SOME CONTINUITY PROPERTIES OF LINEAR TRANSFORMATIONS IN NORMED SPACES

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Let X and Y be normed spaces and let $L(X, Y)$ denote the set of linear transformations (henceforth called "operators") T with domain a linear subspace $D(T)$ of X and range $R(T)$ contained in Y. The restriction of T to a subspace E is denoted by $T \mid E$; by the usual convention $T \mid E = T \mid E \cap D(T)$. For a given linear subspace E the family of infinite dimensional subspaces of E is denoted by $\mathscr{I}(E)$. An operator T is said to have a certain property $\mathbb{P}$ *ubiquitously* if every $E \in \mathscr{I}(X)$ contains an $F \in \mathscr{I}(E)$ for which $T \mid F$ has property $\mathbb{P}$. For example, T is ubiquitously continuous if each $E \in \mathscr{I}(X)$ contains an $F \in \mathscr{I}(E)$ for which $T \mid F$ is continuous. In the present note we shall characterize ubiquitous continuity, isomorphy, precompactness and smallness. A subspace of X is called a *principal* subspace if it is closed and of finite codimension in X. The restriction of an operator to a principal subspace will be called a *principal* restriction. The symbol T will always denote an arbitrary operator in $L(X, Y)$.

LEMMA 1. *If M and E are subspaces of X and if* $\operatorname{codim} E < \infty$, *then* $M = M \cap E \oplus F$ *for some finite dimensional subspace F.*

Proof. The map of $M / M \cap E$ into X / E given by $m + M \cap E \to m + E$ $(m \in M)$ is injective. □

A restriction $T \mid M$ of T is said to be *nontrivial* if $M \cap D(T)$ is infinite dimensional.

LEMMA 2. *The operator T has a principal restriction having a continuous inverse if and only if T has no nontrivial precompact restriction.*

Proof. The "if" part is contained in the Kato–Goldberg result [**3**, p. 80]. For the converse, suppose that M is a principal subspace for which $T \mid M$ has a continuous inverse and let E be a subspace such that $T \mid E$ is precompact. Then $T \mid M \cap E$ is an isomorphism and hence $M \cap E \cap D(T)$ is finite dimensional. Therefore $E \cap D(T)$ is finite dimensional by Lemma 1. □

COROLLARY 3. (See [**2**].) *Any two norms defined on an infinite dimensional linear space are comparable on some infinite dimensional subspace.*

Proof. Consider the appropriate identity map. □

With a given operator T we associate the *graph operator* G of T as follows. Let X_T be the linear space $D(T)$ normed by $\|x\|_T = \|x\| + \|Tx\|$ and define the operator $G : X_T \to X$ by $Gx = x (x \in X_T)$. Observe that T is continuous if and only if G is an isomorphism.

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THEOREM 4. *The operator T is ubiquitously continuous if and only if T is continuous on some subspace of finite codimension.*

Proof. We may clearly suppose $\dim D(T) = \infty$. Suppose T is ubiquitously continuous. Let $E \in \mathcal{I}(D(T))$. There exists $F \in \mathcal{I}(E)$ making $T \mid F$ continuous. Then $G^{-1} \mid F$ is an isomorphism. Consequently G has no nontrivial precompact restriction. Hence by Lemma 2 a principal subspace M of X_T exists for which $G \mid M$ is an isomorhism. Then $T \mid GM$ is continuous. If N is a subspace of X complementary to $D(T)$ then $GM \oplus N$ is a finite codimensional subspace upon which T is continuous.

Conversely let $T \mid E$ be continuous where $\operatorname{codim} E < \infty$ and let $M \in \mathcal{I}(X)$. Then $E \cap M \in \mathcal{I}(X)$ by Lemma 1, and $T \mid E \cap M$ is continuous. Thus T is ubiquitously continuous. □

THEOREM 5. *The following statements are equivalent.*

(i) *T is ubiquitously an isomorphism.*
(ii) *T is an isomorphism on some subspace of finite codimension.*
(iii) *T is continuous on some subspace of finite codimension and T has no nontrivial precompact restriction.*

Proof. The implication (i) ⇒ (iii) is immediate from Theorem 4. Assume (iii). Then by Lemma 2 there exists a finite codimensional subspace M of X for which $T \mid M$ has a continuous inverse. If E is a finite codimensional subspace making $T \mid E$ continuous, then $\operatorname{codim}(E \cap M) < \infty$ (Lemma 1) and $T \mid E \cap M$ is an isomorphism. Hence (iii) ⇒ (ii). The proof that (ii) ⇒ (i) is similar to the corresponding part of the proof of Theorem 4. □

PROPOSITION 6. *Let Z be a subspace of X. For each principal subspace M of Z there exists a principal subspace M_0 of X such that $M = M_0 \cap Z$.*

Proof. Let M be a principal subspace of Z. There exists a finite dimensional subspace F of Z such that $M \oplus F = Z$. Let $x_1, \ldots, x_n$ be a basis for F. Choose $f_1, \ldots, f_n \in X'$ such that $f_i(x_j) = \delta_{ij}$ and $f_i(m) = 0$ for $m \in M (i, j \leq n)$; this is possible since $F \cap \bar{M} = (0)$. Then $M_0 = \bigcap_{i \leq m} f_i^{-1}(0)$ is a principal subspace of X with $M = M_0 \cap Z$. □

LEMMA 7. *If $X = M \oplus N$, where M is a principal subspace, then the projection of X onto M with null space N is bounded.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis for N and let $N_i = \operatorname{sp}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. Since $M + N_i$ is closed for each i (see e.g. [**3**, p. 16]) there exists by the Hahn-Banach theorem an $f_i \in X'$ such that $f_i(x_i) = 1$ and $f_i(x) = 0$ for $x \in M + N_i$. Define $Q = \sum_{i \leq n} f_i \otimes x_i$. Then Q is a bounded projection with range N and null space M, and $P = I - Q$ is the required projection.

COROLLARY 8. *If there exists a principal subspace M for which $T \mid M$ is continuous, then T is continuous.*

A characterisation of bounded semi-Fredholm operators (ϕ_+-operators) between Banach spaces will now be given. An operator T is called a ϕ_+-operator if its null space $N(T)$ is finite dimensional and $R(T)$ is closed.

THEOREM 9. *Let X and Y be Banach spaces and let $T \in L(X, Y)$ be an everywhere defined (injective) operator. The following statements are equivalent.*

(i) *T is a bounded ϕ_+-operator (resp., an isomorphism).*
(ii) *T is bounded and ubiquitously an isomorphism.*
(iii) *T has a principal restriction which is an isomorphism.*

Proof. Assume (ii). Then by Lemma 2 there exists a principal subspace M making $T \mid M$ an isomorphism. Hence (ii) $\Rightarrow$ (iii).

Assume (iii). The implication (iii) $\Rightarrow$ (i) in the case when T is bounded is a well known classical result (cf. [**1**]). Hence, by Corollary 8, (iii) $\Rightarrow$ (i).

Assume (i). Then there exists a principal subspace M such that $X = M \oplus N(T)$. By the Closed Graph Theorem, $T \mid M$ is an isomorphism. Hence (i) $\Rightarrow$ (ii). □

THEOREM 10. *Let T be injective and everywhere defined on a Banach space. If T is bounded, and if T^{-1} is continuous on a subspace of finite codimension, then T is an isomorphism.*

Proof. Let T be bounded and let E be a finite codimensional subspace of Y such that $T^{-1} \mid E$ is continuous. Then $T^{-1}E$ has finite codimension in X, and $T \mid T^{-1}E$ has a continuous inverse. Lemma 1 now implies that T has no nontrivial precompact restriction. Hence, by Lemma 2, there is a principal subspace M of X for which $T \mid M$ is an isomorphism. But M is complete. Therefore TM is complete and hence is a principal subspace of $R(T)$ for which $T^{-1} \mid TM$ is continuous. Therefore T^{-1} is continuous by Corollary 8. □

To show that completeness is essential in Theorem 10 we give an example of a bounded everywhere defined operator which is not an isomorphism, yet has a principal restriction which is an isomorphism. Let f be a discontinuous linear functional with domain X, and let $G: X_f \to X$ be the graph operator associated with f. Then G^{-1} is unbounded. However, $G^{-1} \mid N(f)$ is an isometry. Since $\operatorname{codim} N(f) = 1$, it follows from Lemma 2 and Theorem 5 that G has a principal restriction which is an isomorphism.

COROLLARY 11. *Let X_T be complete. If T is ubiquitously continuous then T is continuous.*

Proof. If T is ubiquitously continuous, then G^{-1} is continuous on some subspace of finite codimension by Theorem 4. Hence, by Theorem 10, G is an isomorphism, or, equivalently, T is continuous. □

We remark that Corollary 11 fails without the completeness assumption; for example, every discontinuous linear functional is ubiquitously continuous.

If X and Y are complete, then X_T is complete if and only if T is closed. Hence we have the following corollary.

COROLLARY 12. *Let X and Y be complete. If T is closed and ubiquitously continuous, then T is continuous.*

Assuming that X and Y are Banach spaces and that T is everywhere defined and injective, L. Drewnowski **[1]** asks whether T is an isomorphism whenever it has the property that for each closed subspace $E \in \mathcal{I}(X)$ there exists $F \in \mathcal{I}(E)$ for which $T \mid \bar{F}$ is an isomorphism. The following example shows that if "closed subspace E" is replaced by "principal subspace E" in the above, then T need not be bounded.

EXAMPLE 13. *There exists an unbounded everywhere defined injective and surjective operator $T: l_2 \to l_2$ such that every principal subspace of l_2 contains an infinite dimensional closed subspace F such that $T \mid F$ is an isomorphism.*

Let M and N_1 be a pair of closed mutually orthogonal infinite dimensional subspaces of l_2 such that $M \dotplus N_1 = l_2$ (where $\dotplus$ denotes the orthogonal sum), and select a dense proper subspace N of N_1 so that $N_1 = K + N$ where K is one-dimensional (N will be the null space of a discontinuous linear functional on N_1). On $l_2 = M \dotplus (K + N)$ define P to be the projection of l_2 onto $M + K$ with null space N. P is unbounded since its null space $N(P)$ is not closed. Let $T = I + P$. Then T is unbounded. Also, $T \mid (M + N)$ is an isomorphism; indeed for $m \in M$, $n \in N$ we have $\|m + n\|^2 = \|m\|^2 + \|n\|^2 \leq \|2m\|^2 + \|n\|^2 = \|2m + n\|^2 = \|T(m + n)\|^2 \leq 4\,\|m + n\|^2$ so $\|m + n\| \leq \|T(m + n)\| \leq 2\,\|m + n\|$. Now let E be a principal subspace. Then $M = M \cap E \oplus W$ where $\dim W < \infty$ by Lemma 1. Hence $F = M \cap E$ is infinite dimensional and has the required property.

We shall now characterize ubiquitously precompact operators. Such an operator will be continuous on a finite codimensional subspace by Theorem 3. An operator T will be called *strictly singular* if there is no infinite dimensional subspace M of $D(T)$ for which $T \mid M$ has a continuous inverse; this is a generalisation of the classical definition (see **[4]**). Any discontinuous linear functional is an example of an unbounded strictly singular operator. We shall call T *ubiquitously small* if for each $\varepsilon > 0$ and each $E \in \mathcal{I}(X)$ there exists $F \in \mathcal{I}(E)$ such that $\|T \mid F\| < \varepsilon$ unless $F \cap D(T)$ is finite dimensional. The theorem below is a generalisation of III.2.1 of **[3]**.

THEOREM 14. *The following statements are equivalent.*

(i) *T is ubiquitously precompact,*

(ii) *T is ubiquitously small.*

(iii) *T is ubiquitously strictly singular.*

(iv) *T is strictly singular.*

Proof. Assume (i). By Theorem 3 there exists a finite codimensional subspace E for which $T \mid E$ is continuous. Since $T \mid E$ is ubiquitously precompact, there exists $F \in \mathcal{I}(E)$ such that $T \mid F$ is precompact. In particular, $T \mid F$ is continuous and precompact on its domain $F \cap D(T)$, and hence ubiquitously small on $F \cap D(T)$ by (**[3]**, loc. cit.). This shows that T is ubiquitously small. Thus (i) $\Rightarrow$ (ii) by Lemma 1. Next assume T is not strictly singular and let $M \in \mathscr{I}(D(T))$ be such that $T|M$ has a continuous inverse. Then $\|Tm\| \geq c\,\|m\|$ for some $c > 0$ and all $m \in M$. Therefore T is not ubiquitously small. Hence (ii) $\Rightarrow$ (iv). Next, assume T is not ubiquitously precompact. Then there exists an infinite dimensional subspace M such that $T|M$ has no precompact restriction to any infinite dimensional subspace of M. Lemma 2 now implies that $T|M \cap N$ has a continuous inverse for some principal subspace N of M. Since evidently $\dim(M \cap D(T)) = \infty$, it follows from Lemma 1 that $T|M$ (and hence T) is not strictly singular. Therefore (iv) $\Rightarrow$ (i). Finally, the equivalence of (i) and (iv) implies immediately the equivalence of (iii) and (iv). □

Theorem 14 implies in particular that the sum of two strictly singular operators is strictly singular, and that every strictly singular operator is continuous on some subspace of finite codimension.

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