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IMAGE AREA AND THE WEIGHTED SUBSPACES OF HARDY SPACES

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ABSTRACT. Let $H^{p,\phi}$ be the subspace of Hardy space H^p consisting of those $f \in H^p(B_n)$ satisfying $\sup_z \phi(|z|)|f(z)| < \infty$, where ϕ is a positive decreasing differentiable function on [0, 1) with $\phi(1-) = 0$. Concerning image area growth, criteria for f to be of $H^{p,\phi}$ are considered extending known results for H^p.

1. **Introduction.** U will denote the open unit disc of the complex plane $\mathbf{C}$ and $B = B_n$ will denote the unit open ball of $\mathbf{C}^n$. For f holomorphic in B and for Ω a subdomain of B, we let

$$\nabla f(z) = \sum_{j=1}^{n} D_j f(z) e_j, z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n,$$

and let

$$A(\Omega, f) = \int_\Omega |\nabla f(z)|^2 \, dm(z),$$

where D_j denotes $\partial/\partial z_j$, e_j denotes the unit vector in $\mathbf{C}^n$ whose j-th component is 1 and $m = m_{2n}$ denotes ordinary Lebesgue measure on $\mathbf{C}^n$ which is topologically identified with the Euclidean space $\mathbf{R}^{2n}$. We simply denote $A(\Omega, f)$ by $A(\rho, f)$ in case $\Omega = \rho B \equiv \{z \in \mathbf{C}^n : |z| < \rho\}$, $0 < \rho \leqq 1$.

Denote by $\mathcal{U}$ the group of all unitary operators on $\mathbf{C}^n$. For a subdomain Ω of B, we say, by definition, that a function f defined in B satisfies "Lusin property with respect to Ω" if

$$\int_{\mathcal{U}} A(U\Omega, f) \, dU < \infty,$$

where $U\Omega = \{Uz : z \in \Omega\}$ and dU denote the Haar measure on $\mathcal{U}$. See [6] and [8] for Lusin property.

It is known that if $f \in H^p(U)$ [2] for some $p : 0 < p \leqq 2$ then $A(\rho, f) = o(1-\rho)^{-2/p}$, $\rho \longrightarrow 1$, and the result breaks down when $p > 2$ [3], [9]. Also known is that if $f \in H^p(U)$ then f satisfies the Lusin property with respect to triangular

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subdomains of U, and conversely [6]. Main concern of this note is to extend these one variable results under the following setting.

Let $\phi(r)$ be a positive decreasing differentiable function defined on [0, 1) with $\phi(0) = 1$, $\phi(1-) = 0$ and extended to B via $\phi(z) = \phi(|z|)$. Let S denote the boundary of B and let $(\chi_\Omega)^\#$ denote the radialization of the characteristic function χ_Ω [3. p. 49], i.e.

$$(\chi_\Omega)^\#(z) = \int_{\mathcal{U}} \chi_\Omega(Uz)\, dU, z \in B.$$

Let $D(\phi)$ be the family of all subdomains Ω of B such that

(1.1) the boundary $\partial\Omega$ of Ω satisfies that $\partial\Omega \cap S = e_1 = (1, 0, \ldots, 0)$, and

(1.2) there exists r_o such that for r, $r_o < r < 1$,

$$(\chi_\Omega)^\#(re_1) \approx \int_r^1 \phi(\rho)\, d\rho,$$

where and hereafter $a(z) \approx b(z)$ means that there are positive constants A and B independent of z of the given domain such that $Aa(z) < b(z) < Ba(z)$. It is not difficult to see that if f satisfies the Lusin property with respect to some $\Omega \in D(\phi)$ then it also satisfies the property with respect to the other $\Omega \in D(\phi)$ [See (3.5)]. We denote $f \in LP(\phi)$ if f satisfies the Lusin property with respect to some $\Omega \in D(\phi)$.

For $0 < p < \infty$, $H^{p,\phi}(B)$ and $A_{p,\phi}(B)$ are defined to be the spaces of those holomorphic functions f in B respectively for which

$$\max\{\|f\|_p, |||f|||_\phi\} < \infty \tag{1.3}$$

and

$$\|f\|_{p,\phi} < \infty, \tag{1.4}$$

where

$$\|f\|_p = \sup_{0 \leqq r < 1} \left\{ \int_S |f(r\zeta)|^p\, d\sigma(\zeta) \right\}^{1/p},$$

$$|||f|||_\phi = \sup_{z \in B} \phi(z)|f(z)|,$$

and

$$\|f\|_{p,\phi} = \left\{ -\int_0^1 d\phi(r) \int_S |f(r\zeta)|^p\, d\sigma(\zeta) \right\}^{1/p}.$$

Here σ denote the rotation invariant probability measure on S.

Our $H^{p,\phi}$ version on the growth of area is as follows.

THEOREM. *Let* $0 < p \leqq 2$. *If*

$$f \in H^{p,\phi}, \tag{1.5}$$

then

$$\int_0^1 \phi(\rho)^{2-p} A(\rho, f)\, d\rho < \infty, \tag{1.6}$$

and

$$f \in LP(\phi^{2-p}). \tag{1.7}$$

Conversely, let $2 \leqq p < \infty$ *and let* f *be holomorphic in* B *with* $f(z) = O(\phi(z)^{-1})$. *Then* (1.6) *or* (1.7) *implies* (1.5).

This theorem, of course, has a corollary concerning little (big) "o" argument. Because the proofs are identical, we state it only for $H^p(B_n)$ case.

COROLLARY. *Let* $0 < p \leqq 2 \leqq q < \infty$, *and let* f *be holomorphic in* B_n.

$$\text{If } f \in H^p(B_n) \quad \text{then } A(\rho, f) = o(1-\rho)^{-1-n(2-p)/p}. \tag{1.8}$$

(1.9) *If* $A(\rho, f) = O(1-\rho)^{-\gamma}$ *for some* $\gamma, 0 < \gamma < 1 + n(2-q)/q$ *then* $f \in H^q(B_n)$.

2. **Weighted subspaces of Hardy spaces.** Note that $H^{p,\phi}$ and $A_{p,\phi}$ are complete topological vector spaces equipped with the translation invariant metric appeared in (1.3) and (1.4).

LEMMA 1. *Let* $0 < p < q < \infty$. *Then there is the continuous inclusion*

$$H^{p,\phi}(B_n) \subset A_{q,\phi^{q-p}}(B_n). \tag{2.1}$$

The case where $n = 1$ and $\phi(r) = (1-r)^\gamma$, $0 < \gamma \leqq 1/p$ was proved by P. Ahern and appeared in [4. Theorem B]. The inclusion (2.1) cannot be improved to a fully Hardy-Littlewood type [5. Theorem 2]. The proof of Lemma 1 presented below is rather elementary but different from that of one variable case of Ahern. We present it for the sake of completeness.

PROOF. Let $f \in H^{p,\phi}$ and $\alpha = q-p$. We may assume f is nonconstant and $f(0) = 0$. Denote $f_\zeta(\lambda) = f(\zeta\lambda)$, $\lambda \in U$ and $f_\zeta(\theta)$ the radial limit of f_ζ at $e^{i\theta}$. Note that $f_\zeta \in H^p(U)$ a.e. ζ. If we set

$$(f_\zeta)(p, \lambda) = |f_\zeta|^{p-2} |(d/d\lambda)(f_\zeta)|^2(\lambda), \lambda \in U, \quad \text{and} \tag{2.2}$$

$$A_p(\rho, f_\zeta) = \int_{\rho U} (f_\zeta)(p, \lambda)\, dm_2(\lambda),$$

then the Green's formula followed by a familiar limiting process gives that

$$\int_0^{2\pi} |f_\zeta(\theta)|^p d\theta = p^2 \int_0^1 \rho^{-1} A_p(\rho, f_\zeta) d\rho, \text{ a.e. } \zeta. \tag{2.3}$$

Since $f \in H^{p,\phi}$,

$$(f_\zeta)(p, \lambda) \geqq [\phi(\lambda)^{-1} |||f|||_\phi]^{-\alpha} (f_\zeta)(q, \lambda), \zeta \in S, \lambda \in U,$$

iot follows from (2.3) that

$$\begin{aligned} \int_0^{2\pi} |f_\zeta(\theta)|^p d\theta &\geqq p^2 |||f|||_\phi^{-\alpha} \int_0^1 \rho^{-1} \phi(\rho)^\alpha A_q(\rho, f_\zeta) d\rho \\ &= -p^2 |||f|||_\phi^{-\alpha} \int_0^1 d\phi(r) \int_0^r \rho^{-1} A_q(\rho, f_\zeta) d\rho \end{aligned} \tag{2.4}$$

a.e. $\zeta \in S$. Now, another application of the Green's formula as in (2.3) gives

$$q^2 \int_0^r \rho^{-1} A_q(\rho, f_\zeta) d\rho = \int_0^{2\pi} |f_\zeta(re^{i\theta})|^q d\theta. \tag{2.5}$$

Inserting (2.5) into the right hand side of (2.4) and then integrating (2.4) with respect to $d\sigma$ therefore gives

$$|||f|||_\phi^\alpha \|f\|_p^p \geqq -\frac{p^2}{q^2} \int_0^1 d\phi^\alpha(r) \int_S |f(r\zeta)|^q d\sigma(\zeta).$$

This completes the proof. □

3. **Proofs.** Our proof depends essentially on the following elementary lemma.

LEMMA 2. *For holomorphic f in B, the following are equivalent.*

$$f \in A_{2,\phi}(B), \tag{3.1}$$

$$\int_0^1 \phi(\rho) A(\rho, f) d\rho < \infty, \tag{3.2}$$

$$f \in LP(\phi). \tag{3.3}$$

PROOF OF LEMMA 2. If f is a monomial z^α, $\alpha \geqq 0$, then it follows from [7. pp. 16–17] that

$$\int_S |f(r\zeta)|^2 d\sigma(\zeta) = |f(0)|^2 + \frac{2(n-1)!}{\pi^n} \int_0^r \rho^{1-2n} A(\rho, f) d\rho. \tag{3.4}$$

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By orthogonality, (3.4) holds for polynomials, and hence for holomorphic f. Now, integrating (3.4) with respect to $d\phi$,

$$\|f\|_{2,\phi} = |f(0)|^2 - \frac{2(n-1)!}{\pi^n}\int_0^1 d\phi(r)\int_0^r \rho^{1-2n}A(\rho, f)d\rho.$$

While, the last integral is

$$\frac{2(n-1)!}{\pi^n}\int_0^1 \rho^{1-2n}\phi(\rho)A(\rho,f)d\rho,$$

so that the equivalence of (3.1) and (3.2) follows.

Next, let $\Omega \in D(\phi)$. By (1.2),

$$\int_0^1 \chi_{\{|z|<\rho\}}(z)\phi(\rho)d\rho = \int_{|z|}^1 \phi(\rho)d\rho$$

$$\approx \int_{\mathcal{U}} \chi_\Omega(Uz)dU, \text{for } |z| \text{ close to } 1,$$

where $\chi_{\{.\}}(z)$ of course denote characteristic functions. Hence it follows that

$$\begin{aligned}\text{(3.5)}\qquad \int_0^1 \phi(\rho)A(\rho, f)d\rho &= \int_0^1 \phi(\rho)d\rho\int_{\rho B}|\nabla f(z)|^2dm(z)\\ &= \int_B |\nabla f(z)|^2\left[\int_{|z|}^1 \phi(\rho)d\rho\right]dm(z)\\ &\approx \int_{\mathcal{U}} dU\int_{U\Omega}|\nabla f(z)|^2dm(z).\end{aligned}$$

The equivalence of (3.2) and (3.3) follows from (3.5). □

PROOF OF THEOREM. The first part follows directly from Lemmas 1 and 2. For the converse, suppose $2 \leqq p < \infty$ and f is holomorphic in B with $|||f|||_\phi < \infty$. It suffices to prove that (1.6) implies (1.5). We may assume $f(0) = 0$. Recall $(f_\zeta)(p,\lambda)$ in (2.2). Since

$$(f_\zeta)(p,\lambda) \leqq [\phi(\lambda)^{-1}|||f|||_\phi]^{p-2}|f_\zeta(\lambda)|^2, \zeta \in S, \lambda \in U,$$

it follows by (2.5) that

$$\begin{aligned}\text{(3.6)}\qquad \int_0^{2\pi}|f_\zeta(re^{i\theta})|^p d\theta &= p^2\int_0^r \rho^{-1}d\rho\int_{\rho U}(f_\zeta)(p,\lambda)dm_2(\lambda)\\ &\leqq p^2|||f|||_\phi^{p-2}\int_0^r \rho^{-1}\phi(\rho)^{2-p}d\rho\int_{\rho U}|(d/d\lambda)f_\zeta(\lambda)|^2dm_2(\lambda).\end{aligned}$$

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Now, it is not difficult to see that

$$\int_S d\sigma(\zeta) \int_{\rho U} |(d/d\lambda)f_\zeta(\lambda)|^2 dm_2(\lambda) = \frac{(n-1)!}{\pi^{n-1}} \frac{A(\rho, f)}{\rho^{2n-2}},$$

so that slice integration makes (3.6) into

$$\|f\|_p^p \leqq C(p,n) \||f\||_\phi^{p-2} \int_0^1 \rho^{1-2n} \phi(\rho)^{2-p} A(\rho, f) d\rho,$$

which completes the proof. □

PROOF OF COROLLARY. Let $\phi(r) = (1-r)^\tau$, $\tau > 0$. Since $A(\rho, f)$ is nondecreasing function of ρ,

$$(3.7) \qquad (1-\rho)^{1+\tau} A(\rho, f) \leqq (1+\tau) \int_\rho^1 (1-r)^\tau A(r, f) dr.$$

If $\tau = n/p$ then $H^{p,\phi}(B_n) = H^p(B_n)$ [7. Theorem 7.2.5], so that all $f \in H^p(B_n)$ satisfy, by Theorem and (3.7), that $A(\rho, f) = o(1-\rho)^{-1-n(2-p)/p}$. This proves (4.1).

For the converse, suppose $A(\rho, f) = O(1-\rho)^{-\gamma}$ for some γ, $0 < \gamma < 1$. Then obviously,

$$(3.8) \qquad \int_0^1 (1-\rho)^{-\delta} A(\rho, f) d\rho < \infty \quad \text{for } \delta < 1 - \gamma.$$

In particular, by (3.4), $f \in H^2(B_n)$, so that

$$(3.9) \qquad f(z) = O(1-|z|)^{-n/2}.$$

Hence it follows from (3.8), (3.9), and Theorem that f is a member of $H^{q_1}(B_n)$, $q_1 = 2 + 2\delta/n$, and this in turn gives $f(z) = O(1-|z|)^{-n/q_1}$. Continuing this way using (3.8), we conclude by induction that $f \in H^q(B_n)$ for all $q < 2n/(n-1+\gamma)$. Now, (1.9) follows. □

4. **Remarks.**

(1) We present the following example as for the sharpness of Theorem 1; If $n = 2$, $\phi(r) = 1 - r$, and $(3/2) < p < 2$, then there exists $f \in H^{p,\phi}$ such that

$$(4.1) \qquad \int_0^1 \phi(\rho)^\alpha A(\rho, f) d\rho = \infty \quad \text{for } \alpha > 2 - p.$$

(So that $f \in LP(\phi^\alpha)$ for $\alpha > 2 - p$).

Fix a p, $3/2 < p < 2$. Let

$$d_k = [k^{1-p}(\log k)^{1/(p-1)}]^{1/(2-p)}, \quad k = 2, 3, \ldots.$$

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Then $\sum d_k < \infty$, and $\sum (d_k/k)^s < \infty$ if and only if $s \geqq 2 - p$. Let $g(z)$ be the Blaschke product formed by $\{1 - d_k\}$. Then it follows from [1. Theorem 6.2] that

$$g' \in H^{p-1}(U) \text{ and } g' \in H^q(U) \text{ for } q > p - 1. \tag{4.2}$$

Since $g'(z) = O(1 - |z|)^{-1}$, g' is a member of $H^{p-1,\phi}(U)$, and so by Lemma 1,

$$g' \in L^p(U), \tag{4.3}$$

Now, let $f(z, w) = g'(z)$, $(z, w) \in B_2$. Then, by [7. p. 15] and (4.3),

$$\|f\|_p^p = \int_U |g'(z)|^p d\nu_1(z) < \infty.$$

Hence $f \in H^{p,\phi}(B_2)$. On the other hand, since

$$\begin{aligned} \|f\|^2_{2,\phi^\alpha} &= \alpha \int_0^1 (1-r)^{\alpha-1} dr \int_U |g'(rz)|^2 d\nu_1(z) \\ &= \alpha \int_U \left[\int_{|z|}^1 (1-r)^{\alpha-1} r^{-2} dr \right] |g'(z)|^2 d\nu_1(z) \\ &\approx \int_U (1-|z|)^\alpha |g'(z)|^2 d\nu_1(z) \end{aligned}$$

for $\alpha > 0$, by (4.2) and [1. Theorem 6.2 a) $\leftrightarrow$ c)], the last integral is finite if and only if $\alpha \leqq 2 - p$. Therefore we conclude (4.1) by Lemma 2.

(2) $f(z) = (1 - z_1)^{-\tau}$ for appropriate positive constants τ show that the exponents of Corollary are best possible.

(3) If $f \in (LH)^p(B_n)$ i.e. $|f|^p$ has pluriharmonic majorants [5. p. 145] then $f(z) = O(1 - |z|)^{-1/p}$, so that by Theorem $A(\rho, f) = o(1 - \rho)^{-2/p}$.

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