On the deformation of the tangent m-plane of a V_n^m

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1. Schouten and van Kampen (1)[1] have studied the deformation of a V_n^m. Applying the methods of that paper to the tangent vectors B_i^λ ($\lambda, \mu, \nu, \ldots = 1, 2, \ldots, n$; $i, j, k, \ldots = 1, 2, \ldots, m$), which exist by hypothesis at all points of a certain region $V_{m'}$ ($m' > m$) of V_n, we shall have

$$(1) \qquad \begin{cases} \overset{1}{d} B_i^\lambda = \quad v^\mu \, dt \, \partial_\mu B_i^\lambda \\ \overset{2}{d} B_i^\lambda = -\Gamma_{\mu\nu}^\lambda B_i^\mu v^\nu \, dt, \qquad \Gamma_{\mu\nu}^\lambda = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \\ \overset{3}{d} B_i^\lambda = \quad B_i^\mu \partial_\mu v^\lambda \, dt \end{cases}$$

whence we define the differentials

$$(2) \qquad \overset{r}{d} - \overset{s}{d} = \overset{(r,s)}{\delta} = \overset{(r,s)}{D} dt \qquad (r, s = 1, 2, 3 \text{ with } r \neq s).$$

In the application of the $\overset{(r,s)}{D}$ to B_i^λ the lower index is treated as an ordinal index only. We shall not be concerned with any extension of the $\overset{(r,s)}{D}$ to indices other than those of the general V_n (see **1**, equ. 3.24).

We here consider the application of $\overset{(r,s)}{D}$ to the m-vector $B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]}$ determining the m-plane tangent to the facet V_n^m at any point. This gives

$$(3) \quad \overset{(r,s)}{D} (B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]}) = (\overset{(r,s)}{D} B_1^{[\lambda_1}) B_2^{\lambda_2} \ldots B_m^{\lambda_m]} + B_1^{[\lambda_1} (\overset{(r,s)}{D} B_2^{\lambda_2}) \ldots B_m^{\lambda_m]} + \ldots + B_1^{[\lambda_1} \ldots . B_{m-1}^{\lambda_{m-1}} (\overset{(r,s)}{D} B_m^{\lambda_m]}).$$

Since the vector $\overset{(r,s)}{D} B_i^\lambda$ will not be perpendicular to the B_i^λ in general, it will have a component in the tangent m-plane, and a component perpendicular to it. Consequently from (3) the m-vector $\overset{(r,s)}{D} B_1^{[\lambda_1} \ldots B_m^{\lambda_m]}$ will have two components respectively parallel and perpendicular to the tangent m-plane. We shall study those components in a few special cases.

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[1] These numbers refer to the list of papers at the end.

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The component of the vector $\overset{(r,s)}{D} B_1^{\lambda_1}$ in the tangent m-plane is $B_\nu^{\lambda_1} \overset{(r,s)}{D} B_1^\nu$, which we may write as

$$(4) \qquad (B_\nu^1 \overset{(r,s)}{D} B_1^\nu)\, B_1^{\lambda_1} + (B_\nu^2 \overset{(r,s)}{D} B_1^\nu)\, B_2^{\lambda_1} + \ldots . + (B_\nu^m \overset{(r,s)}{D} B_1^\nu)\, B_m^{\lambda_1},$$

and on inserting this series in the right-hand side of (3), it is evident that only the first term contributes anything, giving therefore

$$(B_\nu^1 \overset{(r,s)}{D} B_1^\nu)\, B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]}.$$

Treating the vector $\overset{(r,s)}{D} B_2^{\lambda_2}$ in the same way, giving an expression corresponding to (4) and inserting in (3), we shall have

$$(B_\nu^2 \overset{(r,s)}{D} B_2^\nu)\, B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]},$$

so that, on treating all the other vectors in the same way, we conclude that the total component of (3) in the original m-plane can be written in the form of the single m-vector

$$(5) \qquad (B_\nu^i \overset{(r,s)}{D} B_i^\nu) B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]}.$$

Let us now put for brevity

$$(6) \qquad P_j^{\lambda_j} = C_\nu^{\lambda_j} \overset{(r,s)}{D} B_j^\nu ;$$

then the component of (3) perpendicular to the tangent m-vector can be written as the sum of m such m-vectors:—

$$P_1^{[\lambda_1} B_2^{\lambda_2} \ldots . B_m^{\lambda_m]} + B_1^{[\lambda_1} P_2^{\lambda_2} \ldots . B_m^{\lambda_m]} + \ldots . + B_1^{[\lambda_1} \ldots . B_{m-1}^{\lambda_{m-1}} P_m^{\lambda_m]}$$

so that (3) becomes

$$(7) \quad \overset{(r,s)}{D} B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]} = (B_\nu^i \overset{(r,s)}{D} B_i^\nu)\, B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]} + \sum_{j=1}^{m} B_1^{[\lambda_1} \ldots . P_j^{\lambda_j} \ldots . B_m^{\lambda_m]}.$$

2. Let us now take the special case $\overset{(r,s)}{D} = \overset{(1,2)}{D}$, and $v^\lambda = B_\kappa^\lambda u^\kappa$. Then

$$(8) \qquad B_\nu^i \overset{(1,2)}{D} B_i^\nu = B_\nu^i\, D_\kappa B_{(i)}^\nu = \sum_{i=1}^{m} \left\{ \begin{matrix} i \\ i\ \kappa \end{matrix} \right\} u^\kappa$$

where the index (i) is ordinal.

If the metric coefficients of the V_n^m are $b_{ij} = a_{\lambda\mu} B_{ij}^{\lambda\mu}$, with $b = |b_{ij}|$, then it is a well-known fact that

$$(9) \qquad \sum_{i=1}^{m} \left\{ \begin{matrix} i \\ i\ \kappa \end{matrix} \right\} = \frac{\partial \log \sqrt{b}}{\partial \eta^\kappa},$$

where η has the meaning used in 1, so that $u^\kappa dt = d\eta^\kappa$ and

$$(10) \qquad \sum_{i=1}^{m} \left\{ {i \atop i\ \kappa} \right\} = \frac{d \log \sqrt{b}}{dt}.$$

Also for this case

$$(11) \qquad P_j^{\lambda_j} = C_\nu^{\lambda_j} D_\kappa B_{(j)}^\nu = H_{j\kappa}^{\cdot\cdot\lambda} u^\kappa$$

where $H_{j\kappa}^{\cdot\cdot\lambda_j}$ is the first tensor of Eulerian Curvature. Using (8), (10) and (11) we have

$$(12) \qquad \overset{(1,2)}{D} B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]} = \frac{d}{dt} \log \sqrt{b}.\, B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]} + u^\kappa \sum_{j=1}^{m} B_1^{[\lambda_1} \ldots . H_{j\kappa}^{\cdot\cdot\lambda_j} \ldots . B_m^{\lambda_m]}.$$

A sufficient condition for V_n^m to be geodesic is that $H_{j\kappa}^{\cdot\cdot\lambda_j} = 0$, in which case (12) becomes

$$(13) \qquad \overset{(1.2)}{D} B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]} - \frac{d}{dt} \log \sqrt{b}\, B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]} = 0.$$

If, instead of a V_n^m, we have a single curve V_1, and if the displacement $d\xi^\lambda = v^\lambda dt$ is along the tangent, then by taking t to be the parameter of the curve, the $B_1^\lambda = \dfrac{d\xi^\lambda}{dt}$ becomes the (non-unit) tangent vector, so that, if s is the length of arc reckoned from a certain point, then

$$\sqrt{b} = \sqrt{b_{11}} = \sqrt{\left(a_{\lambda\mu} \frac{d\xi^\lambda}{dt} \frac{d\xi^\mu}{dt}\right)} = \frac{ds}{dt},$$

and consequently

$$\frac{d \log \sqrt{b}}{dt} = \frac{d \log \dfrac{ds}{dt}}{dt} = \frac{\dfrac{d^2 s}{dt^2}}{\dfrac{ds}{dt}},$$

and (13) becomes

$$(14) \qquad \frac{d^2 \xi^\mu}{dt^2} + \left\{ {\mu \atop \rho\ \nu} \right\} \frac{d\xi^\rho}{dt} \frac{d\xi^\nu}{dt} - \frac{\dfrac{d^2 s}{dt^2}}{\dfrac{ds}{dt}} \frac{d\xi^\mu}{dt} = 0,$$

which is the well-known form for the equation of geodesic lines when t is not the arc length.

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3. If $\overset{(r,s)}{D} = \overset{(3,2)}{D}$, we shall have $\overset{(3,2)}{D} B_i^\lambda = D_i v^\lambda$, and consequently

$$B_\nu^i \overset{(3,2)}{D} B_i^\nu = B_\nu^i D_i v^\nu = \tfrac{1}{2} b^{ij} B_{ij}^{\lambda\mu} \underset{L}{D} a_{\lambda\mu},$$

where $\underset{L}{D}$ is the "Lie derivative" (1).

If $\overset{m}{d\tau}$ is the element of volume in the local R_m, with (see 1, formula 3.50)

$$\overset{m}{d\tau} = (d\eta)^1 (d\eta)^2 \ldots . (d\eta)^m \sqrt{b},$$

then the deformation of that element of volume is of amount

$$\overset{m}{d\tau} \tfrac{1}{2} b^{ij} B_{ij}^{\lambda\mu} \underset{L}{\delta} a_{\lambda\mu} = \overset{m}{d\tau} B_\nu^i D_i v^\nu dt,$$

so that if $\overset{m}{d\tau'}$ denotes the deformed volume element, we can write

$$\frac{\overset{m}{d\tau'}}{\overset{m}{d\tau}} = 1 + B_\nu^i D_i v^\nu dt, \tag{15}$$

$B_\nu^i D_i v^\nu$ being therefore the "dilatation" at the point. If we are dealing with a curve, whose element of arc is ds and whose unit tangent vector is $i^\lambda = \dfrac{d\xi^\lambda}{ds}$, then if ds' is the element of arc of the deformed curve, the above equation becomes

$$\frac{ds'}{ds} = 1 + \phi dt, \text{ with } \phi = i_\lambda \frac{D}{Ds} v^\lambda. \tag{16}$$

If $C_\nu^{\lambda j} \overset{(3,2)}{D} B_j^\nu = 0$, $(j = 1, 2, \ldots, m)$, then the deformed m-vector $B_1^{[\lambda_1} \ldots . B_m^{\lambda_m]}$ will be parallel to the original one. For a curve, the resulting equations

$$\overset{(3,2)}{D} i^\lambda = \phi i^\lambda \tag{17}$$

determine the parallel-tangent deformations (Hayden, 2).

4. We pass now to the component of the m-vector (3) perpendicular to the tangent plane. For this purpose we remark that if $d\alpha$ is the angle between the two m-vectors v and $v+wdt$, and if $w=w'+w''$, where w' is parallel to v and w'' is perpendicular to it, then

$$\frac{\sin d\alpha}{dt} = \frac{|w''|}{|v|}, \tag{18}$$

where $|v|$ denotes the "measure" (3, Chapter 1) of the m-vector v.

Applying this to the case where v and w have the components

$$v^{\lambda_1 \cdots \lambda_m} = B_1^{[\lambda_1} \dots . B_m^{\lambda_m]} \text{ and } w^{\lambda_1 \cdots \lambda_m} = \overset{(r,s)}{D} B_1^{[\lambda_1} \dots . B_m^{\lambda_m]}$$

we obtain on simplification

$$\frac{\sin d\alpha}{dt} = \sqrt{(b^{ij} a_{\lambda\mu} P_i^\lambda P_j^\mu)}. \tag{19}$$

If $\overset{(r,s)}{D} = \overset{(1,2)}{D}$, and if we are dealing with a curve, with $v^\lambda = \dfrac{d\xi^\lambda}{dt}$ as the tangent vector, the P_i^λ reduces to the left-hand side of equation (14), so that the invariant $\sin d\alpha/dt$ is the first curvature of the curve. For an isolated V_m in general, with $\overset{(r,s)}{D} = \overset{(1,2)}{D}$ and $v^\lambda = B_i^\lambda u^i$, the right-hand side of (19) becomes the square root of the "forma angolare" of Bortolotti, which can therefore be regarded as the first curvature of the V_m in V_n. The particular form taken by (19) in that case has already been given (4, p. 294), together with its expression when $\overset{(r,s)}{D} = \overset{(3,2)}{D}$ and $v^\lambda = C_p^\lambda w^p$.

1. J. A. Schouten and E. R. van Kampen, "Beiträge zur Theorie der Deformation," *Prace Mat. Fiz. Warszawa*, **41** (1933), 1-19.

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3. E. Cartan, "Leçons sur la Géométrie des Espaces de Riemann," *Paris, Gauthier-Villars*, 1928.

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