Fourth Meeting, 12*th February* 1904.

Mr Charles Tweedie, President, in the Chair.

A Basic-sine and cosine with symbolical solutions of certain differential equations.

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The object of this paper is to introduce certain functions analogous to the circular functions. The functions will be denoted by

$$\sin_p(\lambda, x), \ \cos_p(\lambda, x).$$

Formulæ analogous to

$$\sin^2 a + \cos^2 a = 1,$$
$$\cos^2 a - \sin^2 a = \cos 2a,$$
$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$
$$\cos(a \mp b) = \cos a \cos b \pm \sin a \sin b,$$

will be obtained, and the use of the functions in symbolical solutions of certain differential equations exemplified. The connection of the functions with generalised Bessel-functions of order half an odd integer will be shown.

1.

Consider the function

$$E_p(\lambda) = 1 + \frac{\lambda}{[1]!} + \frac{\lambda^2}{[2]!} + \dots + \frac{\lambda^r}{[r]!} + \dots, \qquad (1)$$

$$\lambda = 1 \text{ or } < 1,$$

in which $[r] = \dfrac{p^r - 1}{p - 1}$,

$$[r]! = [1][2][3]\dots[r].$$

If we invert the base p, we obtain

$$E_{\frac{1}{p}}(\lambda) = 1 + \frac{\lambda}{[1]!} + p\frac{\lambda^2}{[2]!} + \dots + p^{r.\overline{r-1}/2}\frac{\lambda^r}{[r]!} + \dots . \qquad (2)$$

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It is well known that

$$(1+\lambda)(1+p\lambda)(1+p^2\lambda)\dots(1+p^{m-1}\lambda)$$

$$=1+\sum_{r=1}^{r=m}\frac{(p^m-1)(p^{m-1}-1)(p^{m-2}-1)\dots(p^{m-r+1}-1)}{(p\ \ -1)(p^2\ \ -1)(p^3\ \ -1)\dots(p^r\ \ \ -1)}p^{r\cdot r-1/2}\lambda^r, \quad (3)$$

$(p<1)$.

When m is infinite this reduces to

$$(1+\lambda)(1+p\lambda)(1+p^2\lambda)\dots \textit{ ad inf.}$$

$$=1+\sum_{r=1}^{r=\infty}(-1)^r\frac{\lambda^r}{(p-1)(p^2-1)(p^3-1)\dots(p^r-1)}p^{r\cdot r-1/2}. \quad (4)$$

If now for λ we substitute $\lambda(1-p)$,
we obtain

$$\{1+\lambda(1-p)\}\{1+p\lambda(1-p)\}\{1+p^2\lambda(1-p)\}\dots \textit{ ad inf.}$$

$$=1+\sum_{r=1}^{r=\infty}\frac{\lambda^r}{[r]!}p^{r\cdot r-1/2} \quad - \quad - \quad - \quad - \quad (5)$$

$$=\mathrm{E}_{\frac{1}{p}}(\lambda).$$

Inverting the base p we obtain

$$\left\{1+\lambda\left(1-\frac{1}{p}\right)\right\}\left\{1+\frac{\lambda}{p}\left(1-\frac{1}{p}\right)\right\}\dots \textit{ ad inf.} \quad - \quad - \quad (6)$$

$$=1+\sum_{r=1}^{r=\infty}\frac{\lambda^r}{[r]!}$$

$$=\mathrm{E}_p(\lambda).$$

The infinite products are convergent only when $p<1$ and $p>1$ respectively, but the series have a much wider range of convergence, for, subject to limitations of the value of λ in $\mathrm{E}_{\frac{1}{p}}(\lambda)$, the series are convergent for all finite values of p.

2.

The expression (3) may be written

$$\frac{1}{[m]!}+\frac{a}{[m-1]![1]!}+p\frac{a^2}{[m-2]![2]!}+\dots\dots+p^{m\cdot m-1/2}\frac{a^m}{[m]!}$$

$$\equiv\frac{(1+a)(1+pa)\dots\dots(1+p^{m-1}a)}{[m]!}. \quad - \quad - \quad - \quad (7)$$

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Forming now the product

$$E_p(a) \,.\, E_{\frac{1}{p}}(b)$$

since the series are absolutely convergent we obtain the series

$$1+\left\{\frac{a}{[1]!}+\frac{b}{[1]!}\right\}+\left\{\frac{a^2}{[2]!}+\frac{ab}{[1]![1]!}+p\frac{b^2}{[2]!}\right\}+\dots$$

$$+\left\{\frac{a^r}{[r]!}+\frac{a^{r-1}b}{[r-1]![1]!}+p\frac{a^{r-2}b^2}{[r-2]![2]!}+\dots+p^{r\,.\,r-1/2}\frac{b^r}{[r]!}\right\}+\dots$$

$$=1+\frac{(a+b)}{[1]!}+\frac{(a+b)(a+pb)}{[2]!}+\frac{(a+b)(a+pb)(a+p^2b)}{[3]!}+\dots. \quad (8)$$

Putting $b=-a$ this gives us

$$E_p(a)\,.\,E_{\frac{1}{p}}(-a)=1, \quad - \quad - \quad - \quad - \quad (9)$$

analogous to

$$e^a \times e^{-a}=1.$$

Putting $b=a$ we obtain

$$E_p(a)E_{\frac{1}{p}}(a)=1+\frac{2a}{[1]!}+\frac{2(1+p)a^2}{[2]!}+\frac{2(1+p)(1+p^2)a^3}{[3]!}+\dots. \quad (10)$$

Considering numbers as formed from a sequence 1, p^0, p^1, p^2,...... we can show that the analogue of 2^2 is $(1+p^0)(1+p^1)$,

„ 2^3 „ $(1+p^0)(1+p^1)(1+p^2)$,

„ 3^2 „ $(1+p^0+p^1)(1+p^0+p^2)$,

....

We therefore write

$$E_p(a)E_{\frac{1}{p}}(a)=1+\frac{(2)_1a}{[1]!}+\frac{(2)_2a^2}{[2]!}+\dots+\frac{(2)_ra^r}{[r]!}+\dots. \quad - \quad (11)$$

This idea of number can be extended to forms $\{x\}_n$ and $(x)_n$ analogous to x^n when n and x are not restricted to integral values.

3.

The functions $\sin_p(a)$, $\cos_p(a)$.

We define these as follows :—

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$$\cos_p(a) = \frac{E_p(ia) + E_p(-ia)}{2}$$

$$= 1 - \frac{a^2}{[2]!} + \frac{a^4}{[4]!} - \ldots\ldots \qquad (12)$$

and

$$\sin_p(a) = \frac{E_p(ia) - E_p(-ia)}{2i}$$

$$= \frac{a}{[1]!} - \frac{a^3}{[3]!} + \frac{a^5}{[5]!} - \ldots\ldots \qquad (13)$$

From these forms we obtain directly

$$\sin_p(a)\sin_{\frac{1}{p}}(a) + \cos_p(a)\cos_{\frac{1}{p}}(a) = 1 \qquad (14)$$

and

$$\cos_p(a)\cos_{\frac{1}{p}}(a) - \sin_p(a)\sin_{\frac{1}{p}}(a) \qquad (15)$$

$$= 1 - \frac{(2)_2 a^2}{[2]!} + \frac{(2)_4 a^4}{[4]!} - \ldots\ldots,$$

where $(2)_r = (1+p^0)(1+p^1)(1+p^2)\ldots(1+p^{r-1})$.

This series is the analogue of the series for $\cos 2a$.

Now

$$\sin_p(a)\cos_{\frac{1}{p}}(b) = \frac{a}{[1]!} - \left\{\frac{a^3}{[3]!} + p\frac{ab^2}{[1]![2]!}\right\} + \ldots,$$

$$\cos_p(a)\sin_{\frac{1}{p}}(b) = \frac{b}{[1]!} - \left\{\frac{a^2 b}{[2]![1]!} + p^3\frac{b^3}{[3]!}\right\} + \ldots.$$

Therefore

$$\sin_p(a)\cos_{\frac{1}{p}}(b) + \cos_p(a)\sin_{\frac{1}{p}}(b)$$

$$= \frac{(a+b)}{[1]!} - \frac{(a+b)(a+pb)(a+p^2b)}{[3]!} + \ldots$$

$$= \mathfrak{S}_p(a, b). \qquad (16)$$

If we denote

$$1 - \frac{(a+b)(a+pb)}{[2]!} + \frac{(a+b)(a+pb)(a+p^2b)(a+p^3b)}{[4]!} - \ldots$$

by $\mathfrak{C}(a, b)$,

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the formulae may be written

$$\sin_p(a)\cos_{\frac{1}{p}}(b) \pm \cos_p(a)\sin_{\frac{1}{p}}(b) = \mathfrak{S}(a, \pm b), \quad (17)$$

$$\cos_p(a)\cos_{\frac{1}{p}}(b) \pm \sin_p(a)\sin_{\frac{1}{p}}(b) = \mathfrak{C}(a, \mp b), \quad (18)$$

$$\{\cos_p(a) + i\sin_p(a)\}\{\cos_{\frac{1}{p}}(b) + i\sin_{\frac{1}{p}}(b)\} = \mathfrak{C}(a, b) + i\mathfrak{S}(a, b), \quad (19)$$

$$\frac{E_p(a) + E_p(-a)}{E_p(a) - E_p(-a)} = \frac{E_p(a)E_{\frac{1}{p}}(a) + 1}{E_p(a)E_{\frac{1}{p}}(a) - 1} \quad (20)$$

$$= \frac{\mathfrak{E}(2, a) + 1}{\mathfrak{E}(2, a) - 1},$$

$\mathfrak{E}(2, a)$ denoting $1 + \frac{(2)_1 a}{[1]!} + \frac{(2)_2 a^2}{[2]!} + \ldots.$

Example :—

$$\frac{1}{[x]} + \frac{1}{[x][x+1]} + \frac{1}{[x][x+1][x+2]} + \cdots$$

$$= E_p(1)\left\{\frac{1}{[x]} - \frac{p}{[1]![x+1]} + \frac{p^3}{[2]![x+2]} - \cdots\right\} \quad (21)$$

The series

$$\frac{1}{[x]} + \frac{1}{[x][x+1]} + \cdots$$

can be expressed as the sum of a number of partial fractions

$$\frac{a_0}{[x]} + \frac{a_1}{[x+1]} + \frac{a_2}{[x+2]} + \ldots\ldots + \frac{a_n}{[x+n]} + \cdots$$

To find the coefficients a, multiply by $[x+n]$

and put $x = -n$; we thus obtain

$$a_n = \frac{1}{[-n][-n+1]\ldots[-3][-2][-1]}\left\{1 + \frac{1}{[1]!} + \frac{1}{[2]!} + \cdots\right\}$$

$$= (-1)^n \frac{p^{n \cdot n+1/2}}{[n]!} E_p(1).$$

The required result is established.

In a similar way we may establish

$$\frac{p^x}{[x]}+\frac{p^{x+1}}{[x][x+1]}+\ldots\ldots+\frac{p^{rx+(r\,.\,r-1)/2}}{[x][x+1]\ldots[x+r-1]}+\ldots$$

$$=E_{\frac{1}{p}}(p)\left\{\frac{1}{[x]}-\frac{1}{[1]\,!\,[x+1]}+\frac{1}{[2]\,!\,[x+2]}-\ldots\right\}. \quad (22)$$

In terms of the generalised Gamma-function * Γ_p, we may write these

$$\frac{1}{\Gamma_p([x+1])}+\frac{1}{\Gamma_p([x+2])}+\ldots=\frac{E_p(1)}{\Gamma_p([x])}\left\{\frac{1}{[x]}-\frac{p}{[1]\,!\,[x+1]}+\ldots\right\}, \quad (23)$$

$$\frac{p^x}{\Gamma_p([x+1])}+\frac{p^{x+1}}{\Gamma_p([x+2])}+\ldots=\frac{E_{1/p}(p)}{\Gamma_p([x])}\left\{\frac{1}{[x]}-\frac{1}{[1]\,!\,[x+1]}+\ldots\right\}, \quad (24)$$

both analogous to a well-known result

$$\frac{1}{\Gamma(x+1)}+\frac{1}{\Gamma(x+2)}+\ldots=\frac{e}{\Gamma(x)}\left\{\frac{1}{x}-\frac{1}{1\,!\,x+1}+\ldots\right\}.$$

4.

If we denote the convergent series

$$1+\frac{\lambda x^{[1]}}{[1]\,!}+\frac{\lambda^2 x^{[2]}}{[2]\,!}+\ldots \quad - \quad - \quad - \quad - \quad (25)$$

by $E_p(\lambda, x)$,

then $$\frac{d}{dx}\,.\,E_p(\lambda, x)=\lambda E_p(\lambda, x^p),$$

$$\frac{d^{[n]}}{dx^{[n]}}E_p(\lambda, x)=\lambda^n E_p(\lambda, x^{p^n}),$$

$D^{[n]}$ denoting $$\frac{d}{d(x^{p^{n-1}})}\left\{\frac{d}{d(x^{p^{n-2}})}\left\{\ldots\left\{\frac{d}{d(a^{p^2})}\left\{\frac{d}{d(x^p)}\left\{\frac{d}{dx}\right\}\right\}\right\}\ldots\right\}\right\}.$$

If $$P_p(x)=\int_0^1 E_p(1, z^{p-x})\,.\,z^{p[x-1]}dz \quad - \quad - \quad - \quad (26)$$

then $$P_p(x+1)=\frac{1}{p^x}[x]\,P_p(x)-\frac{1}{E_{\frac{1}{p}}(p)} \quad - \quad - \quad - \quad (27)$$

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* *Transactions, Royal Society, Edin.*, Vol. XLI., Art. 1.

which may also be written

$$P_p(x+1)+[-x]P_p(x)+E_p(-p)=0. \qquad - \quad - \quad (28)$$

Taking

$$\cos_p(\lambda, x)=\frac{E_p(i\lambda, x)+E_p(-i\lambda, x)}{2}, \qquad - \quad - \quad (29)$$

$$\sin_p(\lambda, x)=\frac{E_p(i\lambda, x)-E_p(-i\lambda, x)}{2i}, \qquad - \quad - \quad (30)$$

$$\frac{d^{[n]}}{dx^{[n]}}\left\{\sin_p(\lambda, x)\right\}=(-1)^{\frac{n-1}{2}}\lambda^n\cos_p(\lambda, x^{p^n}), \quad (n \text{ odd}),$$

$$=(-1)^{\frac{n}{2}}\lambda^n\sin_p(\lambda, x^{p^n}), \quad (n \text{ even}).$$

There must be a kind of periodicity for these functions analogous to that of the circular functions.

5.

Symbolic Solutions.

In the following analysis the functions $\sin_p$ and $\cos_p$ will be used to form symbolical solutions of the differential equation

$$p^{2n+2}\frac{d^{[2]}F}{dx^{[2]}}+\frac{[2n+2]}{x^p}\frac{dF}{dx}+\lambda^2F(x^{p^2})=0. \qquad - \quad - \quad (31)$$

When the base $p=1$ this differential equation reduces to

$$\frac{d^2F}{dx^2}+\frac{2(n+1)}{x}\frac{dF}{dx}+\lambda^2F=0,$$

an equation of great interest in physical investigations (Lamb's *Hydrodynamics*, Arts. 267–309). Various solutions are

$$\psi_{[n]}(x)=(-1)^n\left(\frac{d}{xdx}\right)^n\frac{\sin\lambda x}{\lambda x},$$

$$\Psi_n(x)=(-1)^n\left(\frac{d}{xdx}\right)^n\frac{\cos x\lambda}{x\lambda},$$

$$f_n(x)=(-1)^n\left(\frac{d}{xdx}\right)^n\frac{e^{-ix\lambda}}{x\lambda}.$$

If we integrate the equation

$$p^{2n+2}\frac{d^{[2]}F}{dx^{[2]}}+\frac{[2n+2]}{x^p}\frac{dF}{dx}+\lambda^2F(x^{p^2})=0$$

in series (*Proc. Edin. Math. Soc.*, Vol. XXI., pp. 65 et seq.), we obtain two series which we may denote

$$\psi_{[n]}(\lambda, x)=\frac{1}{[1][3][5]..[2n+1]}\left\{1-\frac{\lambda^2x^{[2]}}{[2][2n+3]}+\frac{\lambda^4x^{[4]}}{[2][4][2n+3][2n+5]}-..\right\} \quad - \quad (32)$$

$$\Psi_{[n]}(\lambda, x)=\frac{[1][3][5]..[2n-1]}{x_1^{[2n+1]}}\cdot\frac{1}{pn^2}\left\{1-\frac{\lambda^2x_1^{[2]}}{[2][1-2n]}+\frac{\lambda^4x_1^{[4]}}{[2][4][1-2n][3-2n]}-..\right\} \quad (33)$$

Operating with $\{D^{[n]}\}$ on the series

$$\Sigma(-1)^r\frac{\lambda^{2r}x^{[2r]}}{[2r+1]}=\frac{\sin_p(\lambda, x^{\frac{1}{p}})}{\lambda x^{\frac{1}{p}}},$$

if $\left\{D^{[n]}\right\}=\left\{\frac{1}{x^{p^{2n-1}}}\frac{d}{d(x^{p^{2n-2}})}\left\{\cdots\cdots\left\{\frac{1}{x^{p^3}}\frac{d}{d(x^{p^2})}\left\{\frac{1}{x^p}\frac{d}{dx}\right\}\right\}\cdots\right\}\right\}$,

then all terms before

$$(-1)^n\frac{\lambda^{2n}x^{[2n]}}{[2n+1]!}$$

are destroyed, while the operations performed on the remaining terms of the series give us

$$(-1)^n\frac{\lambda^{2n}}{[1][3][5]..[2n+1]}\left\{1-\frac{\lambda^2x^{p^{2n}[2]}}{[2][2n+3]}+\frac{\lambda^4x^{p^{2n}[4]}}{[2][4][2n+3][2n+5]}-\ldots\right\}.$$

We see that

$$\psi_{[n]}(x^{p^{2n}}, \lambda)=(-1)^n\lambda^{-2n}\{D^{[n]}\}\cdot\frac{\sin_p(\lambda, x^{\frac{1}{p}})}{\lambda x^{\frac{1}{p}}}, \quad - \quad - \quad (34)$$

and by a change of the variable we may write this

$$\psi_{[n]}(\lambda, x)=(-1)^n\lambda^{-2n}\{\Delta^{[n]}\}\frac{\sin_p(\lambda, x^{p^{-1-2n}})}{\lambda x^{p^{-2-2n}}}, \quad - \quad - \quad (35)$$

where $\{\Delta^{[n]}\}$ is the operator $\left\{\frac{1}{x^{p^{-1}}}\frac{d}{d(x^{p^{-2}})}\left\{\cdots\left\{\frac{1}{x^{p^{1-2n}}}\frac{d}{dx^{p^{-2n}}}\right\}\right\}\right\}$.

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In the same way

$$\Psi_{[n]}(\lambda, x) = (-1)^n \lambda^{-2n}\{\Delta^{[n]}\}\frac{\cos_p(\lambda, x^{p-1-2n})}{\lambda x^{p-1-2n}}, \quad - \quad - \quad (36)$$

$$f_{[n]}(\lambda, x) = \lambda^{2n}\{\Psi_{[n]}(\lambda, x) - \psi_{[n]}(\lambda, x)\}$$

$$= (-1)^n\{\Delta^{[n]}\}\frac{E_p(-i\lambda, x^{p-1-2n})}{\lambda x^{p-1-2n}}, \quad - \quad - \quad (37)$$

may be established.

Recurrence-formula.

The recurrence-formula for the function $\psi_{[n]}$ may easily be established as

$$x\psi'_{[n]}\left(\frac{\lambda}{p}, x\right) + [2n+1]\psi_{[n]}(\lambda, x) = \psi_{[n-1]}(\lambda, x) \quad - \quad - \quad (38)$$

which may also be written in the form

$$-\frac{\lambda^2}{p^2}x^{[2]}\psi_{[n+1]}\left(x^{p^2}, \frac{\lambda}{p}\right) + [2n+1]\psi_{[n]}(\lambda, x) = \psi_{[n-1]}(\lambda, x). \quad (39)$$

6.

Connection with generalised Bessel functions of order half an odd integer.

We define $J_{[n]}(\lambda, x)$

as

$$\sum_{r=0}^{r=\infty}(-1)^r\frac{\lambda^{n+2r}x^{[n+2r]}}{[r]!\,[n+r]!\,(2)_r(2)_{n+r}}. \quad - \quad - \quad (40)$$

The differential equation satisfied by this function may be obtained from (E) page 70, Vol. XXI., *Proc. Edin. Math. Soc.*, by the introduction of a parameter λ (*Trans. R. S. Edin.*, Vol. XLI.). In the following analysis theorems analogous to

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$

$$J_{\frac{3}{2}}(x) = -\left(\frac{2x}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\left(\frac{\sin x}{x}\right),$$

will be obtained :

$$J_{[n]}(\lambda, x) = \sum_{r=0}^{r=\infty}(-1)^r\frac{\lambda^{n+2r}x^{[n+2r]}}{[r]!\,[n+r]!\,(2)_r(2)_{n+r}},$$

in which $[n+r]\,! \;=\; \Gamma_p([n+r+1])$,

$$\Gamma_p([z+1]) = \underset{\kappa=\infty}{\mathrm{L}} \frac{[1][2][3][4]\ldots\ldots\ldots\ldots\ [\kappa]}{[z+1][z+2][z+3]\ldots\ldots[z+\kappa]}[\kappa]^z p^{\frac{z.z+1}{2}},$$

$$(m)_x = (m)^x \frac{\Gamma_{p^x}([m+1])}{\Gamma_p\,([m+1])}.$$

These functions are considered in a paper shortly to be printed (*Proc. R. S. Lond.*). Here we only require the difference equations

$$\frac{1}{[x]} \times \Gamma_p([x+1]) = \Gamma_p([x]),$$

$$(2)_x = (p^x+1) \times (2)_{x-1}.$$

Consider

$$J_{[\frac{1}{2}]}(\lambda,\, x) = \frac{\lambda^{\frac{1}{2}} x^{[\frac{1}{2}]}}{[\frac{1}{2}]\,!\,(2)_{\frac{1}{2}}}$$

$$\left\{1 - \frac{\lambda^2 x^{p^{\frac{1}{2}}[2]}}{[1]\,!\,(2)_1[\frac{3}{2}](p^{\frac{3}{2}}+1)} + \frac{\lambda^4 x^{p^{\frac{1}{2}}[4]}}{[2]\,!\,(2)_2[\frac{3}{2}][\frac{5}{2}](p^{\frac{3}{2}}+1)(p^{\frac{5}{2}}+1)} - \ldots\right\}; \quad (41)$$

we see that the series on the right side of the above reduces by means of the difference equations (38) to the standard form (40).

Since $[1]\,!\,(2)_1 = [2]$,

$[2]\,!\,(2)_2 = [4][2]$,

$\cdots \quad \cdots$

$\cdots \quad \cdots$

and $[\frac{3}{2}] \times (p^{\frac{3}{2}}+1) = [3]$,

$\cdots \quad \cdots$

$\cdots \quad \cdots$

$$J_{[\frac{1}{2}]}(\lambda,\, x) = \frac{\lambda^{\frac{1}{2}} x^{[\frac{1}{2}]}}{[\frac{1}{2}]!(2)_{\frac{1}{2}}}\left\{1 - \frac{\lambda^2 x^{p^{\frac{1}{2}}[2]}}{[1][2][3]} + \frac{\lambda^4 x^{p^{\frac{1}{2}}[4]}}{[1][2][3][4][5]} - \ldots\right\} \quad (42)$$

$$= \frac{\lambda^{\frac{1}{2}} x^{[\frac{1}{2}]}}{[\frac{1}{2}]!(2)_{\frac{1}{2}}} \cdot \frac{\sin_p(\lambda,\, x^{p^{-\frac{1}{2}}})}{\lambda x^{p^{-\frac{1}{2}}}}$$

$$= \frac{\lambda^{-\frac{1}{2}} x^{[-\frac{1}{2}]}}{[\frac{1}{2}]^{\frac{1}{2}}\Gamma_{p^2}[(\frac{3}{2})]} \cdot \sin_p(\lambda,\, x^{p^{-\frac{1}{2}}}). \quad - \quad - \quad - \quad (43)$$

When $p=1$ the function Γ_{p^2} reduces to Euler's Gamma-function

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi}\cdot$$

There is no difficulty in extending

$$J_{\frac{3}{2}}(x) = -\left(\frac{2x}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx}\left(\frac{\sin x}{x}\right)$$

in the form

$$J_{[\frac{3}{2}]}(\lambda, x) = -\frac{\lambda^{\frac{1}{2}} x^{[\frac{1}{2}]}}{[\frac{1}{2}]^{\frac{1}{2}} \Gamma_{p^2}([\frac{3}{2}])} \frac{d}{d(x^{p^{-\frac{1}{2}}})} \left\{ \frac{\sin_p(\lambda, x^{p^{-\frac{3}{2}}})}{\lambda x^{p^{-\frac{3}{2}}}} \right\}; \quad - \quad (44)$$

and generally the formula

$$\left(\frac{\pi}{2x}\right)^{\frac{1}{2}} i^n J_{n+\frac{1}{2}}(x) = P_n\left(\frac{d}{idx}\right)\left(\frac{\sin x}{x}\right)$$

which is due to Lord Rayleigh (*Theory of Sound*, Vol. II., p. 263) may be extended to the functions

$$J_{[n+\frac{1}{2}]}, \quad P_{[n]}, \quad \sin_p, \qquad - \quad - \quad - \quad (45)$$

by means of the following identity

$$1 - \frac{[n][n-1}{[2][2n-1]} \cdot \frac{[2n+2m+1]}{[2n+2m-1]} + p^2 \frac{[n][n-1][n-2][n-3]}{[2][4][2n-1][2n-3]} \cdot \frac{[2n+2m+1]}{[2n+2m-3]} - \ldots$$

$$= p^{\frac{n.\,n-1}{2}} \frac{(2)_{n+m}(2)_n}{(2)_m} \frac{[n+m]![n+2m]![n]![n]!}{[2n+2m]![m]![2n]!}. \quad (46)$$

The general term of the series is

$$(-1)^r p^{r.\,r-1} \frac{[n][n-1][n-2]\ldots\ldots\ldots\ldots[n-2r+1]}{[2][4]\ldots[2r].[2n-1]\ldots[2n-2r+1]} \cdot \frac{[2n+2m+1]}{[2n+2m-2r+1]}.$$

This identity is a particular case of summation of

$$F_p([\alpha][\beta][\gamma][\delta][\epsilon])$$

and is a product of the two following series :— *

$$1 - p^2 \frac{[n][n-1]}{[2][2n-1]} + \ldots + (-1)^r p^{r.\,r+1} \frac{[n][n-1][n-2]..[n-2r+1]}{[2][4]..[2r].[2n-1]..[2n-2r+1]} + \ldots$$

$$= \frac{[n]![n]!(2)_n}{[2n]!}, \quad - \quad - \quad (47)$$

* *Transactions R.S.E.*, Vol. XLI. "Generalised Functions of Legendre and Bessel," Part I. (57), Part II. (8).

$$1 - p^2\frac{[n][n-1]}{[2][2n+2m-1]}p^{2m} + p^4\frac{[n][n-1][n-2][n-3]}{[2][4][2n+2m-1][2n+2m-3]}p^{4m} - \dots$$

$$= p^{\frac{n.n-1}{2}}\frac{[n+m]\,!\,[n+2m]\,!\,(2)_{n+m}}{[2n+2m]\,!\,[m]\,!\,(2)_m}. \qquad (48)$$

Putting $m=0$, $p=1$,

we obtain the following, which I suppose is a known result:

$$\left\{1 - \frac{n.n-1}{2.2n-1} + \frac{n.n-1.n-2.n-3}{2.4.2n-1\ 2n-3} - \dots\right\}^2 = 1 - \frac{n.n-1}{2.2n-1}\cdot\frac{2n+1}{2n-1}$$

$$+ \frac{n.n-1.n-2.n-3}{2.4.2n-1.2n-3}\cdot\frac{2n+1}{2n-3} - \dots; \qquad (49)$$

if c_1, c_2 ... be the coefficients in Legendre's series P_n

$$\{1 + c_1 + c_2 + \dots\}^2 = 1 + c_1\frac{2n+1}{2n-1} + c_2\frac{2n+1}{2n-3} + \dots. \qquad (50)$$

This, however, is outside the range of this paper, and must be left to a paper on $F([\alpha][\beta][\gamma][\delta][\epsilon])$.

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