Canad. Math. Bull. Vol. **40** (2), 1997 pp. 254–256

SUBDIAGONAL ALGEBRAS FOR SUBFACTORS II (FINITE DIMENSIONAL CASE)

KICHI-SUKE SAITO AND YASUO WATATANI

ABSTRACT. We show that finite dimensional subfactors do not have subdiagonal algebras unless the Jones index is one.

1. **Introduction.** In the previous paper [SW] we started to investigate a relation between subdiagonal algebras and subfactors. The notion of subdiagonal algebras was introduced by Arveson [A] to unify several aspects of non-selfadjoint operator algebras. Later Jones [J] created the theory of subfactors.

Let N be a type II_1-factor, G a countable discrete group and $\alpha: G \longrightarrow \mathrm{Aut}\, N$ an outer action. We consider N a subfactor of the crossed product $M = N \rtimes G$. In [SW] we showed that there exists a bijective correspondence between the set of all maximal subdiagonal algebras for $N \subset M$ and the set of all positive cones of total orders on G. Therefore we can regard a subdiagonal algebra as a quantization of (a positive cone of) a total order on a group. A totally ordered group G must be torsion free, in particular the order of G must be infinite, unless $G = \{1\}$. Therefore it is reasonable to conjecture that if N is a subfactor of a finite factor with finite Jones index, then there exist no subdiagonal algebras with respect to the canonical conditional expectation $E: M \longrightarrow N$ determined by the trace unless $M = N$. In [SW] we confirmed the conjecture in the case of subfactor N of a hyperfinite II_1-factor M with $[M : N] \leq 4$. In this paper we shall show that the conjecture is true in the case of finite dimensional subfactors.

2. **Finite dimensional case.** Let M be a finite von Neumann algebra with a faithful normal normalized trace τ. We recall the definition of (σ-weakly closed) subdiagonal algebras by Arveson [1]. Let $\mathcal{A} \ni 1$ be a σ-weakly closed subalgebra of M and E a faithful normal conditional expectation from M onto $N = \mathcal{A} \cap \mathcal{A}^*$ such that $\tau\big(E(x)\big) = \tau(x)$ for $x \in M$. Then $\mathcal{A}$ is called a maximal subdiagonal algebra of M with respect to E if the following conditions are satisfied:

(1) $\mathcal{A} + \mathcal{A}^*$ is σ-weakly dense in M,

(2) $E(xy) = E(x)E(y)$ for $x, y \in \mathcal{A}$,

(3) $\mathcal{A}$ is maximal among those subalgebras of M satisfying (1) and (2).

An analytic crossed product is a typical example of a maximal subdiagonal algebra and has been studied deeply, for example see [LM], [MMS1], [MMS2]. But in this paper

Received by the editors October 17, 1995.
AMS subject classification: Primary 46K50, 46L37.

we consider only finite dimensional algebras. If M is the $n \times n$ full matrix algebra and N is the diagonals, then the set $\mathcal{A}$ of all upper triangular matrices is a maximal subdiagonal algebra. In this case N is abelian. We shall show that the assumption of factorness of N changes the situation completely.

THEOREM 1. *Let $M = M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices and $N = \mathbb{C} \subset M$. Let $E = \mathrm{tr}\colon M \longrightarrow N$ be the normalized trace. Then there exist no maximal subdiagonal algebras of M with respect to E unless $n = 1$.*

PROOF. Suppose that there exists a maximal subdiagonal algebra $\mathcal{A}$ of M with respect to $E = \mathrm{tr}$.

Let $\mathcal{A}_0 = \{x \in \mathcal{A} \mid \mathrm{tr}(x) = 0\}$. Then $M = \mathcal{A}_0 \oplus \mathbb{C}I \oplus \mathcal{A}_0^*$. First we shall show that $x^n = 0$ for any $x \in \mathcal{A}_0$. In fact for $x \in \mathcal{A}_0$, we may assume that x is an upper triangular matrix. Put

$$x = \begin{pmatrix} \alpha_1 & & & * \\ & \alpha_2 & * & \\ & & \ddots & \\ \mathrm{O} & & & \alpha_n \end{pmatrix}$$

For $k = 1, 2, 3, \ldots n$, $\mathrm{tr}(x^k) = \mathrm{tr}(x)^k = 0$, because $E = \mathrm{tr}$ is multiplicative on $\mathcal{A}$. Hence

$$\sum_{i=1}^{n} \alpha_i^k = \mathrm{tr}(x^k) = 0 \quad (\text{for } k = 1, \ldots n).$$

This implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Therefore $x^n = 0$.

Thus each element of $\mathcal{A}_0$ is nilpotent. We regard $\mathcal{A}_0$ as a Lie subalgebra of $M_n(\mathbb{C})$. Then there exists a basis relative to which all elements of $\mathcal{A}_0$ are strictly upper triangular (see, for example, [H; Corollary on page 13]). Consequently,

$$n^2 = \dim M_n(\mathbb{C}) = 1 + 2\dim \mathcal{A}_0 \leq 1 + 2\Big(\frac{n^2 - n}{2}\Big)$$

Thus, $n \leq 1$ and the Theorem 1 follows. ∎

THEOREM 2. *Let M be a finite dimensional factor and N a subfactor of M. Then there exist no maximal subdiagonal algebras of M with respect to the canonical conditional expectation E unless $M = N$.*

PROOF. We may put that $M = M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$, $N = M_m(\mathbb{C}) \otimes \mathbb{C}I$ and $E = \mathrm{id} \otimes \mathrm{tr}\colon M \longrightarrow N$ be the canonical expectation determined by the trace on M. Suppose that there exists a maximal subdiagonal algebra $\mathcal{A}$ of M with respect to E. Let

$$\begin{aligned} \tilde{M} &= N' \cap M = \mathbb{C}I \otimes M_n(\mathbb{C}) \cong M_n(\mathbb{C}) \\ \tilde{N} &= N' \cap N = \mathbb{C}I \otimes \mathbb{C}I \cong \mathbb{C}I \\ \tilde{\mathcal{A}} &= N' \cap \mathcal{A} \\ \tilde{E} &= E|_{\tilde{M}}\colon \tilde{M} \longrightarrow \tilde{N} \end{aligned}$$

https://doi.org/10.4153/CMB-1997-030-8 Published online by Cambridge University Press

Then we can identify $\tilde{E}$ with the trace on $\tilde{M}$. We shall show that $\tilde{A}$ is a maximal subdiagonal subalgebra with respect to $\tilde{E}$.

Since $\tilde{E}$ is a restriction of E, $\tilde{E}$ is multiplicative on $\tilde{A}$.

We have that

$$\tilde{A} \cap \tilde{A}^* = (N' \cap A) \cap (N' \cap A^*) = N' \cap (A \cap A^*) = N' \cap N = \mathbb{C}I = \tilde{N}$$

By definition, we have $\tilde{M} \supset \tilde{A} + \tilde{A}^*$. For $x \in \tilde{M}$, there exist a_1 and $a_2 \in A$ s.t.

$$x = a_1 + a_2^*$$

Then for any unitary $u \in N$

$$x = uxu^* = ua_1u^* + ua_2^*u^*$$

Since N is finite dimensional, the set $U(N)$ of unitaries in N is a compact group. Therefore

$$x = \int_{U(N)} uxu^*\, du = \int_{U(N)} ua_1u^*\, du + \left(\int_{U(N)} ua_2u^*\, du\right)^*$$

Put $\tilde{a}_i = \int_{U(N)} ua_iu^*\, du \quad (i = 1, 2)$, then

$$x = \tilde{a_1} + \tilde{a_2}^* \text{ and } \tilde{a_1}, \tilde{a_2} \in N' \cap A = \tilde{A}.$$

Thus $\tilde{M} = \tilde{A} + \tilde{A}^*$. Therefore $\tilde{A}$ is a maximal subdiagonal algebra. Then $n = 1$ by Theorem 1. Hence $M = N$. ■

ACKNOWLEDGMENT. The authors would like to express their thanks to the referee for his suggestion which shortens significantly the proof of Theorem 1.

REFERENCES

[A] W. B. Arveson, *Analyticity in operator algebras*, Amer. J. Math. **89**(1967), 578–642.
[H] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1972.
[J] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72**(1983), 1–15.
[LM] R. Loebl and P. S. Muhly, *Analyticity and flows in von Neumann algebras*, J. Funct. Anal. **29**(1978), 214–252.
[MMS1] M. McAsey, P. S. Muhly and K.-S. Saito, *Nonselfadjoint crossed products (invariant subspaces and maximality)*, Trans. Amer. Math. Soc. **248**(1979), 381–409.
[MMS2] ______, *Nonselfadjoint crossed products II*, J. Math. Soc. Japan **33**(1981), 485–495.
[SW] K.-S. Saito and Y. Watatani, *Subdiagonal algebras for subfactors*, J. Operator theory **31**(1994), 311–317.

Department of Mathematics
Niigata University
Niigata, 950-21
Japan
e-mail: saito@math.sc.niigata-u.ac.jp

Graduate School of Mathematics
Kyushu University
Ropponmatsu, Fukuoka 810
Japan
e-mail: watatani@math.kyushu-u.ac.jp

https://doi.org/10.4153/CMB-1997-030-8 Published online by Cambridge University Press