f_r\}_{r>k}$ is easily seen to be $\sigma(E', E)$-Cauchy, hence converges to some $g_k \in E'$ which is as required.

Since span $A = \bigcap_1^\infty g_k^{-1}(0)$, span A is closed.

If span A is of finite codimension, the same result holds with only minor alterations in the proof.

Theorem 6. *If F is a subspace of countable codimension of a countably barrelled space E, then F is countably barrelled.*

Proof. Since $\bar{F}$ is countably barrelled by Theorem 3, it is sufficient to consider the case when F is dense in E. Let $U = \bigcap_1^\infty U_n$ be a σ-barrel in F.

Then $\bar{U} \subset \bigcap_1^\infty \bar{U}_n = V$, and each $\bar{U}_n$ is a neighbourhood in E, since F is dense. Since the dual of a countably barrelled space is weakly sequentially complete ((2) Theorem 5), span $\bar{U}$ is closed, hence $\bar{U}$ is absorbent. So V is a σ-barrel, therefore a neighbourhood in E. Since $V \cap F = U$, U is a neighbourhood in F.

Valdivia ((7), Theorem 4) has proved the following result: *Let E be a sequentially complete $\mathcal{DF}$ space. If G is a subspace of E, of infinite countable codimension, then G is a $\mathcal{DF}$ space.* Since a sequentially complete $\mathcal{DF}$ space is countably barrelled, and the property of having a fundamental sequence of bounded sets is inherited by all subspaces, Theorem 6 above is an extension of Valdivia's result.

We now examine the problem of removing the hypothesis that F is closed from the countably quasi-barrelled case of Theorem 3.

We denote by E^+ the set of all sequentially continuous linear functionals on E (see (8)). Note that while the elements of E' are given by closed hyperplanes in E, the elements of E^+ are given by sequentially closed hyperplanes in E.

Lemma 7. *Let E be a locally convex space with $E' = E^+$. Let F be a sequentially closed subspace of E such that for every bounded subset B of E, F is of finite codimension in* span $\{F \cup B\}$. *Then F is closed.*

Proof. Let $\{x_\alpha : \alpha \in A\}$ be a set of points in E linearly independent modulo F, which, together with F, span E. For each $\alpha \in A$, define

$$H_\alpha = F + \text{span}\{x_\beta : \beta \in A, \beta \neq \alpha\}.$$

Then H_α is a hyperplane in E. Let $\{a_n\}$ be a sequence in H_α converging to a_0. Then there are points $x_{\beta_1}, \ldots, x_{\beta_m}$ $(\beta_i \in A)$ such that

$$\{a_n\} \subset F + \text{span}\{x_{\beta_1}, \ldots, x_{\beta_m}\} \subset H_\alpha.$$

Since $F + \text{span}\{x_{\beta_1}, \ldots, x_{\beta_m}\}$ is sequentially closed, $a_0 \in H_\alpha$, which shows that H_α is sequentially closed. Since $E' = E^+$, H_α is closed. But $F = \bigcap_{\alpha \in A} H_\alpha$, so F is closed.

Corollary 8. *Let E be a locally convex space, with $E' = E^+$. If F is a subspace of E such that for every bounded closed absolutely convex set B, $F \cap B$ is closed, and F is of finite codimension in* span $\{F \cap B\}$, *then F is closed.*

A similar result is proved by Valdivia ((7) Lemma 4) assuming that E is a $\mathcal{DF}$ space, instead of $E' = E^+$. The above is not a generalization of Valdivia's result, for there exist $\mathcal{DF}$ spaces E with $E' \neq E^+$ (see (8)).

Theorem 9. *Let E be a countably quasi-barrelled space with $E' = E^+$. If F is a subspace of E such that $\bar{F}$ is of countable codimension in E, and such that F is of finite codimension in* span $\{F \cup B\}$ *for every bounded set B, then F is countably quasi-barrelled.*

Proof. Case 1: F closed. See Theorem 5.

Case 2: F dense in E. Let $U = \bigcap_1^\infty U_n$ be a bornivorous σ-barrel in F. Then $\bar{U} \subset \bigcap_1^\infty \bar{U}_n$. Let $G = \text{span}\, \bar{U}$. We show (i) $\bar{U}$ is bornivorous in G, (ii) $G = E$.

(i) Let B be a bounded subset of G. Since $G \supset F$, there exists a finite-dimensional subspace M of G such that $B \subset F+M = L \subset G$. Now $\bar{U} \cap L$ is the closure of U in L, F is of finite codimension in L and $\bar{U} \cap L$ is absorbent in L. By a result of Valdivia ((7) Lemma 1) $\bar{U} \cap L$ is bornivorous in L, so $\bar{U}$ absorbs B.

(ii) Since $G \supset F$, and F is dense, it is sufficient to prove that G is closed. Let B be a bounded absolutely convex closed subset of E. Then for some α, $G \cap B \subset \alpha \bar{U}$, so $G \cap B$ is closed. By Corollary 8, G is closed.

Thus $\bigcap_1^\infty \bar{U}_n$ is a bornivorous σ-barrel in E, hence a neighbourhood. Therefore $U = \left(\bigcap_1^\infty \bar{U}_n\right) \cap F$ is a neighbourhood in F.

Case 3: Arbitrary F. This follows at once from Cases 1 and 2.

This result is a variation of one of Valdivia, who proves ((7) Theorem 2) that a subspace F of a $\mathcal{DF}$ space E, such that F is of finite codimension in span $\{F \cup B\}$ for every bounded set B, is itself a $\mathcal{DF}$ space. Such a subspace is necessarily of countable codimension, while our result above requires only that the closure of the subspace concerned be of countable codimension.

Corollary. *If E is a countably quasi-barrelled space with $E' = E^+$, and F is a subspace of finite codimension in E, then F is countably quasi-barrelled.*

It is not known whether the condition " $E' = E^+$ " can be omitted from the Corollary.

An easy adaptation of the proof of Theorem 9 yields the following result:

Theorem 10. *Let E be a quasi-barrelled space with $E' = E^+$. If F is a subspace of E such that $\bar{F}$ is of countable codimension in E, and such that F is of finite codimension in* span $\{F \cup B\}$ *for every bounded set B, then F is quasi-barrelled.*

Corollary 11. *Let E be a bornological space. If F is a subspace of E such that $\bar{F}$ is of countable codimension in E, and such that F is of finite codimension in* span $\{F \cup B\}$ *for every bounded set B, then F is quasi-barrelled.*

Proof. A bornological space E is quasi-barrelled and satisfies: $E' = E^+$ **(8)**.

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University of Cape Town
South Africa