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CORRIGENDUM

A spectral refinement of the Bergelson–Host–Kra decomposition and new multiple ergodic theorems – CORRIGENDUM

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Abstract. This is a corrigendum to the paper 'A spectral refinement of the Bergelson–Host–Kra decomposition and new multiple ergodic theorems' [**3**]. Theorem 7.1 in that paper is incorrect as stated, and the error originates with Proposition 7.5, part (iii), which was incorrectly quoted from a paper by Bergelson, Host, and Kra [**1**]. Consequently, this invalidates the proof of Theorem 4.2, which was used in the proofs of the main results in [**3**]. In this corrigendum we fix the problem by establishing a slightly weaker version of Theorem 7.1 (see §2 below) and use it to give a new proof of Theorem 4.2 (see §3 below). This ensures that all main results in [**3**] remain correct. We thank Zhengxing Lian and Jiahao Qiu for bringing this mistake to our attention.

1. *A counterexample to* [**3**, Theorem 7.1]

We begin by presenting the counterexample to [**3**, Theorem 7.1] provided to us by Zhengxing Lian and Jiahao Qiu. We will use common terminology about nilmanifolds and nilsystems as reviewed in [**3**, §3].

THEOREM 7.1. (From [**3**]) *Let $k \in \mathbb{N}$, let X be a connected nilmanifold and let $R : X \to X$ be an ergodic nilrotation. Define $S := R \times R^2 \times \cdots \times R^k$ and*

$$Y_x := \overline{\{S^n(x, x, \ldots, x) : n \in \mathbb{Z}\}} \subseteq X^k. \tag{1.1}$$

For almost every $x \in X$, $\sigma(Y_x, S) = \sigma(X, R)$.

Counterexample. Let $k = 2$ and let (X, R) be the skew-product system given by $R : (x, y) \mapsto (x + \alpha, y + x)$ on $\mathbb{T}^2$ for some irrational α. This system can be realized as an ergodic nilsystem (see [3, Example 7.2]). For any point $(x, y) \in X$ let $Y_{(x,y)}$ be the orbit closure of the diagonal point $(x, y, x, y) \in X^2$ under the map $S = R \times R^2$. Then

$$Y_{(x,y)} = \overline{\left\{\left(x + n\alpha, y + nx + \binom{n}{2}\alpha, x + 2n\alpha, y + 2nx + \binom{2n}{2}\alpha\right) : n \in \mathbb{N}\right\}}$$
$$= (x, y, x, y) + \overline{\left\{\left(n\alpha, nx + \binom{n}{2}\alpha, 2n\alpha, 2nx + 4\binom{n}{2}\alpha - n\alpha\right) : n \in \mathbb{N}\right\}}.$$

If $x, \alpha, 1$ are linearly independent over $\mathbb{Q}$ (which happens almost surely) then it follows that

$$Y_{(x,y)} = (x, y, x, y) + \{(z, w, 2z, \tilde{w}) : z, w, \tilde{w} \in \mathbb{T}\}. \tag{1.2}$$

Therefore the nilsystem $(Y_{(x,y)}, S)$ is isomorphic to the nilsystem $(\mathbb{T}^3, \tau_x)$, where $\tau_x(z, w, \tilde{w}) = (z + \alpha, w + z + x, \tilde{w} + 4z + 2x + \alpha)$. Consider the function $f : \mathbb{T}^3 \to \mathbb{C}$ described by $f(z, w, \tilde{w}) = e(\tilde{w} - 4w)$, where $e(z) := e^{2\pi i z}$. Then

$$f(\tau_x(z, w, \tilde{w})) = e((\tilde{w} + 4z + 2x + \alpha) - 4(w + z + x)) = e(\alpha - 2x) f(z, w, \tilde{w}).$$

This shows that $\alpha - 2x$ is an eigenvalue of the system $(Y_{(x,y)}, S)$, but not of the system (X, R), so $\sigma(Y_{(x,y)}, R \times T^2) \not\subseteq \sigma(X, S)$ for almost every $(x, y) \in X$.

2. *Revised version of* [3, Theorem 7.1]

The above example shows that [3, Theorem 7.1] is not correct as stated. Here is a corrected version.

REVISED THEOREM 7.1. *Let $k \in \mathbb{N}$, let X be a connected nilmanifold and let $R : X \to X$ be an ergodic nilrotation. Define $S := R \times R^2 \times \cdots \times R^k$ and*

$$Y_x := \overline{\{S^n(x, x, \ldots, x) : n \in \mathbb{Z}\}} \subseteq X^k. \tag{2.1}$$

For any $\theta \in [0, 1)$, if $\theta \notin \sigma(X, R)$ then for almost every $x \in X$ we have $\theta \notin \sigma(Y_x, S)$.

Remark 2.1. The difference between the (incorrect) statement of Theorem 7.1 in [3] and the (correct) statement of Revised Theorem 7.1 above is that

'for almost every $x \in X$ and all $\theta \notin \sigma(X, R)$ one has $\theta \notin \sigma(Y_x, S)$'

has been replaced with

'for all $\theta \notin \sigma(X, R)$ and almost all $x \in X$ one has $\theta \notin \sigma(Y_x, S)$'.

In other words, the full measure set of x is now allowed to depend on θ.

Proof of Revised Theorem 7.1. Given a nilpotent Lie group G, denote by $G = G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_s \trianglerighteq \{1_G\}$ its lower central series. For $k \in \mathbb{N}$, define $H^{(1)}(G), \ldots, H^{(k-1)}(G)$ as

$$H^{(i)}(G) := \{(g^{\binom{1}{i}}, g^{\binom{2}{i}}, \ldots, g^{\binom{k}{i}}) : g \in G_i\} \subseteq G^k, \tag{2.2}$$

where $\binom{j}{i} = 0$ for $j < i$, and let $H(G)$ be given by

$$H(G) := H^{(1)}(G)H^{(2)}(G) \cdots H^{(k-1)}(G)G_k^k. \tag{2.3}$$

Also, for a co-compact lattice $\Gamma \subset G$ define $\Delta(G, \Gamma) := H(G) \cap \Gamma^k$. Since $H(G)$ is a rational subgroup of G^k, it follows from [2, Lemma 1.11] that $\Delta(G, \Gamma)$ is a uniform and discrete subgroup of $H(G)$. Define the nilmanifold $Y(G, \Gamma) := H(G)/\Delta(G, \Gamma)$. Note that we can naturally identify $Y(G, \Gamma)$ with a subnilmanifold of $(G/\Gamma)^k$.

For $b \in G$, define $R_b : G/\Gamma \to G/\Gamma$ to be the map $R_b(g\Gamma) = (bg)\Gamma$ and let

$$S_b := R_b \times R_b^2 \times \cdots \times R_b^k. \tag{2.4}$$

For $x = g\Gamma \in G/\Gamma$ define

$$Y_x := \overline{\{S_b^n(x, x, \ldots, x) : n \in \mathbb{Z}\}} \subseteq (G/\Gamma)^k. \tag{2.5}$$

It was shown in [3, Proposition 7.5, part (iv)] that for almost every $x = g\Gamma \in G/\Gamma$ the map $R_{g^{-1}} \times \cdots \times R_{g^{-1}} : (G/\Gamma)^k \to (G/\Gamma)^k$ is an isomorphism from the nilsystem (Y_x, S_a) to the nilsystem $(Y(G, \Gamma), S_{g^{-1}ag})$.

Suppose now that $X = G/\Gamma$ is the system in the statement of the theorem and let $a \in G$ be such that $R = R_a$. Take $\theta \in [0, 1)$. Our goal is to show that if $\theta \notin \sigma(X, R)$ then $\theta \notin \sigma(Y_x, S_a)$ for almost every $x \in X$. Let us first deal with the case when θ is irrational.

Observe that θ is not an eigenvalue of (X, R_a) if and only if the product system $(X, R_a) \times (\mathbb{T}, R_\theta)$ is ergodic, where $R_\theta : t \mapsto t + \theta$ is rotation by θ. Notice that $X \times \mathbb{T} = (G \times \mathbb{R})/(\Gamma \times \mathbb{Z})$ is a nilmanifold too, and hence $(X, R_a) \times (\mathbb{T}, R_\theta)$ is a nilsystem. In accordance with (2.4) and (2.5) let

$$S_{(a,\theta)} = (R_a \times R_\theta) \times (R_a^2 \times R_{2\theta}) \times \cdots \times (R_a^k \times R_{k\theta})$$

and

$$Y_{(x,t)} := \overline{\{S_{(a,\theta)}^n((x, t), \ldots, (x, t)) : n \in \mathbb{Z}\}} \subseteq (X \times \mathbb{T})^k.$$

As was mentioned above, for almost every $(x, t) = (g\Gamma, t) \in X \times \mathbb{T}$, the nilsystem $(Y_{(x,t)}, S_{(a,\alpha)})$ is isomorphic to $(Y(G \times \mathbb{R}, \Gamma \times \mathbb{Z}), S_{(g^{-1}ag,\theta)})$.

We claim that $Y(G \times \mathbb{R}, \Gamma \times \mathbb{Z}) \cong Y(G, \Gamma) \times Y(\mathbb{R}, \Gamma)$. Assuming this claim for now, it follows that

$$\begin{aligned}
(Y_{(x,t)}, S_{(a,\theta)}) &\cong (Y(G \times \mathbb{R}, \Gamma \times \mathbb{Z}), S_{(g^{-1}ag,\theta)}) \\
&\cong (Y(G, \Gamma), S_{g^{-1}ag}) \times (Y(\mathbb{R}, \mathbb{Z}), S_\theta) \\
&\cong (Y(G, \Gamma), S_{g^{-1}ag}) \times (\mathbb{T}, R_\theta) \\
&\cong (Y_x, S_a) \times (\mathbb{T}, R_\theta).
\end{aligned}$$

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Recall that any transitive nilsystem is ergodic. Since $(Y_{(x,t)}, S_{(a,\theta)})$ is transitive by definition, it follows that it is ergodic, which implies that $(Y_x, S_a) \times (\mathbb{T}, R_\theta)$ is ergodic for almost every $x \in X$. However, $(Y_x, S_a) \times (\mathbb{T}, R_\theta)$ can only be ergodic if θ is not in the discrete spectrum of (Y_x, S_a), which finishes the proof that $\theta \notin \sigma(Y_x, S_a)$ for almost every $x \in X$.

It remains to show that $Y(G \times \mathbb{R}, \Gamma \times \mathbb{Z}) \cong Y(G, \Gamma) \times Y(\mathbb{R}, \Gamma)$. Note that $H^{(i)}(\mathbb{R}) = \{0\}^k$ for all $i \geq 2$, so that $H(\mathbb{R}) = \{(t, 2t, \ldots, kt) : t \in \mathbb{R}\}$. More generally, for any G we have $H^{(i)}(G \times \mathbb{R}) = H^{(i)}(G) \times \{0\}^k$ whenever $i \geq 2$. This implies that

$$H(G \times \mathbb{R}) = H(G) \times H(\mathbb{R}).$$

Finally, since

$$\begin{aligned}\Delta(G \times \mathbb{R}, \Gamma \times \mathbb{Z}) &= (H(G) \times H(\mathbb{R})) \cap (\Gamma^k \times \mathbb{Z}^k) \\ &= H(G) \cap \Gamma^k \times H(\mathbb{R}) \cap \mathbb{Z}^k \\ &= \Delta(G, \Gamma) \times \Delta(\mathbb{R}, \mathbb{Z}),\end{aligned}$$

the claim $Y(G \times \mathbb{R}, \Gamma \times \mathbb{Z}) \cong Y(G, \Gamma) \times Y(\mathbb{R}, \Gamma)$ follows.

Lastly, we deal with the case when $\theta = p/q \in (0, 1)$ is rational. Recall that $S_a = R_a \times R_a^2 \times \cdots \times R_a^k$ and $Y_x := \overline{\{S_a^n(x, x, \ldots, x) : n \in \mathbb{Z}\}}$ and that

$$(Y_x, S_a) \cong (Y(G, \Gamma), S_{g^{-1}ag}) \tag{2.6}$$

for all $x = g\Gamma \in X'$, where X' is some full measure subset of X. Observe that (2.6) implies

$$(Y_x, S_a^q) \cong (Y(G, \Gamma), S^q_{g^{-1}ag}), \tag{2.7}$$

for all $x = g\Gamma \in X'$. Then define

$$Y_x^{(q)} := \overline{\{S_a^{qn}(x, x, \ldots, x) : n \in \mathbb{Z}\}} = \overline{\{S_{a^q}^n(x, x, \ldots, x) : n \in \mathbb{Z}\}}.$$

Since X is connected and (X, R_a) is ergodic, the nilsystem (X, R_a^q) is ergodic. This implies that there exists a full measure set $X'' \subset X$ such that for all $x = g\Gamma \in X''$ we have

$$(Y_x^{(q)}, S_a^q) \cong (Y(G, \Gamma), S^q_{g^{-1}ag}). \tag{2.8}$$

Combining (2.7) and (2.8), we see that for any $x \in X' \cap X''$ we have

$$(Y_x, S_a^q) \cong (Y_x^{(q)}, S_a^q).$$

Since $(Y_x^{(q)}, S_a^q)$ is transitive by definition, it must be ergodic, and thus it follows that for all $x \in X' \cap X''$ the system (Y_x, S_a^q) is ergodic. We conclude that $\theta = p/q$ is not an eigenvalue of (Y_x, S_a^q) and this finishes the proof. □

3. *Revised proof of* [3, Theorem 4.2]

In light of the fact that [3, Theorem 7.1] is incorrect, we need to provide a new proof for [3, Theorem 4.2] to ensure that all the main results presented in [3] are still correct. With the same notation as in [3], let us recall the statement of [3, Theorem 4.2].

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THEOREM 4.2. *Let $k \in \mathbb{N}$, let G be an s-step nilpotent Lie group, and let Γ be a uniform and discrete subgroup of G such that $X = G/\Gamma$ is a connected nilmanifold. Let $R : X \to X$ be an ergodic niltranslation on X. Define $S := R \times R^2 \times \cdots \times R^k$ and*

$$Y_{X^\Delta} := \overline{\{S^n(x, x, \ldots, x) : x \in X,\ n \in \mathbb{Z}\}} \subseteq X^k.$$

Then $\sigma(X, R) = \sigma(Y_{X^\Delta}, S)$, where $\sigma(X, R)$ denotes the spectrum of the nilsystem (X, R) and $\sigma(Y_{X^\Delta}, S)$ denotes the spectrum of the nilsystem (Y_{X^Δ}, S).

Proof. Given $\theta \in \sigma(X, R)$, let $f \in L^2(X)$ be an eigenfunction of the system (X, R) with eigenvalue θ. Since the function $\tilde{f} \in L^2(Y_{X^\Delta})$ defined by $\tilde{f}(x_1, \ldots, x_k) = f(x_1)$ is an eigenfunction for the system (Y_{X^Δ}, S) with eigenvalue θ, it follows that $\sigma(X, R) \subseteq \sigma(Y_{X^\Delta}, S)$.

Next we prove the converse inclusion. Let ν be the Haar measure of the nilmanifold Y_{X^Δ} and let ν_x be the Haar measure of the nilmanifold Y_x defined by (2.1). Observe that the sets Y_x are precisely the atoms of the invariant σ-algebra of the system (Y_{X^Δ}, S). Therefore, the measures ν_x form the ergodic decomposition of ν.

Let $\theta \in \sigma(Y_{X^\Delta}, S)$ and let $f \in L^2(Y_{X^\Delta}, \nu)$ be an eigenfunction with eigenvalue θ, that is, for almost every $y \in Y_{X^\Delta}$ we have $Sf(y) = e(\theta)f(y)$. Since f cannot be 0ν-almost everywhere, there exists a positive measure set of $x \in X$ for which the restriction of f to the system (Y_x, ν_x, S) is not the zero function. But for any such x, the restriction of f to the system (Y_x, ν_x, S) is an eigenfunction with eigenvalue θ. This implies that $\theta \in \sigma(Y_{X^\Delta}, S)$ for all such x. Finally, by invoking Revised Theorem 7.1, we conclude that $\theta \in \sigma(X, R)$, finishing the proof. □

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