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COVERING THEOREMS FOR CLASSES OF UNIVALENT FUNCTIONS

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1. Introduction. Let $\mathscr{S}$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in $U = \{z : |z| < 1\}$. $\mathscr{S}^*$ and $\mathscr{C}$ will denote the collection of $f \in \mathscr{S}$ that map U onto a domain that is respectively starlike with respect to the origin and convex.

In [**4**, p. 85] Hayman used Steiner symmetrization to solve a problem, a special case of which is the following. If $0 \leqq x < \frac{1}{2}$, what is the minimum of the linear measure of $\{w : \operatorname{Re} w = x\} \cap f(U)$ for $f \in \mathscr{S}$ (if $x > \frac{1}{2}$ the solution is trivially 0)? In this paper we use Steiner symmetrization [**4**, p. 68] to solve this problem for the classes $\mathscr{S}^*$ and $\mathscr{C}$.

We also solve the following covering problem for the class $\mathscr{C}$. Let $R(\phi) = \{w : \arg w = \phi\}$ and let $l(\phi)$ denote the linear measure of $R(\phi) \cap f(U)$. What is the minimum of $l(\phi_1) \cdot l(\phi_2)$ $(0 \leqq \phi_1 \leqq \phi_2 < 2\pi)$ for $f \in \mathscr{C}$? The solution is complicated by the fact that (except in the case $\phi_1 = \phi_2$ and $\phi_2 = \phi_1 + \pi$) methods of symmetrization that preserve $\mathscr{C}$ are of no use for this particular problem. If $\phi_1 = \phi_2$ our result reduces to a well-known result due to Löwner [**8**], and if $\phi_2 = \phi_1 + \pi$, it reduces to a result due to Strohhäcker [**10**].

In addition to Hayman's result mentioned above, the results of this paper are similar in spirit to [**5**] and [**6**].

2. Covering of vertical segments. In order to simplify the statement of the following theorem, we introduce the function

$$F(\lambda, \mu, s) = \int_0^1 \frac{1 - (1 - st)^{\mu}}{t^{\lambda}} dt \tag{2.1}$$

where s is a real number, $\mu > 0$ and $\lambda < 2$. $F(\lambda, \mu, s)$ is closely related to the Incomplete Beta Function [**3**, p. 104]

$$B(p, q, s) = \int_0^s t^{p-1}(1 - t)^{q-1} dt \qquad (\operatorname{Re} p > 0, \operatorname{Re} q > 0).$$

In fact it is easy to show that

$$F_s(\lambda, \mu, s) = \mu s^{\lambda-2} B(2 - \lambda, \mu, s).$$

We will have occasion to use the following easily proved fact about $F(\lambda, \mu, s)$.

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LEMMA 1. *If* $|s| \leqq 1$, $\tau < 1$, $\sigma > 0$, *then*

$$\sum_{n=1}^{\infty} \binom{\sigma}{n} \frac{1}{n-\tau} s^n = -F(1+\tau, \sigma, -s).$$

We will also need the following known result.

LEMMA 2. *Let D be a domain starlike with respect to $w = 0$. If D^* is the domain obtained from D by Steiner symmetrization, then D^* is also starlike with respect to $w = 0$.*

The proof of this lemma follows immediately from the observation that if D is starlike and $l(x)$ is the linear measure of $D \cap \{\text{Re}\, w = x\}$, $0 < x < \infty$, then $l(x)/x$ is a decreasing function of x.

We now state the main result of this section.

THEOREM 1. *Let $0 < x < \frac{1}{2}$ and let $l(x)$ denote the linear measure of $f(U) \cap \{w : \text{Re}\, w = x\}$. Then,*

$$\min_{f \in \mathscr{S}^*} l(x) = \frac{a^\alpha}{2} (1-a)^{\frac{1}{2}} \sin \alpha\pi \left[\frac{1}{\alpha} + F(1+\alpha, \tfrac{1}{2}+\alpha, 1)\right] \tag{2.2}$$

where (α, a) is the unique solution in $(0, \frac{1}{2}) \times (0, 1)$ of the equations

$$\begin{aligned} x &= \frac{a^\alpha}{4} (1-a)^{\frac{1}{2}} \cos \alpha\pi \left[\frac{1}{\alpha} + F(\alpha+1, \alpha+\tfrac{1}{2}, 1)\right] \\ 0 &= \frac{1}{\alpha} + F\left(\alpha+1, \alpha+\tfrac{1}{2}, \frac{a}{1-a}\right). \end{aligned} \tag{2.3}$$

Notes. 1. If $x = 0$ the extremal function for this problem is $f(z) = z/(1-z^2)$ since, as Hayman has shown [**4**, p. 85], this function is extremal for the class $\mathscr{S}$.

2. As we will show, the extremal function for this problem maps U onto a domain symmetric with respect to the real axis whose boundary in the upper half-plane consists of a radial and a vertical slit to ∞ emanating from the point $(x, x \tan \alpha\pi)$.

Proof of Theorem 1. For $0 < x < \frac{1}{2}$, let $D(x, y)$ denote the domain symmetric with respect to the real axis whose boundary in the upper half-plane consists of a radial and a vertical slit to ∞ emanating from the point (x, y). Let $r(y)$ denote the conformal mapping radius [**4**, p. 79] of $D(x, y)$ with respect to 0 (in the sequel we write m.r. $D(x, y) = r(y)$). It follows from the Principle of Subordination and the Carathèodory Kernel Theorem that $r(y)$ is a strictly increasing continuous function of y. Moreover, $\lim_{y\to 0} r(y) = 2x < 1$ and $\lim_{y\to+\infty} r(y) = +\infty$. Thus there exists a unique value of $y = y(x)$, such that $r[y(x)] = 1$. The corresponding domain, which we denote $D(x)$, is then the image of U under a function $g \in \mathscr{S}^*$. We claim that g is the extremal function for (2.2). Indeed, let $f(z)$ be an extremal function for this problem and let $D = f(U)$. Let $l = 2\rho$ denote the linear measure of $D \cap \{w : \text{Re}\, w = x\}$. If

D^* is the domain obtained from the Steiner symmetrization of D, $D^* \cap \{w : \operatorname{Re} w = x\}$ consists of a single segment of length 2ρ that is symmetric with respect to the real axis. Since D^* is starlike with respect to 0 and Steiner symmetric with respect to the real axis,

$$D^* \subset D(x, \rho).$$

It follows from a result of Polya-Szegö [**4**, p. 81] and the Principle of Subordination that

$$1 = \text{m.r. } D(x) = \text{m.r. } D \leqq \text{m.r. } D^* \leqq \text{m.r. } D(x, \rho)$$

and hence $D(x) \subset D(x, \rho)$. But 2ρ is the extremal value for (2.2). This is possible only if $D(x) = D(x, \rho)$ and hence $D(x)$ is an extremal domain for (2.2). It remains to determine explicitly the function $g(z)$.

We begin by determining the map of the upper half-plane onto the infinite triangle whose "sides" are the real axis, the radial slit to ∞ and the vertical slit to ∞ emanating from the point $e^{i\alpha\pi}$ where $0 < \alpha < \frac{1}{2}$. It follows from the Schwarz-Christofel formula [**9**, p. 189] that

$$f(z) = -Ce^{i\alpha\pi} \int_0^z \frac{(1-z)^{\frac{1}{2}+\alpha}}{(z-a)^{1+\alpha}}\, dz, \tag{2.4}$$

where $C > 0$ and a, $0 < a < 1$, are constants depending on α to be determined, maps the half-plane $\operatorname{Im} z > 0$ onto the above triangle with $f(0) = 0, f(a) = \infty$, $f(1) = e^{i\alpha\pi}$ and $f(\infty) = \infty$.

The constants C and a are determined as follows. Since $f(1) = e^{i\alpha\pi}$, we have from (2.4)

$$-\frac{1}{C} = \int_0^1 \frac{(1-z)^{\frac{1}{2}+\alpha}}{(z-a)^{1+\alpha}}\, dz, \tag{2.5}$$

where the path of integration is contained in the closed upper half-plane avoiding the point $z = 1$ and is otherwise arbitrary. Any choice of $C > 0$ and a, $0 < a < 1$, satisfying (2.5) determines a map (2.4) of the upper half-plane onto the triangle and since for a function of the form (2.4) where C satisfies (2.5), $f(\infty) = \infty$, $f(0) = 0$ and $f(1) = e^{i\alpha\pi}$, there is only one such map, i.e., there is a unique solution for $C > 0$ and a, $0 < a < 1$, to (2.5). In order to place (2.5) in a more convenient form, we choose a specific path of integration, namely, the interval from 0 to $a - \epsilon$ ($\epsilon > 0$ small and positive) the semicircular arc from $a - \epsilon$ to $a + \epsilon$ and the interval from $a + \epsilon$ to 1. Letting I_1, I_2 and I_3 denote the integral over each of these intervals respectively, we have

$$-1/C = I_1 + I_2 + I_3.$$

Let c, $0 \leqq c < a$ be chosen so that $c > 2a - 1$. After an elementary calculation we have

$$I_1 = \left[\int_0^c + \int_c^{a-\epsilon} \right] \frac{(1-x)^{\frac{1}{2}+\alpha}}{(x-a)^{1+\alpha}}\, dx$$

$$= \int_0^c \frac{(1-x)^{\frac{1}{2}+\alpha}}{(x-a)^{1+\alpha}}\, dx - (1-a)^{\frac{1}{2}} e^{-i\alpha\pi} \left[\sum_{n=0}^{\infty} \binom{\frac{1}{2}+\alpha}{n} \frac{1}{n-\alpha} \left(\frac{a-c}{1-a}\right)^{n-\alpha} \right]$$

$$- (1-a)^{\frac{1}{2}+\alpha} e^{-i\alpha\pi} \frac{1}{\alpha\epsilon^{\alpha}} + O(\epsilon^{1-\alpha}).$$

Applying Lemma 1 and a similar calculation for I_2 and I_3, we obtain

$$-\frac{1}{C} = \int_0^c \frac{(1-x)^{\frac{1}{2}+\alpha}}{(x-a)^{1+\alpha}}\, dx$$

$$+ (1-a)^{\frac{1}{2}} \left[\frac{a-c}{1-a}\right]^{-\alpha} e^{-i\alpha\pi} \left[\frac{1}{\alpha} + F\left(1+\alpha, \tfrac{1}{2}+\alpha, -\frac{a-c}{1-a}\right)\right]$$

$$- (1-a)^{\frac{1}{2}} \left[\frac{1}{\alpha} + F(1+\alpha, \tfrac{1}{2}+\alpha, 1)\right].$$

Since the right-hand side of the above equation is continuous in c for $c < a$, the equation also holds for $c = 0$. Setting $c = 0$ and taking real and imaginary parts in the resulting equation, we obtain

$$0 = \frac{1}{\alpha} + F\left(1+\alpha, \tfrac{1}{2}+\alpha, -\frac{a}{1-a}\right) \tag{2.6}$$

$$\frac{1}{C} = (1-a)^{\frac{1}{2}} \left[\frac{1}{\alpha} + F(1+\alpha, \tfrac{1}{2}+\alpha, 1)\right]. \tag{2.7}$$

As noted above these equations uniquely determine (a, C) on $(0, 1) \times (0, \infty)$.

The function (2.4) maps the interval $(-\infty, a)$ onto the real axis. By the Schwarz Reflection Principle, $f(z)$ maps the plane slit along $[a, +\infty)$ onto $D(\cos \alpha\pi, \sin \alpha\pi)$. If $h(z) = 4az/(1+z)^2$ then $f \circ h(z)$ maps U onto $D(\cos \alpha\pi, \sin \alpha\pi)$. Hence

$$g(z) = (a^{\alpha}/4C) f[h(z)] \tag{2.8}$$

belongs to $\mathscr{S}^*$ and maps U onto $D(x)$ where

$$\begin{aligned} x &= \frac{a^{\alpha}}{4C} \cos \alpha\pi \\ &= \frac{a^{\alpha}}{4} (1-a)^{\frac{1}{2}} \cos \alpha\pi \left[\frac{1}{\alpha} + F(1+\alpha, \tfrac{1}{2}+\alpha, 1)\right]. \end{aligned} \tag{2.9}$$

It is clear that given x, $0 < x < \frac{1}{2}$, there is a unique pair $(a, \alpha) \in (0, 1) \times (0, \frac{1}{2})$ that satisfies (2.6) and (2.9). Indeed, a solution $(a, \alpha) \in (0, 1) \times (0, \frac{1}{2})$ to (2.6) and (2.9) determines a function in $\mathscr{S}^*$ that maps U onto $D(x)$ which, as noted in the beginning of the proof, determines α uniquely. Finally it is clear from (2.7) and (2.8) that

$$2 \operatorname{Im} \frac{a^\alpha}{4c} f(1) = \frac{a^\alpha}{2} (1-a)^{\frac{1}{2}} \sin \alpha\pi \left[\frac{1}{\alpha} + F(1+\alpha, \tfrac{1}{2}+\alpha, 1) \right]$$

is the extreme value for $\min_{f \in \mathscr{S}^*} l(x)$ and the proof is complete.

We now consider the above problem for the class $\mathscr{C}$. Before stating the theorem, we introduce the function

$$f_a(z) = a\left[1 - \left(\frac{1-z}{1+z} \right)^{1/2a} \right] \tag{2.10}$$

where $\frac{1}{2} \leqq a < +\infty$. $f_a \in \mathscr{C}$ and maps U onto an "infinite wedge" that is symmetric with respect to the real axis and has its vertex at the point a. The angular opening at a is $\pi/2a$. When $a = \frac{1}{2}$, the wedge degenerates to a half-plane and when a tends to $+\infty$, $f_a(z)$ approaches $\frac{1}{2} \log[(1+z)/(1-z)]$. Incorporating this value of a into the definition (2.10) we can state

THEOREM 2. *If* $0 \leqq x < \frac{1}{2}$,

$$\inf_{f \in \mathscr{C}} l(x) = (a - x) \tan(\pi/4a) \tag{2.11}$$

where a is the unique solution of

$$(2a/\pi) \sin(\pi/2a) = (1 - x/a) \tag{2.12}$$

on $(\frac{1}{2}, \infty]$.

The proof follows the lines of the proof of Theorem 1 and consequently the details will be omitted. We note that one first shows, using Steiner symmetrization, that for given x, $0 \leqq x < \frac{1}{2}$, a function of the form (2.10) is the extremal function for (2.11). An elementary calculation then shows that the value of a that yields the extremal value (2.11) is the unique solution to (2.12).

3. Covering of radial segments. Let $f(z) \in \mathscr{C}$, $R(\phi) = \{w : \arg w = \phi\}$ and $l(\phi)$ denote the linear measure of $R(\phi) \cap f(U)$. We consider the following question: What is the minimum over the class $\mathscr{C}$ of $l(\phi_1) \cdot l(\phi_2)$ $(0 \leqq \phi_1 \leqq \phi_2 \leqq 2\pi)$?

It will be more convenient for us to reformulate this problem in an equivalent way, namely: Let $f(z) \in \mathscr{C}$, with $R(\phi)$ and $l(\phi)$ defined as above. What is the minimum over the class $\mathscr{C}$ of $l(\phi) \cdot l(-\phi)$ for $0 \leqq \phi \leqq \pi/2$?

With $f_a(z)$ as defined in § 2, we have the following theorem.

THEOREM 3. *Let* $f(z) \in \mathscr{C}$ *and* $\phi \in [0, \pi/2]$. *If*

$$0 \leqq \phi \leqq \tan^{-1}(2/\pi)$$

then

$$l(\phi) \cdot l(-\phi) \geqq (1/4)\sec^2 \phi$$

with equality for $f(z) = z/(1+z)$. *If* $\tan^{-1}(2/\pi) < \phi \leqq \pi/2$ *then* $l(\phi)\cdot l(-\phi)$ *is minimized by* $f_a(z)$ *where* $a = a(\phi)$ *is the unique solution of the equation*

$$\tan\phi = \frac{1-\cos\left(\frac{\pi}{2a}\right)}{\frac{\pi}{2a}-\sin\left(\frac{\pi}{2a}\right)}. \tag{3.1}$$

Proof. Using the principle of subordination, we may deduce that the extremal function for this problem is an "infinite wedge." Using the transformation $g(z) = -\overline{f(-\bar{z})}$ we conclude that the vertex of the "infinite wedge" must be in the right half-plane. Our first aim is to show that for each ϕ, there exists an extremal function among the functions $f_a(z)$, ($a \geqq \frac{1}{2}$).

Denote the polar coordinates of the vertex of the "infinite wedge" by $(|L|, x)$ (so the vertex is the point $L = |L|e^{ix}$). Let the upper and lower sides of the wedge form angles α and β, respectively, with the segment joining the origin with the vertex. If $l_1 = l(\phi)$, $l_2 = l(-\phi)$ denote the linear measures defined as above, then

$$l_1 l_2 = \frac{|L|^2\sin\alpha\sin\beta}{\sin(\phi - x+\alpha)\sin(\phi+x+\beta)}. \tag{3.2}$$

If we "fix" α, β, $|L|$ and let x vary in the interval $-\pi/2 < x < \pi/2$, we find by a trivial calculation that for an extremal function

$$x = (\alpha-\beta)/2. \tag{3.3}$$

The condition (3.3) implies that there exists an extremal function $f(z)$ of the form

$$f(z) = g(z)/g'(0)$$

where

$$g(z) = L - \left[\frac{1+[(\zeta+z)/(1+\bar{\zeta}z)]e^{i\theta}}{1-[(\zeta+z)/(1+\bar{\zeta}z)]e^{i\theta}}\right]^{1/2a} \tag{3.4}$$

for some L, a, θ and ζ such that $|\zeta| < 1$, $0 \leqq \theta < 2\pi$ and $\frac{1}{2} \leqq a$.

From (3.2) and (3.3) we have for $t = (\alpha+\beta)/2 = \pi/4a$,

$$l_1\cdot l_2 = \frac{|L|^2\sin(t+x)\sin(t-x)}{\sin^2(\phi+t)}. \tag{3.5}$$

Since $g(0) = 0$, it follows from (3.4) that

$$L = \left(\frac{1+\zeta e^{i\theta}}{1-\zeta e^{i\theta}}\right)^{1/2a}, \quad g'(0) = -\frac{1}{a}\left(\frac{1+\zeta e^{i\theta}}{1-\zeta e^{i\theta}}\right)^{1/2a-1}\frac{(1-|\zeta|^2)e^{i\theta}}{(1-\zeta e^{i\theta})^2}. \tag{3.6}$$

Denoting $\zeta e^{i\theta} = re^{i\mu}$ we obtain after an easy calculation

$$l_1(f)\cdot l_2(f) = \frac{(\sin^2 t - \sin^2 x)a^2}{\sin^2(\phi+t)\cos^2(2ax)}. \tag{3.7}$$

Since $\beta > 0$, $x = (\alpha - \beta)/2 < (\alpha + \beta)/2 = t = \pi/4a$. Let

$$T(x) = \frac{\sin^2 t - \sin^2 x}{\cos^2(2ax)}, \quad 0 \leqq x < t.$$

It is not hard to show $T(x) \geqq T(0)$ $(0 < x < t)$ and hence for the extremal we may assume that $x = 0$ which together with (3.4), implies there exists an extremal function of the form $f_a(z)$. From (3.7) we have for this extremal function,

$$l_1 = l_2 = \frac{\pi}{4t} \frac{\sin t}{\sin(\phi + t)} = l(t).$$

This formula holds even if $a = \infty$; i.e., $t = 0$ if we interpret the right hand side as a limit. If we set

$$y(t) = 4/\pi l(t)$$

then the problem of minimizing $l(t)$ for $0 \leqq t \leqq \pi/2$ (or $\frac{1}{2} \leqq a \leqq \infty$) is equivalent to maximizing the function

$$y(t) = \frac{t \sin(\phi + t)}{\sin t} \quad (0 \leqq t \leqq \pi/2).$$

It is readily seen that if $\tan \phi \leqq 2/\pi$, $y'(t) > 0$ on $(0, \pi/2)$ and hence $y(t)$ assumes its maximum at $t = \pi/2$ and hence for $a = \frac{1}{2}$. For this value of a, $f_a(z) = z/(1 + z)$ and $l_1 = l_2 = \frac{1}{2} \sec \phi$. This proves the first assertion of the theorem.

If $\tan \phi > 2/\pi$, then it can be shown that there exists a unique $t_0 \in (0, \pi/2)$ such that $y'(t_0) = 0$. Moreover, $y'(t) > 0$ for $0 < t < t_0$ and $y'(t) < 0$ for $t_0 < t < \pi/2$. Thus t_0 is a unique maximum for $y(t)$ on $[0, \pi/2]$. It follows that $l_1 = l_1(a)$ has a unique maximum at the point $a = \pi/4t_0$, where a is the unique solution of

$$\tan \phi = \frac{1 - \cos(\pi/2a)}{\pi/2a - \sin(\pi/2a)}.$$

This completes the proof of the theorem.

Remarks. 1. Theorem 3 extends two results for the class $\mathscr{C}$. The case $\phi = 0$ is the well-known result due to Löwner [**8**] that for every function $f(z) \in \mathscr{C}$, $f(U) \supset \{|w| < \frac{1}{2}\}$. The case $\phi = \pi/2$ generalizes the result due to Strohhäcker [**10**] that if η and ϵ are the boundary points of $f(U)$ that lie on a line through the origin, then $\max(|\eta|, |\epsilon|) \geqq \pi/4$. Indeed, if $\phi = \pi/2$, the solution of (3.1) is $a = \infty$ which implies that the extremal function is

$$f_\infty(z) = \tfrac{1}{2} \log \frac{1 + z}{1 - z}.$$

2. It is perhaps worth noting that the corresponding problems for the classes $\mathscr{S}$ and $\mathscr{S}^*$ follow quite easily from results in [**1**; **2**; **6**].

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