DISCONJUGACY CONDITIONS FOR THE THIRD ORDER LINEAR DIFFERENTIAL EQUATION

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1. Introduction. An n^{th} order homogeneous linear differential equation is said to be disconjugate on the interval I of the real line in case no non-trivial solution of the equation has more than $n - 1$ zeros (counting multiplicity) on I. It is the purpose of this paper to establish several necessary and sufficient conditions for disconjugacy of the third order linear differential equation

$$(1.1) \qquad L[x] \equiv x''' + p_2 x'' + p_1 x' + p_0 x = 0,$$

where $p_i(t)$ is continuous on the compact interval $[a,b]$, $i = 0, 1, 2$. Lasota [1], Mathsen [2], [3], and Jackson [4] have all given sufficient conditions to insure disconjugacy of (1.1). Recently, Hartman [5] gave necessary and sufficient conditions for disconjugacy of the general homogeneous n^{th} order linear differential equation. We shall study disconjugacy of (1.1) by considering the corresponding Ricatti equation

$$(1.2) \qquad u'' = -3uu' - p_2 u' - u^3 - p_2 u^2 - p_1 u - p_0 ,$$

which is obtained from (1.1) by the substitution $u = x'/x$. Section 2 is devoted to definitions and a preliminary result for the general nonlinear second order differential equation

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(1.3) $$x'' = f(t,x,x') ,$$

where $f(t,x,x')$ is continuous on the set

$$S = \{(t,x,x') : a \leq t \leq b , |x| + |x'| < + \infty\} .$$

2. A function $\alpha \in C^{(2)}[a,b]$ is said to be a <u>lower solution</u> of (1.3) in case $\alpha'' \geq f(t,\alpha,\alpha')$ on $[a,b]$. Similarly, $\beta \in C^{(2)}[a,b]$ is said to be an <u>upper solution</u> of (1.3) in case $\beta'' \leq f(t,\beta,\beta')$ on $[a,b]$. In [6] a solution x of (1.3) is said to have property (B) on the interval I in case there exists a sequence x_n of solutions of (1.3) such that:

(i) $x_n \to x$ and $x_n' \to x'$ uniformly on $[a,b]$;

(ii) $x - x_n \neq 0$ and has the same sign for all $n \geq 1$ and $a \leq t \leq b$;

(iii) $|x' - x_n'| \leq C|x - x_n|$ for all $n \geq 1$ and $a \leq t \leq b$, where C is a constant independent of n and t.

LEMMA 2.1. <u>Let</u> $f(t,x,x')$ <u>and the partial derivative functions</u> $f_{x'}(t,x,x')$ and $f_x(t,x,x')$ <u>be continuous on</u> S <u>and let</u> $x(t)$ <u>be a solution of</u> (1.3) <u>on</u> $[a,b]$. <u>Then</u> $x(t)$ <u>has property</u> (B) <u>on</u> $[a,b]$ <u>if and only if the equation</u>

(1.4) $$x'' = f_{x'}(t,x(t),x'(t))x' + f_x(t,x(t),x'(t))x$$

<u>is disconjugate on</u> $[a,b]$.

<u>Proof</u>. Assume first that $x(t)$ has property (B) on $[a,b]$ and let x_n be a sequence of solutions of (1.3) as in the definition of property (B). To be specific, assume $x(t) - x_n(t) > 0$ for all

$n \geq 1$ and $a \leq t \leq b$. Define $\Delta_n(t)$ by

$$\Delta_n(t) = (x(t) - x_n(t))/(x(a) - x_n(a)), \quad a \leq t \leq b .$$

Then $\Delta_n(a) = 1$, $|\Delta_n'(a)| \leq C$, where C is the constant occurring in the definition of property (B), and $\Delta_n(t) > 0$ on $[a,b]$. Also,

$$\Delta_n'' = f_{x'}(t,x(t),x'(t))\Delta_n' + f_x(t,x(t),x'(t))\Delta_n + p_n(t)\Delta_n' + q_n(t)\Delta_n ,$$

where p_n, $q_n \to 0$ uniformly on $[a,b]$. Hence, by standard convergence theorems (see for example [9], Theorem 3.2, p. 14), a subsequence of the sequence Δ_n converges to a solution $Z(t)$ of (1.4) satisfying $Z(a) = 1$, $Z'(a) = C_o$ where $|C_o| \leq C$, and $Z(t) \geq 0$ on $[a,b]$. Since initial value problems for (1.4) have unique solutions it follows that $Z(t) > 0$ on $[a,b)$ and this implies that (1.4) is disconjugate on $[a,b]$.

Conversely, if $x(t)$ is a solution of (1.3) such that (1.4) is disconjugate on $[a,b]$, let $Z(t)$ be a solution of (1.4) with $Z(t) > 0$ on $[a,b]$. For each $n \geq 1$, let $x_n(t)$ be a solution of the IVP:

$$y'' = f(t,y,y') , \quad y(a) = x(a) + Z(a)/n , \quad y'(a) = x'(a) + Z'(a)/n .$$

It follows that there is an $N \geq 1$ such that $n \geq N$ implies $x_n(t) \in C^{(2)}[a,b]$. For $n \geq N$ define $Z_n(t)$ by

$$Z_n(t) = n(x_n(t) - x(t)) , \quad t \in [a,b].$$

Then $Z_n(t)$ satisfies $Z_n(a) = Z(a)$, $Z_n'(a) = Z'(a)$, and

$$Z_n'' = f_{x'}(t,x(t),x'(t))Z_n' + f_x(t,x(t),x'(t))Z_n + p_n(t)Z_n' + q_n(t)Z_n ,$$

where p_n, $q_n \to 0$ uniformly on $[a,b]$. Therefore $Z_n(t) \to Z(t)$ uniformly on $[a,b]$, so there is an $N_1 \geq N$ such that $Z_n(t) > 0$ for $n \geq N_1$. Let $\delta > 0$ be such that $Z_n(t) \geq \delta$ on $[a,b]$ for $n \geq N_1$ and let $M = \max \{|Z_n'(t)| : n \geq N_1 , a \leq t \leq b\}$. Setting $C = (M + 1)/\delta$ it follows that $\{x_n(t)\}_{n=N_1}^{\infty}$ is the desired sequence in the definition of property (B).

The necessity of Lemma 2.1 is due to Knobloch [6] and the sufficiency is due to Reid [8]. The proof given above is, however, independent. Sufficient conditions for the existence of solutions of (1.3) possessing property (B) are given in [6] and [7].

3. The Cauchy function $K(t,s)$ for equation (1.1) is defined as follows: for $s \in [a,b]$ $x(t) \equiv K(t,s)$ is the solution of the IVP

$$L[x] = 0 , \quad x(s) = x'(s) = 0 , \quad x''(s) = 1 .$$

The next Theorem, which is the main result, also includes statements of relatively well-known and previously published results.

THEOREM 3.1. _The following statements are equivalent_:

(a) $L[x] = 0$ _is disconjugate on_ $[a,b]$;

(b) _The Cauchy function_ $K(t,s)$ _satisfies_ $K(t,s) > 0$ _for all_ $s, t \in [a,b]$, $s \neq t$;

(c) _There exists a lower solution_ $\alpha \in C^{(2)}[a,b]$ _of_ (1.2) _and an upper solution_ $\beta \in C^{(2)}[a,b]$ _of_ (1.2) _with_ $\alpha(t) < \beta(t)$ _on_ $[a,b]$;

(d) *There is a solution* $u(t)$ *of* (1.2) *which has property* (B) *on* $[a,b]$;

(e) *There is a solution* $u(t)$ *of* (1.2) *which is such that the variational equation*

$$x'' + (3u + p_2)x' + (3u^2 + 2p_2u + p_1 + 3u')x = 0 \tag{1.5}$$

is disconjugate on $[a,b]$;

(f) *There exists a sequence of solutions* y_n *of* (1.1) *and a limit solution* y_o *of* (1.1) *such that* $y_n > 0$, $y_o > 0$ *on* $[a,b]$ *for all* $n \geq 1$, $y_n \to y_o$, $y'_n \to y'_o$, $y''_n \to y''_o$ *uniformly on* $[a,b]$, *and such that* $y_n - y_o \neq 0$ *and* $y_ny'_o - y'_ny_o \neq 0$ *and has the same sign for all* $n \geq 1$ *and* $t \in [a,b]$;

(g) *There exist solutions* $x_1(t)$, $x_2(t)$ *of* (1.1) *with* $x_1 > 0$, $x_2 > 0$ *and* $x_1x'_2 - x_1x'_2 \neq 0$ *on* $[a,b]$.

Proof. That (a) is equivalent to (b) is well-known. (See for example [2], Lemma, p. 630). Hartman [6] states that (a) is equivalent to (g). Jackson [4] showed that (c) implies (a). Note also that (d) is equivalent to (e) by Lemma 2.1. Therefore, we shall prove (b) =>(c) =>(d) =>(f) =>(g) =>(b).

(b) implies (c). We may extend the definition of p_o , p_1 , and p_2 to a slightly larger interval $[a - \delta,b]$ so that $p_i(t) \in C[a - \delta,b]$, $i = 0, 1, 2$. Since $K(t,s) > 0$ for s , $t \in [a,b]$, $s \neq t$, a continuity argument implies that we may choose $\delta > 0$ sufficiently small so that $K(t,s) > 0$ for s , $t \in [a - \delta,b]$, $s \neq t$. Let $y_o(t) \equiv K(t,a - \delta)$ and let $y_1(t)$ be the solution of

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the IVP:

$$L[x] = 0 \ , \ x(a - \delta) = 0 \ , \ x'(a - \delta) = 1 \ , \ x''(a - \delta) = 0 \ .$$

Since $y_o(t) > 0$ on $[a,b]$ there is a $k > 0$ such that

$$y_2(t) \equiv ky_o(t) + y_1(t) > 0 \quad \text{on} \quad [a,b] \ .$$

We claim that $y_o y_2' - y_o' y_2 \neq 0$ on $[a,b]$. For if not, let $t_o \in [a,b]$ be such that $y_o y_2' - y_o' y_2 = 0$ at $t = t_o$. Define

$$h(t) = y_o(t)y_2(t_o) - y_o(t_o)y_2(t) \ , \ t \in [a - \delta, b] \ .$$

Then we have $h(t_o) = h'(t_o) = 0$, so that $h(t)$ is a constant multiple of $K(t,t_o)$. But $h(a - \delta) = 0$, which is a contradiction. Therefore, we may assume $y_o y_2' - y_o' y_2 > 0$ on $[a,b]$. Then $\beta \equiv y_2'/y_2$ and $\alpha \equiv y_o'/y_o$ are upper and lower solutions of (1.2), respectively, with $\alpha < \beta$ on $[a,b]$.

<u>(c) implies (d)</u>. Let α, $\beta \in C^{(2)}[a,b]$ be lower and upper solutions of (1.2), respectively, with $\alpha < \beta$ on $[a,b]$. The right hand side of (1.2) is $\leq C_1 + C_2|u'|$ for all $a \leq t \leq b$, $\alpha(t) \leq u \leq \beta(t)$, and all $|u'| < +\infty$, where C_1, C_2 are real constants, and this is sufficient to imply that (1.2) satisfies all hypotheses of Theorem 3 of [6]. Alternatively, one may apply Theorem 4.1 of [7] with a suitably chosen auxiliary function $g(x)$. In either case, we conclude the existence of a solution $u(t)$ of (1.2) which has property (B) on $[a,b]$ and satisfies $\alpha(t) \leq u(t) \leq \beta(t)$ on $[a,b]$.

(d) implies (f). Let $u_o(t)$ be a solution of (1.2) with property (B) on [a,b] and let u_n be the associated sequence. To be specific, assume $u_n > u_o$ on [a,b] for all $n \geq 1$. Let $r_n = 1 + 1/n$, $n \geq 1$, and define

$$y_n(t) = r_n(\exp(\int_a^t u_n(s)ds)) \ , \ y_o(t) = \exp(\int_a^t u_o(s)ds) \ .$$

Then $y_n \to y_o$, $y_n' \to y_o'$, and $y_n'' \to y_o''$ uniformly on [a,b]. Also, $y_n - y_o > 0$ on [a,b] for all $n \geq 1$ and, finally,

$$y_n'/y_n - y_o'/y_o = u_n - u_o > 0 \quad \text{on } [a,b] \text{ for all } n \geq 1 \ .$$

(If $u_n < u_o$ on [a,b], take $r_n = 1 - 1/n$, $n \geq 1$).

(f) implies (g). This is obvious.

(g) implies (b). Let x_1, x_2 be solutions of (1.1) with $x_1 > 0$, $x_2 > 0$ and $x_1x_2' - x_1'x_2 < 0$ on [a,b]. Let $a \leq t_1 \leq b$ and define

$$y(t,t_1) = x_2(t)u(t) \ ,$$

where

$$u(t) = \int_{t_1}^t v(s)w(s)ds \ , \ v(s) = (x_1(s)/x_2(s))' \ ,$$

$$w(s) = \int_{t_1}^s \{A(r)/(v(r))^2\}dr \ , \ A(r) = \exp\,(-\int_{t_1}^r (3(x_2'/x_2) + p_2)dq) \ .$$

Then $y(t,t_1)$ is a solution of (1.1) and we have $A(r) > 0$ and $v(r) > 0$ for $a \leq r \leq b$. Hence $w(s) = 0$ if and only if $s = t_1$.

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Since $y(t_1,t_1) = 0 = y'(t_1,t_1)$ and $y(t,t_1) \neq 0$ if $t \neq t_1$, we conclude that $y(t,t_1)$ is a constant multiple of $K(t,t_1)$. Therefore, $K(t,t_1) > 0$ for $a \leq t_1$, $t \leq b$, $t \neq t_1$, and this completes the proof.

As a consequence of the preceding Theorem, we note that if (1.1) has a positive solution $x(t)$ then disconjugacy of (1.5) on $[a,b]$ with $u = x'/x$ implies disconjugacy of (1.1) on $[a,b]$. One can therefore establish sufficient conditions for disconjugacy of (1.1) by using known sufficient conditions for disconjugacy of the general second order linear equation

(1.6) $$x'' + px' + qx = 0 \text{ , } p, q \in C[a,b] \text{ .}$$

LEMMA 3.2. <u>Equation (1.6) is disconjugate on</u> $[a,b]$ <u>if and only if there is a continuously differentiable function</u> $r(t)$ <u>such that</u>

$$r' + r^2 + pr + q \leq 0 \quad \text{on} \quad [a,b] \text{ .}$$

<u>Proof</u>. See [9], p. 362, Theorem 7.2 and the following remark.

COROLLARY 3.3. <u>Assume (1.1) has a solution</u> $x(t) > 0$ <u>on</u> $[a,b]$. <u>Let</u> $u = x'/x$. <u>Then (1.1) is disconjugate on</u> $[a,b]$ <u>in case there is a continuously differentiable function</u> r <u>such that</u>

(1.7) $$r' + r^2 + (3u + p_2)r + (3u^2 + 3u' + 2p_2u + p_1) \leq 0$$

<u>on</u> $[a,b]$.

With $r \equiv 0$, Corollary 3.3 is Lemma 2.3 of [3]. Note also that if (1.1) is disconjugate on $[a,b]$ then there is a solution $u(t)$ of (1.2) and a function $r \in C^{(1)}[a,b]$ such that (1.7) holds on $[a,b]$.

As an example of Theorem 3.1, we next consider the equation

$$L[x] = x''' - (3/t)x'' + (k/t)x' - (k/t^2)x = 0$$

on the interval $[1,+\infty)$. One solution of (1.8) is $x(t) = t$ so that in this case, with $u = 1/t$, equation (1.5) becomes

$$(1.9) \qquad x'' + (k/t - 6/t^2)x = 0 .$$

The well-known Lyapunov criterion (see [9], Corollary 5.1, p. 346) implies that (1.9) is disconjugate on $[1,T]$ provided

$$(1.10) \qquad \int_1^T (k/t - 6/t^2)^+ dt \leq 4/(T - 1) ,$$

where $(k/t - 6/t^2)^+ = \max\{k/t - 6/t^2, 0\}$. If $k > 0$, let $T_1 = \max\{1, 6/k\}$. Then the integrand in (1.10) is non-negative on $[T_1,T]$. Therefore (1.9) is disconjugate on $[1,T]$ as long as

$$(T - 1)/4 \{k \ln(T/T_1) - 6(1/T_1 - 1/T)\} \leq 1 .$$

For example, if $k = 1/10$, then (1.9) and hence (1.8) are disconjugate on $[1,T]$ where $T > 120$. The conditions given in [1], for example, imply disconjugacy of (1.8) on $[1,T_0]$ where $T_0 > 1$ and

$$3(T_0 - 1)/4 + (T_0 - 1)^2/10\pi^2 + (T_0 - 1)^3/20\pi^2 \leq 1 .$$

This implies that $T_0 < 7/3$. Additional disconjugacy results for (1.1) can be obtained in an analogous manner by using other known sufficient conditions for disconjugacy of (1.6).

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