SOME REMARKS ON THE EXCEPTIONAL SIMPLE LIE GROUP $\mathfrak{F}_4$

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1. Let $\mathfrak{C}$ be the Cayley algebra of dimension 8 over the field R of real numbers and let $\mathfrak{J}$ be the set of all 3×3 Hermitian matrices

$$X=\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \tag{1}$$

with coefficients in $\mathfrak{C}$. We define the multiplication in $\mathfrak{J}$ by

$$X\circ Y=\frac{1}{2}(XY+YX).$$

Then $\mathfrak{J}$ becomes a distributive algebra over R. A non-singular linear transformation α of $\mathfrak{J}$ is said to be an automorphism of $\mathfrak{J}$, if

$$\alpha(X\circ Y)=\alpha X\circ\alpha Y$$

for all $X, Y\in\mathfrak{J}$. The group $\mathfrak{A}$ of all the automorphisms of $\mathfrak{J}$ is compact and the connected component containing the identity of $\mathfrak{A}$ is the exceptional simple compact group $\mathfrak{F}_4$.[1] Denote by E_i the matrix (1) with $\xi_i=1$, all remaining terms zero. Let $\mathfrak{N}$ be the subgroup of $\mathfrak{F}_4$ consisting of all automorphisms α such that $\alpha E_i=E_i$ for $i=1, 2, 3$ and let $\mathfrak{H}_i$ $(i=1, 2, 3)$ be the subgroups of $\mathfrak{F}_4$ consisting of all $\alpha\in\mathfrak{F}_4$ such that $\alpha E_i=E_i$. Then the left coset spaces $\mathfrak{F}_4/\mathfrak{H}_i$ are homomorphic to the set Π of all irreducible idempotents of $\mathfrak{J}$ and Π is geometrically the "*plan projectif des octaves.*"[2]

In this note we prove the following two theorems.

THEOREM 1. *$\mathfrak{N}$ is connected and isomorphic to the universal covering group $\widetilde{SO(8)}$ of the proper orthogonal group $SO(8)$ of 8 dimensional euclidean space.*

THEOREM 2. *$\mathfrak{H}_i$ are connected and isomorphic to the universal covering group $\widetilde{SO(9)}$ of the proper orthogonal group $SO(9)$ of 9 dimensional euclidean space.*

Theorem 2 gives a proof of a result anounced by A. Borel.[3]

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1) See, Chevalley-Schafer [2] and Freudenthal [3].

2) See, Freudenthal [3] §7 and Hirsch [4].

3) See, Borel [1], Théorème 1.

2. *Proof of Theorem 1.* Let F_i^a be the matrix (1) with $x_i = a$ and all numbers except x_i zero. Then $E_i \circ F_i^a = 0$, $E_j \circ F_i^a = \frac{1}{2} F_i^a$ if $i \neq j$. Let $\alpha \in \mathfrak{N}$. Then $E_i \circ \alpha F_i^a = 0$ and $E_j \circ \alpha F_i^a = \frac{1}{2} \alpha F_i^a$. It follows that

$$\alpha F_i^a = F_i^{\alpha_i a}, \quad (i = 1, 2, 3),$$

where α_i are the linear transformtions of $\mathfrak{C}$.

Now $F_i^a \circ F_i^b = (a, b)(E_j + E_k)$,[4] where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$, implies

$$(\alpha_i a, \alpha_i b) = (a, b).$$

Denote by $O(8)$ the group of all linear transformations of $\mathfrak{C}$ which leave the positive definite bilinear form (a, b) invariant. (i.e. orthogonal transformations of $\mathfrak{C}$.) Further $F_1^{2x} \circ F_2^{2y} = F_3^{2(\overline{xy})}$, $F_2^{2x} \circ F_3^{2y} = F_1^{2(\overline{xy})}$ and $F_3^{2x} \circ F_1^{2y} = F_2^{2(\overline{xy})}$ imply

$$(2) \qquad \begin{cases} \alpha_1(x)\alpha_2(y) = \kappa\alpha_3(xy), \\ \alpha_2(x)\alpha_3(y) = \kappa\alpha_1(xy), \\ \alpha_3(x)\alpha_1(y) = \kappa\alpha_2(xy), \end{cases}$$

where $\kappa\alpha_i(x) = \overline{\alpha_i(\bar{x})}$. Let γ be the orthogonal transformation of $\mathfrak{C}$ defined by $\gamma x = \bar{x}$ for all $x \in \mathfrak{C}$. Then $\kappa\alpha_i = \gamma\alpha_i\gamma$ and $\alpha_i \to \kappa\alpha_i$ is an automorphism of $O(8)$. We shall show that $\alpha_i \in SO(8)$ i.e. $\det. \alpha_i = 1$.

LEMMA 1. (*Principle of Triality.*)[5] *For every $\theta \in SO(8)$, there exist θ_1 and θ_2 in $SO(8)$ such that*

$$\theta(x)\theta_1(y) = \theta_2(xy)$$

for all $x, y \in \mathfrak{C}$. If there exist the other θ_1' and θ_2' in $SO(8)$ such that $\theta(x)\theta_1'(y) = \theta_2'(xy)$, then $\theta_1' = \pm\theta_1$ and $\theta_2' = \pm\theta_2$. The same holds also, if we start from θ_1 or θ_2 instead of θ.

LEMMA 2. *Let θ_i be in $O(8)$ and let*

$$(3) \qquad \theta_1(x)\theta_2(y) = \kappa\theta_3(xy)$$

for all $x, y \in \mathfrak{C}$. Then $\theta_2(x)\theta_3(y) = \kappa\theta_1(xy)$ and $\theta_3(x)\theta_1(y) = \kappa\theta_2(xy)$ for all $x, y \in \mathfrak{C}$.

Proof. Multiplying the both sides of (3) by $\overline{\theta_1(x)}/|x|^2$,[6] we have

$$\theta_2(y) = \frac{1}{|x|^2} \overline{\theta_1(x)}\ \overline{\theta_3(\overline{xy})}.$$

[4] The positive definite bilinear form (a, b) on $\mathfrak{C}$ is defined by $(a, b) = Re(a\bar{b})$, where $Re x = \frac{1}{2}(x + \bar{x})$.

[5] See, Freudenthal [3] p. 16.

[6] $|x|^2 = (x, x) = x \cdot \bar{x} = \bar{x} \cdot x$. In the following proof, we use the formulae $|\bar{x}| = |x|$, $|xy| = |x||y|$, $\bar{x}(xa) = (\bar{x}x)\, a$ and $(a\bar{x})x = a(\bar{x}x)$. See, Freudenthal [3] p. 7.

Analogously we have

$$\frac{1}{|y|^2}\theta_2(y)\cdot\theta_3(\bar{y}\bar{x}) = \overline{\theta_1(x)}\,.$$

Let $\bar{x} = yz$. Then

$$\frac{1}{|y|^2}\theta_2(y)\theta_3(\bar{y}(yz)) = \overline{\theta_1(\overline{yz})}\,.$$

Hence $\theta_2(y)\theta_3(z) = \kappa\theta_3(yz)$.

Lemma 3. *Let $\theta_i \in O(8)$ $(i = 1, 2, 3)$ and $\theta_1(x)\theta_2(y) = \kappa\theta_3(xy)$ for all $x, y \in \mathfrak{C}$. Then $\theta_i \in SO(8)$ $(i = 1, 2, 3)$.*

Proof. Suppose that θ_1 is not in $SO(8)$. For every $\eta_1 \in SO(8)$, there exist η_2 and η_3 in $SO(8)$ such that

$$\eta_1\theta_1(x)\eta_2\theta_2(y) = \kappa\eta_3\cdot\kappa\theta_3(xy) = \kappa(\eta_3\cdot\theta_3)(xy)\,.$$

Let us choose η_1 such that $\eta_1\theta_1 = \gamma$, where $\gamma x = \bar{x}$ for all $x \in \mathfrak{C}$. Then

(4) $$\bar{x}\zeta_2(y) = \kappa\zeta_3(xy)$$

for all $x, y \in \mathfrak{C}$, where $\zeta_2 = \eta_2\theta_2$ and $\zeta_3 = \eta_3\theta$. Putting $x = 1$ in (4), we have $\zeta_2(y) = \kappa\zeta_3(y)$. Hence $\zeta_2 = \kappa\zeta_3$ and

(5) $$\bar{x}\zeta_2(y) = \zeta_2(xy)\,.$$

Putting $y = 1$ in (5), we have

(6) $$\zeta_2(x) = \bar{x}\zeta_2(1)\,.$$

Let $\zeta_2(1) = a$. Then $a \neq 0$. It follows from (5) and (6) that $\bar{x}(\bar{y}a) = (\bar{y}\bar{x})a$. Hence $x(ya) = (yx)a$ for all $x, y \in \mathfrak{C}$. It follows that $a = 0$ and this is a contradiction. Hence $\theta_1 \in SO(8)$. We may prove analogously that θ_2 and θ_3 are also in $SO(8)$.

Thus α_i $(i = 1, 2, 3)$ in (2) are in $SO(8)$. Thus if $\alpha \in \mathfrak{N}$, then

(7) $$\alpha X = \begin{pmatrix} \xi_1 & \alpha_3(x_3) & \kappa\alpha_2(\bar{x}_2) \\ \kappa\alpha_3(\bar{x}_3) & \xi_2 & \alpha_1(x_1) \\ \alpha_2(x_2) & \kappa\alpha_1(\bar{x}_1) & \xi_3 \end{pmatrix},$$

where X is the matrix (1) and α_i's satisfy the relations (2).

Conversely let α_1 be an arbitrary element in $SO(8)$ and let α_2 and α_3 be the elements in $SO(8)$ such that $\alpha_1(x)\alpha_2(y) = \kappa\alpha_3(xy)$ for all $x, y \in \mathfrak{C}$ (cf. Lemma 1). Then the relations (2) hold for these α_i's by Lemma 2. Now we define the linear transformation $\alpha(\alpha_1, \alpha_2, \alpha_3)$ of $\mathfrak{J}$ by (7). For every $\alpha_1 \in SO(8)$ we have thus two linear transformations $\alpha(\alpha_1, \alpha_2, \alpha_3)$ and $\alpha(\alpha_1, -\alpha_2, -\alpha_3)$ (cf. Lemma 1). We may easily verify that these linear transformations are the automorphisms of $\mathfrak{J}$ and form a closed subgroup $\mathfrak{M}$ of the group $\mathfrak{A}$ of all automorphisms of $\mathfrak{J}$. It is clear that every automorphism in $\mathfrak{M}$ leaves fixed the elements E_i $(i=1, 2, 3)$ and $\mathfrak{M} \geqq \mathfrak{N}$. The mapping $f_1(\alpha(\alpha_1, \alpha_2, \alpha_3)) = \alpha_1$ is a homomorphism of $\mathfrak{M}$ onto $SO(8)$ and the kernel of f_1 consists of $\alpha(1, 1, 1)$ and $\alpha(1, -1, -1)$.[7] Let $\mathfrak{M}_0$ be the connected component of $\mathfrak{M}$ containing the identity. Then $f_1(\mathfrak{M}_0) = SO(8)$. Since $f_1^{-1}(\alpha_1) = \{\alpha(\alpha_1, \alpha_2, \alpha_3), \alpha(\alpha_1, -\alpha_2, -\alpha_3)\}$, at least one of $\alpha(\alpha_1, \alpha_2, \alpha_3)$ and $\alpha(\alpha_1, -\alpha_2, -\alpha_3)$ is in $\mathfrak{M}_0$. We shall prove that $\mathfrak{M} = \mathfrak{M}_0$. Suppose, on the contrary, that $\mathfrak{M} \neq \mathfrak{M}_0$. Since $\mathfrak{M}_0 \cup \alpha(1, -1, -1)\mathfrak{M}_0 = \mathfrak{M}$, $\mathfrak{M}$ consists of two connected components and $\alpha(1, -1, -1) \overline{\in} \mathfrak{M}_0$. Now $\alpha(-1, 1, -1)$ and $\alpha(-1, -1, 1)$ belong to the distinct components of $\mathfrak{M}$, for otherwise $\alpha(-1, 1, -1)\ \alpha(-1, -1, 1) = \alpha(1, -1, -1)$ is in $\mathfrak{M}_0$. Let, for example, $\alpha(-1, -1, 1) \in \mathfrak{M}_0$. Let $f_3(\alpha(\alpha_1, \alpha_2, \alpha_3)) = \alpha_3$. Then f_3 is also a homomorphism of $\mathfrak{M}$ onto $SO(8)$ and the kernel of f_3 is $\{\alpha(1, 1, 1), \alpha(-1, -1, 1)\}$. Hence f_3 is a local isomorphism and $f_3(\mathfrak{M}_0) = SO(8)$. By assumption the kernel of f_3 is contained in $\mathfrak{M}_0$ and hence $\mathfrak{M} = \mathfrak{M}_0$ and this is a contradiction. Hence $\mathfrak{M} = \mathfrak{M}_0$. Moreover we have shown that $\mathfrak{M}$ is a two sheeted covering group of $SO(8)$. Hence $\mathfrak{M}$ is isomorphic to the universal covering group $\widetilde{SO(8)}$ of $SO(8)$. Since $\mathfrak{M}$ is connected, $\mathfrak{M}$ is contained in $\mathfrak{F}_4$ and each automorphism in $\mathfrak{M}$ leaves fixed the elements E_i. Hence $\mathfrak{M} \leqq \mathfrak{N}$. Since we have already shown that $\mathfrak{M} \geqq \mathfrak{N}$, we have $\mathfrak{M} = \mathfrak{N}$ and this completes the proof of Theorem 1.

3. *Proof of Theorem 2.* Since the subgroups $\mathfrak{H}_i$ of $\mathfrak{F}_4$ are conjugate to each other in $\mathfrak{F}_4$,[8] it is sufficient to consider the group $\mathfrak{H}_1$. The derivation δ of $\mathfrak{J}$ such that $\delta E_1 = 0$ may be represented uniquely as the sum of two derivations

$$\delta = \tilde{A} + \Delta,$$

where $\Delta E_i = 0$ $(i=1, 2, 3)$ and

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}, \quad a \in \mathfrak{C},$$

and $\tilde{A}X = [A, X] = AX - XA$. Conversely for each such a metrix A, $\tilde{A}$ is a derivation of $\mathfrak{J}$ such that $\tilde{A}E_1 = 0$.[9] Since Δ's form the Lie algebra of the group $\mathfrak{N}$, dim. $\{\Delta\} = 28$ and dim. $\{\tilde{A}\} = 8$, where $\{\Delta\}$ and $\{\tilde{A}\}$ denote the linear spaces consisting of Δ's and $\tilde{A}$'s respectively. Hence the derivations which maps E_1 to 0 form a Lie algebra of dimensions 36 and this is the Lie algebra of $\mathfrak{H}_1$. Hence dim. $\mathfrak{H} = 36$. Now let Π be the set of all irreducible idempotents of $\mathfrak{J}$.[10] Further let Π_1 be the set of all $X \in \Pi$ such that $E_1 \circ X = 0$. Then an element $X \in \mathfrak{J}$ is in Π_1 if and only if

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7) We denote by 1 and -1 the identity transformation and the transformation defined by $x \to -x$ respectively.

8) For, there exist α and β in $\mathfrak{F}_4$ such that $\alpha E_1 = E_2$ and $\beta E_1 = E_3$. See, Freudenthal [3] p. 27. This fact is also proved in the following.

9) Chevalley-Schafer [2] and Freudenthal [3] p. 20.

10) See, Freudenthal [3] §5. Note that the set Π is invariant under the transformations of $\mathfrak{F}_4$.

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$$X=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \tag{8}$$

where $\xi_2=\xi_2^2+x_1\bar{x}_1$, $\xi_2+\xi_3=1$. Then $\xi_3=\xi_3^2+x_1\bar{x}_1$. Hence $1=\xi_2^2+\xi_3^2+2x_1\bar{x}_1$. Now the bilinear form $(X, Y)=Sp(X\circ Y)$ defined on $\mathfrak{J}$ is positive definite and invariant under the transformations of $\mathfrak{F}_4$.[11] Let $\|X\|^2=(X, X)$. If X is the matrix (1), then $\|X\|^2=\sum_{i=1}^{3}\xi_i+2\sum_{i=1}^{3}x_i\bar{x}_i$. Hence if $X\in\Pi_1$, then $\|X\|=1$. Now let $\mathfrak{J}_1$ be the 10 dimensional linear subspace of $\mathfrak{J}$ consisting of the matrices of the form (8), and let S^9 be the set of all $X\in\mathfrak{J}_1$ such that $\|X\|=1$. Then S^9 is a 9 dimensional sphere and Π_1 is the intersection of S^9 and the hyper-plane $\xi_2+\xi_3=1$ in $\mathfrak{J}_1$. Hence Π_1 is an 8 dimensional sphere. Let $\alpha\in\mathfrak{H}_1$. Then $\alpha(E_1\circ X)=E_1\circ\alpha X$, hence $\alpha(\Pi_1)=\Pi_1$. Thus α induces a transformation R_α of the sphere Π_1. Since α is an orthogonal transformation of $\mathfrak{J}$, R_α is an isometric transformation of Π_1 and hence a (proper or improper) rotation. Thus $g(\alpha)=R_\alpha$ is a homomorphism of $\mathfrak{H}_1$ into the group $O(9)$. Let $\mathfrak{D}$ be the kernel of g. Since each $\alpha\in\mathfrak{D}$ leaves fixed the elements E_i, α is contained in $\mathfrak{N}$. Hence $\alpha(\in\mathfrak{D})$ is of the form $\alpha=\alpha(\alpha_1, \alpha_2, \alpha_3)$ (see §1) and

$$\alpha X=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & \alpha_1(x_1) \\ 0 & \kappa\alpha_1(\bar{x}_1) & \xi_3 \end{pmatrix}=X$$

for all $X\in\Pi_1$. We see easily that $\alpha_1=1$ and hence $\mathfrak{D}$ is the finite group of order 2. Since $\dim.\mathfrak{H}_1=\dim.O(9)=36$, the component $\mathfrak{H}_1^0$ containing the identity is mapped by g onto $SO(9)$. As $\mathfrak{H}_1^0\supset\mathfrak{N}\supset\mathfrak{D}$ by Theorem 1, $\mathfrak{H}_1^0$ is a two-sheeted covering group of $SO(9)$ and hence it is isomorphic to the universal covering group $\widetilde{SO(9)}$ of $SO(9)$. We may easily see that if $\mathfrak{H}_1\neq\mathfrak{H}_1^0$, then the order of the group $\mathfrak{H}_1/\mathfrak{H}_1^0$ is 2 and $g(\mathfrak{H}_1)=O(9)$. Now the mapping

$$X\to RX=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_3 & x_1 \\ 0 & \bar{x}_1 & \xi_2 \end{pmatrix}$$

is an improper rotation of the sphere Π_1. If $\mathfrak{H}_1\neq\mathfrak{H}_1^0$, there exists $\alpha\in\mathfrak{H}_1$ such that $\alpha X=RX$ for all $X\in\Pi_1$. Then $\alpha E_1=E_1$, $\alpha E_2=E_3$ and $\alpha E_3=E_2$. Since $g(\mathfrak{H}_1^0)=SO(9)$ and $SO(9)$ is transitive on Π_1, there exists $\beta\in\mathfrak{H}_1^0$ such that $\beta E_2=E_3$. $\beta(E_1\circ E_3)=E_1\circ\beta E_3=0$, $\beta(E_2\circ E_3)=E_3\circ\beta E_3=0$ and $\beta E_3\circ\beta E_3=\beta E_3$ imply $\beta E_3=E_2$. Then $\beta^{-1}\alpha E_i=E_i$ for $i=1, 2, 3$. Thus $\beta^{-1}\alpha\in\mathfrak{N}\cap\mathfrak{H}_1^0$. Hence $\alpha\in\mathfrak{H}_1^0$ and this is a contradiction. Thus $\mathfrak{H}_1$ is connected and isomorphic to $\widetilde{SO(9)}$.

Remark. The group of all automorphisms of $\mathfrak{J}$ is not connected. For example,

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[11] See, Freudenthal [3], §4.

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$$X=\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \rightarrow \alpha X=\begin{pmatrix} \xi_1 & x_2 & \bar{x}_3 \\ \bar{x}_2 & \xi_3 & x_1 \\ x_3 & \bar{x}_1 & \xi_2 \end{pmatrix}$$

is an automorphism of $\mathfrak{J}$. α is an improper orthogonal transformation of $\mathfrak{J}$ and hence $\alpha \in \mathfrak{F}_4$.

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