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ON COMMUTATIVITY OF BANACH ALGEBRAS WITH DERIVATIONS

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Abstract

The aim of this paper is to discuss the commutativity of a Banach algebra A via its derivations. In particular, we prove that if A is a unital prime Banach algebra and A has a nonzero continuous linear derivation $d : A \to A$ such that either $d((xy)^m) - x^m y^m$ or $d((xy)^m) - y^m x^m$ is in the centre of A for an integer $m = m(x, y)$ and sufficiently many x, y, then A is commutative. We give examples to illustrate the scope of the main results and show that the hypotheses are not superfluous.

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1. Introduction and results

This research has been motivated by the work of Yood [16]. Throughout, we let A denote a Banach algebra over the complex field with identity e, $Z(A)$ denote the centre of A and M be a closed linear subspace of A. Recall that an algebra A is said to be prime if for any $a, b \in A$, $aAb = (0)$ implies $a = 0$ or $b = 0$, and A is semiprime if for any $a \in A$, $aAa = (0)$ implies $a = 0$. For any $x, y \in A$, the symbol $[x, y]$ will denote the commutator $xy - yx$. We shall use several times the readily established fact that if the polynomial $p(t) = \sum_{r=0}^{n} b_r t^r \in A[t]$ lies in M for infinitely many values of the real variable t, then each b_r lies in M.

A linear mapping $d : A \longrightarrow A$ is said to be a derivation on A if $d(xy) = d(x)y + x\,d(y)$ holds for all $x, y \in A$. In [10, Theorem 2], Posner proved that if a prime ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. The analogous result was obtained for automorphisms [9]. Many authors have generalised Posner's result in the setting of rings and algebras (see [2, 7, 11–13], where further references can be found). Considerable attention has been paid to commutativity theorems for rings and algebras (see, for example, [6, Ch. 3] and [3, Ch. 2]), where again further references can be found. Our results on commutativity for Banach algebras take a different direction.

In [4, 5], Herstein proved that a ring R is commutative if it has no nonzero nilpotent ideal and there is a fixed integer $n > 1$ such that $(xy)^n = x^n y^n$ for all $x, y \in R$

(see also [1]). In the case of Banach algebras, Yood [16] sharpened these results. More precisely, he proved the following result:

THEOREM 1.1. *Suppose that there are nonempty open subsets G_1 and G_2 of A such that for each $x \in G_1$ and $y \in G_2$ there is an integer $n = n(x, y) > 1$ where either $(xy)^n - x^n y^n$ or $(xy)^n - y^n x^n$ lies in M. Then $[x, y] \in M$ for all $x, y \in A$.*

This result motivated us to prove the following theorems.

THEOREM 1.2. *Let A be a unital prime Banach algebra and $d : A \to A$ be a nonzero continuous linear derivation. Suppose that there are open subsets G_1, G_2 of A such that either $d((xy)^m) - x^m y^m \in Z(A)$ or $d((xy)^m) - y^m x^m \in Z(A)$ for each $x \in G_1$ and $y \in G_2$ and an integer $m = m(x, y) > 1$. Then A is commutative.*

THEOREM 1.3. *Let A be a unital prime Banach algebra and $d : A \to A$ be a nonzero continuous linear derivation. Suppose that there are open subsets G_1, G_2 of A such that either $d((xy)^m) - d(x^m)d(y^m) \in Z(A)$ or $d((xy)^m) - d(y^m)d(x^m) \in Z(A)$ for each $x \in G_1$ and $y \in G_2$ and an integer $m = m(x, y) > 1$. Then A is commutative.*

THEOREM 1.4. *Let A be a unital prime Banach algebra and $d : A \to A$ be a nonzero continuous linear derivation. Suppose that there is an open subset G_1 of A such that either $d(x^m) - x^m \in Z(A)$ or $d(x^m) + x^m \in Z(A)$ for each $x \in G_1$ and an integer $m = m(x) > 1$. Then A is commutative.*

2. Proofs of the theorems

PROOF OF THEOREM 1.2. Fix $x \in G_1$. For each n we define the set $U_n = \{y \in A \mid d((xy)^n) - x^n y^n \notin Z(A) \text{ and } d((xy)^n) - y^n x^n \notin Z(A)\}$. We claim that U_n is open. To show that U_n is open we prove that its complement, U_n^c, is closed. For this, we take a sequence $(z_k) \in U_n^c$ such that $z_k \to z$ as $k \to \infty$ and prove that $z \in U_n^c$. Since $z_k \in U_n^c$, either

$$d((xz_k)^n) - x^n z_k^n \in Z(A) \tag{2.1}$$

or

$$d((xz_k)^n) - z_k^n x^n \in Z(A). \tag{2.2}$$

From (2.1), since d is continuous,

$$\lim_{k\to\infty} (d((xz_k)^n) - x^n z_k^n) = d\Big(\Big(x \lim_{k\to\infty} z_k\Big)^n\Big) - x^n \lim_{k\to\infty} z_k^n = d((xz)^n) - x^n z^n$$

is in $Z(A)$ and, similarly, from (2.2), we see that $d((xz)^n) - z^n x^n$ is in $Z(A)$. This implies that $z \in U_n^c$ and so U_n^c is closed and U_n is open.

By the Baire category theorem, if every U_n is dense then their intersection is also dense, which contradicts the existence of G_2. Hence, there exist a positive integer r such that U_r is not dense and a nonempty open set G_3 in the complement of U_r such

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that either $d((xy)^r) - x^r y^r \in Z(A)$ or $d((xy)^r) - y^r x^r \in Z(A)$ for all $y \in G_3$. Take $v_0 \in G_3$ and $w \in A$. For sufficiently small real t, $v_0 + tw \in G_3$ and either

$$d((x(v_0 + tw))^r) - x^r(v_0 + tw)^r \in Z(A) \tag{2.3}$$

or

$$d((x(v_0 + tw))^r) - (v_0 + tw)^r x^r \in Z(A). \tag{2.4}$$

Thus at least one of (2.3) and (2.4), say (2.3), is valid for infinitely many t. The expression $d((x(v_0 + tw))^r) - x^r(v_0 + tw)^r$ can be written as

$$\begin{aligned} &d(A_{r,0}(x, v_0, w)) - x^r B_{r,0}(v_0, w) \\ &\quad + d(A_{r-1,1}(x, v_0, w)) - x^r B_{r-1,1}(v_0, w)t \\ &\quad + \cdots \\ &\quad + d(A_{1,r-1}(x, v_0, w)) - x^r B_{1,r-1}(v_0, w)t^{r-1} \\ &\quad + d(A_{0,r}(x, v_0, w)) - x^r B_{0,r}(v_0, w)t^r. \end{aligned}$$

Let i, j be nonnegative integers. If $i + j = r$, then $A_{i,j}(x, v_0, w)$ is the sum of all terms in which xv_0 appears exactly i times and xw appears exactly j times in the expansion of $d(x(v_0 + tw)^r)$. Similarly, $B_{i,j}(v_0, w)$ is the sum of all terms in which v_0 appears exactly i times and w appears exactly j times in the expansion of $(v_0 + tw)^r$. The above expression is a polynomial in t and the coefficient of t^r is just $d((xw)^r) - x^r w^r$. Hence $d((xw)^r) - x^r w^r \in Z(A)$. We have therefore shown that, given $x \in G_1$, there is a positive integer r depending on w such that for each $w \in A$ either $d((xw)^r) - x^r w^r \in Z(A)$ or $d((xw)^r) - w^r x^r \in Z(A)$.

Next, fix $y \in A$ and for each positive integer k, set $V_k = \{v \in A \mid d((vy)^k) - v^k y^k \notin Z(A) \text{ and } d((vy)^k) - y^k v^k \notin Z(A)\}$. Each V_k is open (as shown above). By the Baire category theorem, if each V_k is dense then so is their intersection, which contradicts the existence of the open set G_1. Hence there exist an integer $m > 1$ and a nonempty open subset G_4 in the complement of V_m. If $x_0 \in G_4$ and $y \in A$, then $x_0 + tu \in G_4$ for all sufficiently small real t and either

$$d(((x_0 + tu)y)^m) - (x_0 + tu)^m y^m \in Z(A)$$

or

$$d(((x_0 + tu)y)^m) - y^m(x_0 + tu)^m \in Z(A)$$

for each $u \in A$ and $x_0 \in G_4$. Arguing as before, we see that either $d((uy)^m) - u^m y^m \in Z(A)$ or $d((uy)^m) - y^m u^m \in Z(A)$ for each $u \in A$.

Now let S_k, $k > 1$, be the set of $y \in A$ such that for each $w \in A$ either $d((wy)^k) - w^k y^k \in Z(A)$ or $d((wy)^k) - y^k w^k \in Z(A)$. By what we have shown, the union of S_k is A. It is easily seen that each S_k is closed. Again, by the Baire category theorem, some S_l, $l > 1$, must have a nonempty open subset G_5. For $z \in A$, $y_0 \in G_5$ and all sufficiently small real t, either

$$d((w(y_0 + tz))^l) - w^l(y_0 + tz)^l \in Z(A)$$

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or

$$d((w(y_0 + tz))^l) - (y_0 + tz)^l w^l \in Z(A).$$

By the same arguments as before, for each $w, z \in A$, either $d((wz)^l) - w^l z^l \in Z(A)$ or $d((wz)^l) - z^l w^l \in Z(A)$. Since A is unital, for all real t, either

$$d(((e + tx)y)^n) - (e + tx)^n y^n \in Z(A)$$

or

$$d(((e + tx)y)^n) - y^n(e + tx)^n \in Z(A)$$

for all $x, y \in A$. Taking the coefficient of t in the expansion of the above equations, we get either

$$d\Big(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}\Big) - nxy^n \in Z(A) \tag{2.5}$$

or

$$d\Big(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}\Big) - ny^n x \in Z(A) \tag{2.6}$$

for all $x, y \in A$. Again, starting with $d((y(e + tx))^n)$ instead of $d(((e + tx)y)^n)$, we see that either

$$d\Big(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}\Big) - nxy^n \in Z(A) \tag{2.7}$$

or

$$d\Big(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}\Big) - ny^n x \in Z(A) \tag{2.8}$$

for all $x, y \in A$. Then at least one of the pairs of equations {(2.5), (2.7)}, {(2.5), (2.8)}, {(2.6), (2.7)} and {(2.6), (2.8)} must hold.

On combining the equations in these pairs, we get either $d([x, y^n]) \in Z(A)$ for all $x, y \in A$ or $d([x, y^n]) \pm n[x, y^n] \in Z(A)$ for all $x, y \in A$. Replacing y by $e + ty$ in the last expressions and using the same arguments as we have used above, we obtain $d([x, y]) \in Z(A)$ for all $x, y \in A$ or $d([x, y]) \pm n[x, y] \in Z(A)$ for all $x, y \in A$.

If we assume that $d([x, y]) \in Z(A)$ for all $x, y \in A$, then by [2, Theorem 2.2], since A is prime, we conclude that A is commutative.

Suppose, instead, that

$$d([x, y]) - n[x, y] \in Z(A) \quad \text{for all } x, y \in A.$$

This can be written as

$$[d([x, y]) - n[x, y], z] = 0 \quad \text{for all } x, y, z \in A \tag{2.9}$$

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which implies $[d([x,y]),z] - n[[x,y],z] = 0$ for all $x,y,z \in A$, that is,

$$[[d(x),y],z] + [[x,d(y)],z] - n[[x,y],z] = 0 \quad \text{for all } x,y,z \in A.$$

Replacing y by $[y,w]$ in the above expression,

$$[[d(x),[y,w]],z] + [[x,d([y,w])],z] - n[[x,[y,w]],z] = 0 \quad \text{for all } w,x,y,z \in A,$$

that is, $[[d(x),[y,w]],z] + [[x,d([y,w]) - n[y,w]],z] = 0$ for all $w,x,y,z \in A$. An application of (2.9) then yields $[[d(x),[y,w]],z] = 0$ for all $w,x,y,z \in A$, that is,

$$[d(x),[y,w]] \in Z(A) \quad \text{for all } w,x,y \in A.$$

In view of [8, Theorem 2], we have either $[y,w] \in Z(A)$ for all $y,w \in A$ or $A \subseteq Z(A)$. In both cases, A must be commutative.

Finally, in a similar manner, we can prove the result for the case in which $d([x,y]) + n[x,y] \in Z(A)$ for all $x,y \in A$. This completes the proof of the theorem. □

The proof of Theorem 1.3 is the same as that of Theorem 1.2 and we omit the details. The proof of Theorem 1.4 uses a simpler version of the same technique.

Proof of Theorem 1.4. First set $U_n = \{x \in A \mid d(x^n) - x^n \notin Z(A) \text{ and } d(x^n) + x^n \notin Z(A)\}$. By applying the Baire category theorem to the sets U_n we deduce, by reasoning as above, that there is a positive integer r such that either $d(y^r) - y^r \in Z(A)$ or $d(y^r) + y^r \in Z(A)$ for all $y \in A$. Since A is unital, then for infinitely many real t we have either

$$d((e+ty)^n) - (e+ty)^n \in Z(A)$$

or

$$d((e+ty)^n) - (e+ty)^n \in Z(A)$$

for all $y \in A$. The coefficient of t in the above equations is $d(y) - y$ or $d(y) + y$. Hence, either $d(y) - y \in Z(A)$ or $d(y) + y \in Z(A)$ for all $y \in A$. If we suppose that $d(y) - y \in Z(A)$ for all $y \in A$ then $[d(y),z] = [y,z]$ for all $y,z \in A$. In particular, for $y = z$,

$$[d(z),z] = 0 \quad \text{for all } z \in A.$$

Hence, by Posner's result [10], A is commutative. Replacing $d = -d$ deals with the alternative case. This proves the theorem.

The following example demonstrates that it is essential for A to be prime in the hypotheses of Theorems 1.2 and 1.3 (in the case where $A = G_1 = G_2$).

Example 2.1. Let $\mathbb{F}$ be any field, and consider

$$A = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a_{12}, a_{13}, a_{23} \in \mathbb{F} \right\}.$$

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Clearly, A is a Banach algebra under the norm $\|A\| = \max_k \sum_{j=1}^{3} |a_{jk}|$ for all $a_{jk} \in \mathbb{F}$ but not prime. Define a map $d : A \longrightarrow A$ by

$$d\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is straightforward to check that d is a nonzero continuous derivation on A and, for $n > 1$, $d((xy)^n) - x^n y^n \in Z(A)$ or $d((xy)^n) - y^n x^n \in Z(A)$ and $d((xy)^n) - d(x^n)d(y^n) \in Z(A)$ or $d((xy)^n) - d(y^n)d(x^n) \in Z(A)$ hold for all $x, y \in A$. However, A is not commutative.

3. Applications

In this section we will discuss some applications of Theorem 1.2.

3.1. Let $\mathbb{C}$ be the field of complex numbers, let

$$M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{C} \right\}$$

be a noncommutative unital prime algebra of all 2×2 matrices over $\mathbb{C}$ with the usual matrix addition, and define matrix multiplication as follows:

$$A \times_K B = KAB \quad \text{for all } A, B \in M \text{ where } K = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \text{ and } |\lambda| > 1.$$

For $A = (\alpha_{jk}) \in M$, define $\|A\| = \max_k \sum_{j=1}^{2} |\alpha_{jk}|$. Then M is a normed linear space. Further, define a map $d : M \to M$ by

$$d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix} \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M.$$

Since M is finite-dimensional, it is straightforward to check that d is a nonzero continuous linear derivation on M. Observe that

$$G_1 = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \middle| \, t \in \mathbb{R} \right\} \quad \text{and} \quad G_2 = \left\{ \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

are open subsets of M such that $d((A \times_K B)^m) - A^m \times_K B^m \in Z(M)$ or $d((A \times_K B)^m) - B^m \times_K A^m \in Z(M)$ for all $A \in G_1$ and $B \in G_2$. Hence, it follows from Theorem 1.2 that M is not a Banach algebra under the defined norm.

3.2. Define M, G_1, G_2 and matrix multiplication in the same way as above. Take the Frobenius norm $\|A\|_F$ on M defined by

$$\|A\|_F = \left(\sum_{i,j=1}^{2} |\alpha_{ij}|^2 \right)^{1/2} \quad \text{for all } A = (\alpha_{ij}) \in M.$$

Then, M is a normed linear space under the defined norm. Next, let $d : M \to M$ be the inner derivation of M determined by $e_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, that is, $d(A) = A \times_K e_{11} - e_{11} \times_K A$ for

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all $A \in M$. Since M is finite-dimensional, it is easily seen that d is a nonzero continuous derivation on M. Also, for any $A \in G_1$ and $B \in G_2$, either $d((A \times_K B)^m) - A^m \times_K B^m \in Z(M)$ or $d((A \times_K B)^m) - B^m \times_K A^m \in Z(M)$. Hence, in view of Theorem 1.2, we conclude that M cannot be made into a Banach algebra under the defined norm.

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