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ORDER BOUNDED WEIGHTED COMPOSITION OPERATORS

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Abstract

Let ϕ and ψ be analytic maps on the open unit disk D such that $\phi(D) \subset D$. Such maps induce a weighted composition operator $C_{\phi,\psi}$ acting on weighted Banach spaces of type H^∞ or on weighted Bergman spaces, respectively. We study when such operators are order bounded.

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1. Introduction

For an analytic self-map ϕ of the unit disk D in the complex plane the classical *composition operator* C_ϕ is defined by

$$C_\phi : H(D) \to H(D), \quad f \mapsto f \circ \phi,$$

where $H(D)$ denotes the set of all analytic functions on D. Multiplying with a map $\psi \in H(D)$ induces the so-called *weighted composition operator*

$$C_{\phi,\psi} : H(D) \to H(D), \quad f \mapsto \psi(f \circ \phi)$$

which is a combination of C_ϕ with the pointwise multiplication operator

$$M_\psi : H(D) \to H(D), \quad f \mapsto \psi \cdot f.$$

(Weighted) composition operators occur naturally in various problems and therefore have been widely investigated. For an overview of results in the classical setting of the Hardy space H^2 as well as for an introduction to composition operators, we refer the reader to the excellent monographs by Cowen and MacCluer [6] and Shapiro [11]. Order bounded composition operators have been studied by several authors; see, for example, [8–10] and the references therein. Motivated by the work of Hunziker [10], Hibschweiler [9] characterized order bounded composition operators acting on weighted Bergman spaces and weighted Banach spaces of holomorphic functions, generated by the so-called *standard weights*. One of the aims of this article is to generalize her results to the setting of more generally weighted spaces. The other aim is to continue the research we started in [12, 13]. There we characterized bounded, compact and Schatten class weighted composition operators acting on weighted Bergman spaces generated by the absolute value of holomorphic functions. We will show that these results remain true for Bergman spaces generated by a quite general class of radial weights.

2. Setting

2.1. Weighted spaces. We say that a function $v : D \to]0, \infty[$ which is bounded and continuous is a *weight*. For such a weight v we define

$$H_v^\infty := \Big\{ f \in H(D);\ \|f\|_v := \sup_{z \in D} v(z)|f(z)| < \infty \Big\}.$$

Endowed with the weighted sup-norm $\|\cdot\|_v$, this is a Banach space. We refer to such a space as *weighted Banach space of holomorphic functions*. These spaces arise naturally in several problems related, for example, to complex analysis, spectral theory, Fourier analysis, partial differential and convolution equations. Concrete examples may be found in [4]. Weighted Banach spaces of holomorphic functions have been studied deeply in [3], but also in [2, 5].

The formulation of results on weighted spaces often requires the so-called *associated weights*. For a weight v, its associated weight $\tilde{v}$ is defined as follows:

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H(D),\ \sup_{z \in D} v(z)|f(z)| \le 1\}}, \quad z \in D.$$

The concept of associated weights was implicitly introduced by Anderson and Duncan in [1]. Bierstedt, Bonet and Taskinen thoroughly studied associated weights in [3].

During his studies of Hilbert spaces of analytic functions Bergman mainly studied spaces of analytic functions which are square-integrable over the given domain with respect to the Lebesgue area or volume measure. Later attention was drawn to more general spaces. The reproducing kernel, the *Bergman kernel function*, played an important role. In this article we are interested in *weighted Bergman spaces* given by

$$A_{v,p} := \Big\{ f \in H(D);\ \|f\|_{v,p} := \Big(\int_D |f(z)|^p v(z)\, dA(z) \Big)^{1/p} < \infty \Big\}, \quad 1 \le p < \infty,$$

endowed with norm $\|\cdot\|_{v,p}$. This means that the usual Bergman space is denoted by $A_{1,2}$.

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2.2. The weights. In this article we consider the following class of weights. Let ν be a holomorphic function on D, nonvanishing, strictly positive on $[0, 1[$ and satisfying $\lim_{r\to 1} \nu(r) = 0$. Then we define the weight v as

$$v(z) := \nu(|z|^2), \quad \text{for every } z \in D. \tag{2.1}$$

Examples include all the famous and popular weights.

(a) *Standard weights*. The function $\nu_\alpha(z) = (1 - z)^\alpha$, $\alpha \geq 1$, yields the weight $v_\alpha(z) = (1 - |z|^2)^\alpha$.

(b) *Exponential weights*. We consider

$$\nu(z) = e^{-1/(1-z)^\alpha}, \quad \alpha \geq 1,$$

and obtain

$$v(z) = e^{-1/(1-|z|^2)^\alpha}.$$

(c) *Logarithmic weights*. With $\nu(z) = (1 - \log(1 - z))^\beta$, $\beta < 0$, we get $v(z) = (1 - \log(1 - |z|^2))^\beta$.

For a fixed point $a \in D$ we introduce a function $\nu_a(z) := \nu(\bar{a}z)$ for every $z \in D$. Since ν is holomorphic on D, so is the function ν_a. In particular, we will often assume that there is a constant $C > 0$ with

$$\sup_{a\in D} \sup_{z\in D} \frac{v(z)}{|\nu_a(z)|} \leq C. \tag{2.2}$$

The following weights satisfy (2.2).

(a) For $v(z) = 1 - |z|^2$,

$$\frac{v(z)}{|\nu_a(z)|} = \frac{1 - |z|^2}{|1 - \bar{a}z|} \leq \frac{1 - |z|^2}{1 - |z|} \leq 1 + |z| \leq 2$$

for every $z \in D$.

(b) Consider $v(z) = 1/(1 - \log(1 - |z|^2))$. This weight has the desired property since

$$|1 - \log(1 - \bar{a}z)| \leq 1 - \log(1 - |z|), \quad \text{for every } z \in D,$$

and the function $(1 - \log(1 - |z|))/(1 - \log(1 - |z|^2))$ is continuous and tends to 1 if $|z| \to 1$.

2.3. The operators. This section is devoted to giving the necessary definitions as well as an overview of results already obtained.

Let X be a Banach space of functions analytic in D and let $q > 0$. Moreover, let μ be a positive measure on the unit circle. The operator $T : X \to L^q(\mu)$ is said to be *order bounded* if there exists $h \in L^q(\mu)$, $h \geq 0$, such that the inequality

$$|T(f)(e^{i\Theta})| \leq h(e^{i\Theta})$$

holds almost everywhere with respect to μ, for all $f \in X$ with $\|f\|_X \leq 1$.

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Recall that, for $p \geq 1$, the *Hardy space* H^p consists of all functions f analytic in the disk with

$$\|f\|_{H^p}^p = \frac{1}{2\pi} \sup_{0 \leq r < 1} \int_0^{2\pi} |f(r(e^{i\Theta}))|^p \, d\Theta < \infty.$$

Equipped with norm $\|\cdot\|_{H^p}$ this is a Banach space. Moreover, if $f \in H^p$, then the boundary function f^* defined by

$$f^*(e^{i\Theta}) = \lim_{r \to 1-} f(re^{i\Theta})$$

exists almost everywhere with respect to the Lebesgue measure m.

Now, for $\beta > 0$, Hunziker [10] showed that the operator $C_\phi : H^p \to L^{\beta_p}(m)$ is order bounded for some $p \geq 1$ if and only if the function $1/(1 - |\phi^*|)$ belongs to $L^\beta(m)$. Later Hibschweiler (see [9]) proved that if $\alpha \geq -1$, $q > 0$, $\psi \in L^q(m)$ and $|\phi^*(e^{i\Theta})| < 1$ almost every (a.e.) $[m]$, the operator $C_{\phi,\psi} : A_{v_\alpha,p} \to L^q(m)$ is order bounded if and only if $\psi/(1 - |\phi^*|)^{\alpha+2/p} \in L^q(m)$. Moreover, for $\alpha > 0$ and $q > 0$, $C_{\phi,\psi} : H^\infty_{v_\alpha} \to L^q(m)$ is order bounded if and only if $\psi/v \circ \phi^* \in L^q(m)$.

3. Results

3.1. Order boundedness for radial weights. For $z \in D$ and $f \in A_{v,p}$, let $E_z(f) = f(z)$.

Lemma 3.1. *Let v be a weight of the type defined in (2.1) such that*

$$\sup_{a \in D} \sup_{z \in D} \frac{v(z)}{|v_a(z)|} \leq M < \infty.$$

Then there exist positive constants C_1, C_2, depending only on v and p, such that

$$\frac{C_1}{v(z)^{1/p}(1 - |z|^2)^{2/p}} \leq \|E_z\| \leq \frac{C_2}{v(z)^{1/p}(1 - |z|^2)^{2/p}} \|f\|_{v,p}.$$

Proof. By [12, Lemma 1] we can find a constant $C_2 > 0$ such that

$$|f(z)| \leq \frac{C_2}{v(z)^{1/p}(1 - |z|^2)^{2/p}} \|f\|_{v,p}$$

for every $z \in D$. Obviously, this is the upper estimate. In order to prove the lower estimate, we consider the function

$$f_z(w) = \frac{1}{v_z(w)^{1/p}(1 - \bar{z}w)^{2/p}}, \quad \text{for every } w \in D.$$

Then

$$\begin{aligned} \|f_z\|_{v,p}^p &= \int_D \frac{v(w)}{|v_z(w)|(1 - \bar{z}w)^2} \, dA(w) \\ &\leq M \int_D \frac{1}{(1 - \bar{z}w)^2} \, dA(w) = M \end{aligned}$$

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for every $z \in D$. Hence

$$\|E_z\| \geq \frac{f_z(z)}{\|f_z\|_{v,p}} \geq \frac{1}{M^{1/p}}|f_z(z)| = \frac{1}{M^{1/p}v(z)^{1/p}(1-|z|^2)^{2/p}}$$

and the claim follows. □

The proof is inspired by [9].

Theorem 3.2. *Let v be a weight as defined in (2.1) and $\psi \in L^q(m)$. Let ϕ be an analytic self-map of the disk such that $|\phi^*(e^{i\Theta})| < 1$ a.e. $[m]$. For fixed p, $1 \leq p < \infty$, the following are equivalent.*

(a) $C_{\phi,\psi} : A_{v,p} \to L^q(m)$ *is order bounded.*
(b) $\psi/(v(|\phi^*|)^{1/p}(1-|\phi^*|^2)^{2/p}) \in L^q(m)$.

Proof. Let us first assume that (b) holds. Since $|\phi^*(e^{i\Theta})| < 1$ a.e. $[m]$, the previous lemma gives a constant C, depending only on v and p, such that

$$|f(\phi^*(e^{i\Theta}))| \leq \frac{C}{v(\phi^*(e^{i\Theta}))^{1/p}(1-|\phi^*(e^{i\Theta})|^2)^{2/p}}$$

a.e. $[m]$ for all f with $\|f\|_{v,p} \leq 1$. Next, we consider the function

$$h(e^{i\Theta}) := \frac{C|\psi(e^{i\Theta})|}{v(\phi(e^{i\Theta}))^{1/p}(1-|\phi^*(e^{i\Theta})|^2)^{2/p}}.$$

Then $h \in L^q(m)$, by hypothesis, and the previous inequality implies that

$$|\psi(e^{i\Theta})||f(\phi^*(e^{i\Theta}))| \leq h(e^{i\Theta}) \quad \text{a.e. } [m].$$

Thus, $C_{\phi,\psi} : A_{v,p} \to L^q(m)$ is order bounded.

Next, let us assume that $C_{\phi,\psi} : A_{v,p} \to L^q(m)$ is order bounded. Then we can find an $h \in L^q(m)$, $h \geq 0$, with

$$|h(e^{i\Theta})| \geq |\psi(e^{i\Theta})||f(\phi^*(e^{i\Theta}))| \quad \text{a.e. } [m]$$

for every $f \in A_{v,p}$ with $\|f\|_{v,p} \leq 1$. An application of the previous lemma now yields

$$\begin{aligned}|h(e^{i\Theta})| &\geq |\psi(e^{i\Theta})||f(\phi^*(e^{i\Theta}))| \geq |\psi(e^{i\Theta})| \sup\{|E_{\phi^*(e^{i\Theta})}(f)|; \|f\|_{v,p} \leq 1\} \\ &= |\psi(e^{i\Theta})|\|E_{\phi^*(e^{i\Theta})}\| \geq C\frac{|\psi(e^{i\Theta})|}{v(\phi(e^{i\Theta}))^{1/p}(1-|\phi^*(e^{i\Theta})|^2)^{2/p}}\end{aligned}$$

a.e. $[m]$. We can conclude that (b) holds. □

Lemma 3.3. *Let v be an arbitrary weight. For $z \in D$, let $E_z(f) = f(z)$ for $f \in H_v^\infty$. Then*

$$\|E_z\| = \frac{1}{\tilde{v}(z)}.$$

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PROOF. Fix $z \in D$. Then for $f \in H_v^\infty$,

$$|E_z(f)| = \frac{f(z)\tilde{v}(z)}{\tilde{v}(z)} \le \frac{\|f\|_v}{\tilde{v}(z)},$$

since, by [3], H_v^∞ and $H_{\tilde{v}}^\infty$ are isometrically isomorphic. Hence $\|E_z\| \le 1/\tilde{v}(z)$ (see [3]). For the lower estimate take a function $f_z \in H_v^\infty$ with $\|f_z\|_v \le 1$ such that $|f_z(z)| = 1/\tilde{v}(z)$. Then

$$\|E_z\| \ge \frac{|f_z(z)|}{\|f_z\|_v} \ge \frac{1}{\tilde{v}(z)}$$

and the claim follows. □

The proof is inspired by [9].

THEOREM 3.4. *Let v be a weight and $0 \neq \psi \in L^q(m)$. Let ϕ be an analytic self-map of the disk such that $|\phi^*(e^{i\Theta})| < 1$ a.e. $[m]$. For fixed p, $1 \le p < \infty$, the following are equivalent.*

(a) *$C_{\phi,\psi} : H_v^\infty \to L^q(m)$ is order bounded.*
(b) *$\psi/v \circ \phi^* \in L^q(m)$.*

PROOF. First, we assume that $C_{\phi,\psi} : H_v^\infty \to L^q(m)$ is order bounded, that is, there exists a positive function $h \in L^q(m)$ such that

$$|h(e^{i\Theta})| \ge |\psi(e^{i\Theta})||f(\phi^*(e^{i\Theta}))| \quad \text{a.e. } [m]$$

for every $f \in H_v^\infty$ with $\|f\|_v \le 1$. Hence an application of the previous lemma yields

$$\begin{aligned} |h(e^{i\Theta})| &\ge |\psi(e^{i\Theta})| \sup\{|E_{\phi^*(e^{i\Theta})}(f)|;\ \|f\|_v \le 1\} \\ &= |\psi(e^{i\Theta})|\|E_{\phi^*(e^{i\Theta})}\| \ge \frac{|\psi(e^{i\Theta})|}{\tilde{v}(\phi^*(e^{i\Theta}))} \end{aligned}$$

and (b) holds.

Conversely, let us suppose that $\psi/v \circ \phi^* \in L^q(m)$ and $f \in H_v^\infty$ with $\|f\|_v \le 1$. Since $|\phi^*(e^{i\Theta})| < 1$ a.e. $[m]$, by Lemma 3.3,

$$|C_{\phi,\psi}(f)(e^{i\Theta})| \le |\psi(e^{i\Theta})|\|E_{\phi^*(e^{i\Theta})}\|\|f\|_v \le \frac{|\psi(e^{i\Theta})|}{\tilde{v}(\phi^*(e^{i\Theta}))}$$

a.e. $[m]$. Finally, take $h := |\psi|/\tilde{v} \circ \phi^*$ and conclude that the operator must be order bounded. □

3.2. Bounded, compact and Schatten class composition operators on weighted Bergman spaces.

REMARK. Weights v of the type defined above can be written as $v(z) = f(|z|)$, $z \in D$, where $f \in H(D)$ is a function whose Taylor series (at 0) has nonnegative coefficients. By [3, Corollary 1.6] such weights my be written as

$$v(z) = M(f, z) := \max\{|f(\lambda z)|;\ |\lambda| = 1\}.$$

In the following we will write $f_\lambda(z) = f(\lambda z)$.

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Now, before we can characterize when weighted composition operators are bounded, compact or Schatten class, we need some further information related to the geometry of the unit disk.

For $a, z \in D$, let $\sigma_a(z)$ be the Möbius transformation of D which interchanges 0 and a, that is,

$$\sigma_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

Moreover,

$$-\sigma_a'(z) = \frac{1 - |a|^2}{(1 - \overline{a}z)^2}, \quad z \in D.$$

Next, let $K_a(z) = 1/(1 - \overline{a}z)^2$ be the Bergman kernel and

$$k_a(z) = -\sigma_a'(z) = \frac{1 - |a|^2}{(1 - \overline{a}z)^2} = (1 - |a|^2)K_a(z)$$

the normalized Bergman kernel in $A_{1,2}$ so that $\|k_a\|_{1,2} = 1$. For an analytic self-map ϕ of D and weights v, w on D the *weighted* (ϕ, v)*-Berezin transform* of w is given by

$$[B_{\phi,v}(w)](a) = \int_D |\sigma_a'(\phi(z))|^2 \frac{w(z)}{v(\phi(z))}\, dA(z).$$

The formulation of results on composition operators acting on weighted Bergman spaces requires the Carleson measure. Thus, let μ be a positive Borel measure on D. Then μ is called a *Carleson measure* on the Bergman space if there is a constant $C > 0$ such that, for any $f \in A_{1,2}$,

$$\int_D |f(z)|^2\, d\mu(z) \le C\|f\|_{1,2}^2.$$

Finally, the *Carleson square* is defined by

$$S(I) = \left\{ z \in D;\ 1 - |I| \le |z| < 1,\ \frac{z}{|z|} \in I \right\},$$

where I is an arc of the unit circle ∂D.

The following result is well known. The form stated below is taken from [7].

THEOREM 3.5 [7, Theorem A]. *Let μ be a positive Borel measure on D. Then the following statements are equivalent.*

(a) *There is a constant $C_1 > 0$ such that for any $f \in A_{1,2}$,*

$$\int_D |f(z)|^2\, d\mu(z) \le C_1\|f\|_{1,2}^2.$$

(b) *There is a constant $C_2 > 0$ such that, for any arc $I \in \partial D$,*

$$\mu(S(I)) \le C_2|I|^2.$$

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(c) *There is a constant $C_3 > 0$ such that, for every $a \in D$,*

$$\int_D |\sigma_a'(z)|^2 \, d\mu(z) \le C_3.$$

Theorem 3.6. *Let v be a weight as defined in (2.1), w an arbitrary weight, ϕ an analytic self-map of D and $\psi \in H(D)$. Then the weighted composition operator $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$ is bounded if and only if $B_{\phi,v}(w|\psi|^2) \in L^\infty(D)$.*

Proof. By definition, the operator $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$ is bounded if and only if there is $C > 0$ such that for every $f \in A_{v,2}$ the following inequality holds:

$$\int_D |f(\phi(z))|^2 |\psi(z)|^2 w(z) \, dA(z) \le C \int_D |f(z)|^2 v(z) \, dA(z).$$

Since there is obviously an $h \in H(D)$ with $v(z) := \max_{|\lambda|=1} |h_\lambda(z)|$ for every $z \in D$, this holds if and only if

$$\int_D |f(\phi(z))|^2 |\psi(z)|^2 w(z) \, dA(z) \le C \int_D |f(z)|^2 |h_\lambda(z)| \, dA(z) \tag{3.1}$$

for every $\lambda \in C$ with $|\lambda| = 1$. From now on we can proceed exactly as in the proof of Theorem 2 in [12]; for the sake of completeness we repeat this here. Note that $f \in A_{h_\lambda,2}$ if and only if $g := h_\lambda^{1/2} \in A_{1,2}$. Thus, (3.1) is equivalent to the condition that there is a constant $C > 0$ such that for every $g \in A_{1,2}$,

$$\int_D \frac{|g(\phi(z))|^2}{v(\phi(z))} |\psi(z)|^2 w(z) \, dA(z) \le C \int_D |g(z)|^2 \, dA(z).$$

Let

$$d\nu_{v,w,\psi}(z) = |\psi(z)|^2 \frac{w(z)}{v(\phi(z))} \, dA(z)$$

and let $\mu_{v,w,\psi} = \nu_{v,w,\psi} \circ \phi^{-1}$ be the pull-back measure induced by ϕ. If we change the variable $s = \phi(z)$, then

$$\begin{aligned}\int_D |g(\phi(z))|^2 |\psi(z)|^2 \frac{w(z)}{v(\phi(z))} \, dA(z) &= \int_D |g(\phi(z))|^2 \, d\nu_{v,w,\psi}(z) \\ &= \int_D |g(s)|^2 \, d\mu_{v,w,\psi}(s).\end{aligned}$$

Thus, (3.1) is equivalent to

$$\int_D |g(s)|^2 \, d\mu_{v,w,\psi}(s) \le C \int_D |g(s)|^2 \, dA(s).$$

By Theorem 3.5 this holds if and only if

$$\sup_{a \in D} \int_D |\sigma_a'(s)|^2 \, d\mu_{v,w,\psi}(s) < \infty.$$

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Changing the variable back to z yields the inequality

$$\sup_{a\in D}\int_D |\sigma_a'(\phi(z))|^2|\psi(z)|^2\frac{w(z)}{v(\phi(z))}\,dA(z)<\infty,$$

and the claim follows. □

Analogously we also obtain the following theorems.

THEOREM 3.7. *Let v be a weight as defined in (2.1) and w be an arbitrary weight. For an analytic self-map ϕ of D and a map $\psi\in H(D)$ the weighted composition operator $C_{\phi,\psi}:A_{v,2}\to A_{w,2}$ is compact if and only if*

$$\lim_{|a|\to 1}\sup[B_{\phi,v}(|\psi|^2w)](a)=0.$$

THEOREM 3.8. *Let v be a weight as defined in (2.1) and w be an arbitrary weight. Moreover, let the operator $C_{\phi,\psi}:A_{v,2}\to A_{v,2}$ be compact. Then $C_{\phi,\psi}\in S_p$ if and only if $B_{\phi,v}(|\psi|^2v)\in L^{p/2}(D,d\lambda)$, where $d\lambda(z)=(1-|z|^2)^{-2}\,dA(z)$ is the Möbius invariant measure on D.*

3.3. Order boundedness: weights which are the absolute value of a holomorphic function.

LEMMA 3.9. *Let u be a bounded holomorphic function without any zeros on D and $v=|u|$ with $\sup_{z\in D}v(z)\le M<\infty$. Then for $f\in H_v^\infty$ and $z\in D$,*

$$|f(z)|\le\frac{M}{v(0)^{1/p}(1-|z|^2)^{2/p}v(z)^{1/p}}\|f\|_{v,p}.$$

PROOF. Let $\alpha\in D$ be an arbitrary point and consider the map

$$T_\alpha:A_{v,p}\to A_{v,p},\qquad T_\alpha f(z)=f(\sigma_\alpha(z))\sigma_\alpha'(z)^{2/p}u(\sigma_\alpha(z))^{1/p}.$$

Then a change of variables yields

$$\begin{aligned}\|T_\alpha f\|_{v,p}^p&=\int_D v(z)|f(\sigma_\alpha(z))|^p|\sigma_\alpha'(z)|^2v(\sigma_\alpha(z))\,dA(z)\\&\le\sup_{z\in D}v(z)\int_D|f(\sigma_\alpha(z))|^p|\sigma_\alpha'(z)|v(\sigma_\alpha(z))\,dA(z)\\&\le M\int_D|f(\sigma_\alpha(z))|^p|\sigma_\alpha'(z)|v(\sigma_\alpha(z))\,dA(z)\\&\le M\int_D v(t)|f(t)|^p\,dA(z)=M\|f\|_{v,p}^p.\end{aligned}$$

Now put $g(z):=T_\alpha(f(z))$ for every $z\in D$. By the mean-value property,

$$v(0)|g(0)|^p\le\int_D v(z)|g(z)|^p\,dA(z)=\|g\|_{v,p}^p\le M\|f\|_{v,p}^p.$$

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Hence

$$v(0)|g(0)|^p = v(0)|f(\alpha)|^p(1-|\alpha|^2)^2 v(\alpha) \le M\|f\|_{v,p}^p.$$

Thus,

$$|f(\alpha)| \le M^{1/p} \frac{\|f\|_{v,p}}{v(0)^{1/p}(1-|\alpha|^2)^{2/p} v(\alpha)^{1/p}}.$$

Since α was arbitrary, the claim follows. □

We can now give the following sufficient condition for the boundedness of an operator $C_{\phi,\psi} : A_{v,p} \to A_{w,p}$.

THEOREM 3.10. *Under the assumptions of Lemma 3.9, the operator $C_{\phi,\psi} : A_{v,p} \to A_{w,p}$ is bounded if*

$$\sup_{z\in D} \frac{|\psi(z)| w(z)^{1/p}}{(1-|\phi(z)|^2)^{2/p} v(\phi(z))^{1/p}} < \infty.$$

The proof is the same as that of [12, Proposition 1]. What was needed was Lemma 3.9.

We close this section by characterizing when weighted composition operators acting on weighted Bergman spaces generated by a weight given as the absolute value of a holomorphic function are order bounded. We will not give the proof here since with an application of Lemma 3.9 everything is analogous to the proofs given in Section 3.1.

THEOREM 3.11. *Let u be a bounded holomorphic function without any zeros on D and $v = |u|$ with $\sup_{z\in D} v(z) \le M < \infty$ and $\psi \in L^q(m)$. Let ϕ be an analytic self-map of the disk such that $|\phi^*(e^{i\Theta})| < 1$ a.e. $[m]$. For fixed p, $1 \le p < \infty$, the following are equivalent.*

(a) $C_{\phi,\psi} : A_{v,p} \to L^q(m)$ *is order bounded.*
(b) $\psi/(v(|\phi^*|)^{1/p}(1-|\phi^*|^2)^{2/p}) \in L^q(m)$.

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