Canad. Math. Bull. Vol. **45** (4), 2002 pp. 525–536

Some Factorizations in Universal Enveloping Algebras of Three Dimensional Lie Algebras and Generalizations

This paper is dedicated to Robert V. Moody on the occasion of his 60th birthday

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Abstract. We introduce the notion of Lie algebras with plus-minus pairs as well as regular plus-minus pairs. These notions deal with certain factorizations in universal enveloping algebras. We show that many important Lie algebras have such pairs and we classify, and give a full treatment of, the three dimensional Lie algebras with plus-minus pairs.

1 Introduction

All of our algebras will be over a field F of characteristic zero. We begin by recalling the well known fact that if L is a Kac-Moody Lie algebra with the usual Chevalley generators $\{e_i, f_i \mid 1 \leq i \leq l\}$ satisfying $L = [L, L]$ and l is finite, then every L-module on which the elements $e_i, f_i, 1 \leq i \leq l$ act locally nilpotently is integrable in the sense that the elements $h_i = [e_i, f_i], 1 \leq i \leq l$ are simultaneously diagonalizable. (*cf.* [10] Ex. 6.31, p. 585, or [11]). In other words, every weakly integrable module for such an algebra is integrable. The usual proof of this fact uses that the three dimensional Lie algebra with basis e_i, f_i, h_i is isomorphic to the Lie algebra sl_2 (so $\mathfrak{g}$ is generated by sl_2-triples) together with the result which says that if V is any module for sl_2 on which the standard generators e, f of sl_2 act locally nilpotently then the element $h = [e, f]$ is diagonalizable on V. One can see this last fact as follows. We denote by $M(W)$ the maximal integrable submodule of an sl_2-module W. In general, $M\big(W/M(W)\big) = 0$ for all sl_2-modules W. If a vector v of an sl_2-module W satisfies that $fv = 0$, $e^n v \neq 0$ and $e^{n+1}v = 0$, then we obtain $h(e^n v) = n(e^n v)$, since

Received by the editors October 23, 2001; revised March 13, 2002.
The first author was supported by an NSERC grant. The third author was supported by an NSERC Postdoctoral Fellowship in 2002.
AMS subject classification: 17B05, 17B35, 17B67, 17B70.

$fe^n = e^n f - ne^{n-1}(h+n-1)$ and

$$\begin{aligned} h(e^n v) &= (ef - fe)e^n v \\ &= efe^n v \\ &= e\big(e^n f - ne^{n-1}(h+n-1)\big)\, v \\ &= -ne^n(h+n-1)v \\ &= -n(h-n-1)e^n v \\ &= -nh(e^n v) + n(n+1)e^n v. \end{aligned}$$

This implies that $M(V)$ is nontrivial for every nonzero sl_2-module V on which the elements e and f are locally nilpotent operators. Therefore, $M(V) = V$ for such an sl_2-module V, that is, V is integrable.

We want to indicate another approach to the above fact about sl_2-modules which uses a factorization in the universal enveloping algebra. This method appears to be new and was the starting point of this paper. For any Lie algebra $\mathfrak{g}$ we let $U(\mathfrak{g})$ denote its universal enveloping algebra. Then one knows that for the algebra $\mathrm{sl}_2 = Fe \oplus Fh \oplus Ff$ we have the factorization

$$U(\mathrm{sl}_2) = U(Fe)\, U(Ff)\, U(Fe). \tag{1}$$

Using this it is easy to see that if V is an sl_2-module on which both e and f act locally nilpotently then any vector $v \in V$ generates a finite dimensional submodule. Thus h acts semisimply on this submodule and so we obtain h acts semisimply on V. Also, the proof of the factorization (1) is quite straightforward and follows easily from the following formula in $U(\mathrm{sl}_2)$:

$$f(e^i f^j e^k) = \frac{j-i+1}{j+1} e^i f^{j+1} e^k + \frac{i}{j+1} e^{i-1} f^{j+1} e^{k+1} + i(j-i+1)e^{i-1} f^j e^k, \tag{2}$$

for all $i > 0$, $j, k \geq 0$. This formula can be established using the following: for any $k \geq 0$ we have

(A_k) $\quad fef^k = \frac{k}{k+1} ef^{k+1} + \frac{1}{k+1} f^{k+1} e + kf^k,$
(B_k) $\quad f^k ef = \frac{1}{k+1} ef^{k+1} + \frac{k}{k+1} f^{k+1} e + kf^k,$

which is proved using $fe^i = e^i f - ie^{i-1}(h+i-1)$ for $i \geq 1$ and induction.

Next let $\mathfrak{H}$ be the three dimensional Heisenberg Lie algebra with a basis x, y, z satisfying $[x, y] = z$, $[x, z] = [y, z] = 0$. Then, in $U(\mathfrak{H})$, we obtain the following factorization

$$U(\mathfrak{H}) = U(Fx)\, U(Fy)\, U(Fx). \tag{3}$$

The proof of this is much like the sl_2 case. It follows easily from the following formula in $U(\mathfrak{H})$

$$y(x^i y^j x^k) = \frac{j-i+1}{j+1} x^i y^{j+1} x^k + \frac{i}{j+1} x^{i-1} y^{j+1} x^{k+1}, \tag{4}$$

for all $i > 0$, $j, k \geq 0$. Note that (4) is proved by establishing for any $k \geq 0$ we have

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(A_k') $yxy^k = \frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}y^{k+1}x$,
(B_k') $y^kxy = \frac{1}{k+1}xy^{k+1} + \frac{k}{k+1}y^{k+1}x$,

which in turn is proved using $yx^i = x^iy - ix^{i-1}z$ for $i \geq 1$ and induction. Thus, the picture is similar for both algebras sl_2 and $\mathfrak{H}$ in that they both have a pair of subalgebras P, M satisfying $P + M$ is not the whole algebra and $U(P)\,U(M)\,U(P)$ is the whole enveloping algebra. This prompts the following definition which singles out those Lie algebras having this type of factorization in their universal enveloping algebras.

Definition 1.1 (i) A Lie algebra L is said to have a *plus-minus pair* if it has two subalgebras P, M satisfying $P + M \neq L$ and

$$U(L) = U(P)\,U(M)\,U(P).$$

In this case we say L has a plus-minus pair (P, M).

(ii) Let (P, M) be a plus-minus pair of L. We say this is a *regular plus-minus pair* if $P \cap M = (0)$ and there is an automorphism σ of L of order two satisfying $\sigma(P) = M$. Note that in this case we then have $U(L) = U(P)\,U(M)\,U(P) = U(M)\,U(P)\,U(M)$.

It is clear that both Lie algebras sl_2 and $\mathfrak{H}$ have regular plus-minus pairs. Moreover if L is any three dimensional Lie algebra with a plus-minus pair (P, M) then each of P and M must be one dimensional. Indeed, $P + M$ cannot be 3 dimensional as $P+M \neq L$. Thus, $P+M$ is two dimensional and so if one of P, M is 2 dimensional then $P+M$ is a subalgebra of L and so $U(P+M) \neq U(L)$ but $U(P)\,U(M)\,U(P) \subseteq U(P+M)$ so (P, M) cannot be a plus-minus pair. Letting $P = Fx$, $M = Fy$ we have that

$$U(L) = \sum_{i,j,k \geq 0} Fx^iy^jx^k. \tag{5}$$

Moreover, the following result, which extends the situation discussed in the sl_2 case, is quite clear. Let L be a three dimensional Lie algebra with a plus-minus pair (P, M) where $P = Fx$, $M = Fy$. Let V be any L-module on which the action of the elements x, y is locally finite. Then any finitely generated submodule of V is finite dimensional. Thus, one is led to ask just which three dimensional Lie algebras have plus-minus pairs.

In Section 2 we will extend the methods used in the proofs for the sl_2 and $\mathfrak{H}$ cases above and show that any three dimensional Lie algebra which is generated by two elements has a plus-minus pair. Then we go on to see that there are only two isomorphism classes of three dimensional Lie algebras which do not have plus-minus pairs. We also go on to study, when the base field F is algebraically closed, which of these algebras have regular plus-minus pairs and are able to give a complete list of these. Here we use some results from [6]. In the third and final section of this paper we go on to investigate plus-minus pairs, or similar factorizations, in the universal enveloping algebras, of Borcherds Lie algebras as well as in some $\mathbf{Z}^n$-graded Lie algebras which satisfy some extra conditions.

Thanks go to the referee for simplifying the proof of Theorem 2.3 and other helpful comments.

2 Three Dimensional Case

In this section we begin by showing a three dimensional Lie algebra has a plus-minus pair if and only if it is generated by two elements. We then go on to investigate some special cases as well as regular plus-minus pairs when the base field is algebraically closed.

Throughout we let L be a three dimensional Lie algebra unless mentioned otherwise. If (P, M) is a plus-minus pair for L then we know that each of P, M is one dimensional so we let $P = Fx$, $M = Fy$. If x and y do not generate L then it must be that $P + M$ is a proper subalgebra of L and so since $U(P)\,U(M)\,U(P) \subseteq U(P + M)$ we get a contradiction. Thus L is generated by x and y so is two-generated. We want to establish the converse of the above result.

We define subspaces U_k for $k \geq 0$ of $U(L)$ by saying

$$U_k = \sum_{0 \leq m \leq k} (Fxy^m + Fy^m x + Fy^m). \tag{6}$$

Notice that $U_0 = Fx + F1$ and that $U_k \subseteq U_{k+1}$ for all $k \geq 0$.

Lemma 2.1 *Let L be an arbitrary three-dimensional Lie algebra and x, y any two elements of L. For $k \geq 0$ the following statements hold,*

(A_k) $yxy^k \equiv \frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}y^{k+1}x \mod U_k$,
(B_k) $y^kxy \equiv \frac{1}{k+1}xy^{k+1} + \frac{k}{k+1}y^{k+1}x \mod U_k$,
(C_k) $yU_k \subseteq U_{k+1}$, $U_k y \subseteq U_{k+1}$.

Proof We prove this by induction on k noting that for $k = 0$ both (A_0) and (B_0) are clear. Next, we show $(\mathrm{A}_0), \dots, (\mathrm{A}_k), (\mathrm{B}_0), \dots, (\mathrm{B}_k)$ imply (C_k). Indeed, by definition we have that

$$yU_k = \sum_{0 \leq m \leq k} (Fyxy^m + Fy^{m+1}x + Fy^{m+1})$$

and so by $(\mathrm{A}_0), \dots, (\mathrm{A}_k)$ we get that this is contained in U_{k+1}. Similarly we have

$$U_k y = \sum_{0 \leq m \leq k} (Fxy^{m+1} + Fy^m xy + Fy^{m+1})$$

and so by $(\mathrm{B}_0), \dots, (\mathrm{B}_k)$ we get this is contained in U_{k+1}. Hence (C_k) holds.

Next we show that (A_k), (B_k), (C_k) imply (A_{k+1}), (B_{k+1}). We assume $k \geq 1$ as when $k = 0$ the fact that L is three dimensional implies that (A_1) and (B_1) hold. Now $yxy^{k+1} = (yxy^k)y$ so that (A_k) implies that the difference

$$yxy^{k+1} - \Big(\frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}y^{k+1}x\Big)\,y \in U_k y.$$

But (C_k) implies that $U_k y \subseteq U_{k+1}$ so we get that

$$yxy^{k+1} \equiv \Big(\frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}y^{k+1}x\Big)\,y \mod U_{k+1}.$$

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Similarly, using (B_k) and (C_k) we get that

$$y^{k+1}xy \equiv y\Big(\frac{1}{k+1}xy^{k+1} + \frac{k}{k+1}y^{k+1}x\Big) \mod U_{k+1}.$$

Thus, we finally get

$$yxy^{k+1} \equiv \frac{k}{k+1}xy^{k+2} + \frac{1}{(k+1)^2}yxy^{k+1} + \frac{k}{(k+1)^2}y^{k+2}x \mod U_{k+1}$$

which implies that

$$\frac{k(k+2)}{(k+1)^2}yxy^{k+1} \equiv \frac{k}{k+1}xy^{k+2} + \frac{k}{(k+1)^2}y^{k+2}x \mod U_{k+1}.$$

Therefore, we obtain

$$yxy^{k+1} \equiv \frac{k+1}{k+2}xy^{k+2} + \frac{1}{k+2}y^{k+2}x \mod U_{k+1},$$

and we see that (A_{k+1}) holds.

Using a similar type of argument we have that

$$\begin{aligned} y^{k+1}xy &= y(y^kxy) \\ &\equiv y\Big(\frac{1}{k+1}xy^{k+1} + \frac{k}{k+1}y^{k+1}x\Big) \mod U_{k+1} \\ &\equiv \frac{1}{k+1}yxy^{k+1} + \frac{k}{k+1}y^{k+2}x \mod U_{k+1} \\ &\equiv \frac{1}{k+1}\Big(\frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}y^{k+1}x\Big)\,y + \frac{k}{k+1}y^{k+2}x \mod U_{k+1} \\ &\equiv \frac{k}{(k+1)^2}xy^{k+2} + \frac{1}{(k+1)^2}y^{k+1}xy + \frac{k}{k+1}y^{k+2}x \mod U_{k+1} \end{aligned}$$

and

$$\frac{k(k+2)}{(k+1)^2}y^{k+1}xy \equiv \frac{k}{(k+1)^2}xy^{k+2} + \frac{k}{k+1}y^{k+2}x \mod U_{k+1}.$$

Therefore, we obtain

$$y^{k+1}xy \equiv \frac{1}{k+2}xy^{k+2} + \frac{k+1}{k+2}y^{k+2}x \mod U_{k+1},$$

and we see that (B_{k+1}) holds. This completes our induction. ■

We apply this lemma in proving the following theorem.

Theorem 2.2 *Let L be a three dimensional Lie algebra. Then L has a plus-minus pair if and only if L is two generated. Moreover, if x and y generate L then (P, M) is a plus-minus pair for L where $P = Fx$, $M = Fy$.*

Proof We need only show L has a plus-minus pair if L is generated by two elements x, y. Let $z = [x, y]$. Now we want to show $U(L) = \sum_{i,j,k\geq 0} Fx^i y^j x^k$. Put $\mathfrak{X} = \sum_{i,j,k\geq 0} Fx^i y^j x^k \subseteq U(L)$ and let U_k be defined as above. Clearly $x\mathfrak{X} \subseteq \mathfrak{X}$, $\mathfrak{X}x \subseteq \mathfrak{X}$ and $U_k \subseteq \mathfrak{X}$ for all $k \geq 0$. We claim

$$y(x^\ell y^m x^n) \in \mathfrak{X},$$
$$z(x^\ell y^m x^n) \in \mathfrak{X}.$$

and show this by induction on ℓ. If $\ell = 0$, then we see $y(y^m x^n) \in \mathfrak{X}$ and using (A_m) we get

$$\begin{aligned} z(y^m x^n) &= (xy - yx)(y^m x^n) \\ &= xy^{m+1}x^n - yxy^m x^n \\ &\in Fxy^{m+1}x^n + (Fxy^{m+1} + Fy^{m+1}x + U_m)x^n \subseteq \mathfrak{X}. \end{aligned}$$

Let $\ell > 0$. Then, we obtain, using our inductive assumption, that

$$\begin{aligned} y(x^\ell y^m x^n) &= (xy - z)(x^{\ell-1} y^m x^n) \\ &\in x\mathfrak{X} + \mathfrak{X} \subseteq \mathfrak{X} \end{aligned}$$

and, letting $[z, x] = ax + by + cz$ for $a, b, c \in F$, we also get using our inductive assumption that

$$\begin{aligned} z(x^\ell y^m x^n) &= (xz + ax + by + cz)(x^{\ell-1} y^m x^n) \\ &\in x\mathfrak{X} + \mathfrak{X} + \mathfrak{X} + \mathfrak{X} \subseteq \mathfrak{X}. \end{aligned}$$

Hence, $y\mathfrak{X} \subseteq \mathfrak{X}$. Since $\mathfrak{X}$ is a left ideal of $U(L)$ containing 1, we obtain $\mathfrak{X} = U(L)$. Therefore, (P, M) with $P = Fx$ and $M = Fy$ is a plus-minus pair for L. ■

If our three dimensional Lie algebra L is abelian it clearly does not have a plus-minus pair. Also, we let $\mathfrak{g}$ be the three dimensional Lie algebra with basis x, y, z satisfying

$$[x, y] = 0, \quad [x, z] = x, \quad [y, z] = y.$$

Then for any elements $a, b, c, \alpha, \beta, \gamma \in F$ we have the very special identity

$$[ax + by + cz, \alpha x + \beta y + \gamma z] = \gamma(ax + by + cz) - c(\alpha x + \beta y + \gamma z).$$

This clearly implies that $\mathfrak{g}$ is not two generated so does not have a plus-minus pair. Our next result shows that these are the only two kinds of three dimensional Lie algebras which do not have plus-minus pairs.

Theorem 2.3 *Let L be a three dimensional Lie algebra which is not two generated. Then L is either abelian or is isomorphic to the algebra $\mathfrak{g}$ above.*

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Proof Assume L is not abelian. Choose a 1-dimensional subspace Fz of L which is not an ideal. Every 2-dimensional subspace of L is a subalgebra. Hence there exist x, y in L such that $\{x, y, z\}$ is a basis of L and $[z, x] = ax$, $[z, y] = by$ for some a, b in F. As $[z, x + y]$ belongs to $Fx + Fy$ and $F(x + y) + Fz$, we must have $a = b$. As Fz is not an ideal, a is not 0. We may assume that $a = 1$. As $[x, y] = [x + z, y] - y$ belongs to $Fx + Fy$ and $F(x + z) + Fy$, we deduce that $[x, y]$ is in Fy. Similarly, it is in Fx. Hence $[x, y] = 0$ and L is isomorphic to $\mathfrak{g}$. ■

The special case when $L = L_{-1} \oplus L_0 \oplus L_1$ is a three graded Lie algebra of dimension three with a plus-minus pair will be used in the final section of this work so will be discussed now. Put $L_1 = Fx$, $L_{-1} = Fy$, $L_0 = Fz$. We can assume first that $[x, y]$ is either 0 or z. Suppose $[x, y] = 0$. If $[x, z] = [y, z] = 0$, then L is abelian so has no plus-minus pair. If $[x, z] = 0$ and $[y, z] \neq 0$, then we can also suppose $[z, y] = y$ and hence, $P = F(x + y)$ and $M = Fz$ give a plus minus pair. If $[x, z] \neq 0$ and $[y, z] = 0$, then we can suppose $[z, x] = x$ and hence, again $P = F(x + y)$ and $M = Fz$ becomes a plus-minus pair. If $[x, z] = ax$ and $[y, z] = by$ with $ab \neq 0$, then we can suppose $a = 1$. In this case, $P = F(x + y)$ and $M = Fz$ give a plus-minus pair when $b \neq 1$. Otherwise we have $[x, z] = x$ and $[y, z] = y$ and there is no plus-minus pair. Next we suppose $[x, y] = z$. If $[x, z] = [y, z] = 0$, then L is a Heisenberg Lie algebra, and hence, L has a plus-minus pair. If $[x, z] = ax$ and $[y, z] = by$ with $a \neq 0$ or $b \neq 0$, then $0 = [z, z] = \big[[x, y], z\big] = \big[[x, z], y\big] + \big[x, [y, z]\big] = a[x, y] + b[x, y] = (a + b)z$ and $a + b = 0$. Put $x' = x$, $y' = -2y/a$, $z' = -2z/a$. Then, $[x', y'] = -2[x, y]/a = -2z/a = z'$, $[z', x'] = -[x', z'] = 2[x, z]/a = 2x = 2x'$ and $[z', y'] = -[y', z'] = -4[y, z]/(a^2) = 4y/a = -2y'$. This means that L is isomorphic to sl_2. Therefore we obtain the following result which gives a characterization of sl_2 and $\mathfrak{H}$.

Proposition 2.4 *Let $L = L_1 \oplus L_0 \oplus L_{-1}$ be a three graded Lie algebra of dimension three with $\dim L_{\pm 1} = \dim L_0 = 1$.*

(1) *If L has a plus-minus pair, then L is isomorphic to one of $s\ell_2$, $\mathfrak{H}$ and $K(a, b)$, where $K(a, b) = Fx \oplus Fy \oplus Fz$ is the Lie algebra having the relations: $[x, y] = 0$, $[x, z] = ax$, $[y, z] = by$ with $a \neq b$.*
(2) *If L has (L_1, L_{-1}) for a plus-minus pair, then L is isomorphic to either sl_2 or $\mathfrak{H}$.*

Remark If $a = b$ is non-zero then we have $K(a, b) = K(a, a) \simeq K(1, 1)$ and this is nothing but our algebra $\mathfrak{g}$ of Theorem 2.3 which does not have a plus-minus pair.

Next we will briefly discuss isomorphism classes among the Lie algebras $K(a, b)$. For this we will freely use the classification in Jacobson's book [6] on page 12 where he classifies the three dimensional Lie algebras having a two dimensional derived algebra. This is listed there as (d) of his general classification. We have $K(0, c) \simeq K(c, 0) \simeq K(0, 1)$ for nonzero $c \in F$. Thus if a or b is 0 then $K(a, b) \simeq K(0, 1)$. Next we suppose that both a, b are nonzero. Then, we also see $K(a, b) \simeq K(a/b, 1)$. The only isomorphisms between the algebras $K(c, 1)$ for c non-zero are $K(c, 1) \simeq K(1/c, 1)$ and none of these are isomorphic to $K(0, c)$. Thus, the isomorphism classes of the Lie algebras $K(a, b)$, having plus-minus pairs, are parametrized by the set

$$\mathfrak{P}(F) = \big\{ \{u, u^{-1}\} \mid u \in F, u \neq 0, 1 \big\} \cup \big\{ \{0\} \big\}.$$

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Here the isomorphism class of $K(0, a)$ corresponds to $\{0\}$ while that of $K(a, 1)$ to $\{a, a^{-1}\} = \{a^{-1}, a\}$ for $a \in F$, $a \neq 0$.

We next assume that F is an algebraically closed field of characteristic 0, and will study the three dimensional Lie algebras over F having a regular plus-minus pair. Let L be such an algebra and let (P, M) be a regular plus-minus pair of L. Then we can choose nonzero elements $x \in P$ and $y \in M$ as well as an involutive automorphism σ of L such that $[x, y] \neq 0$ and $\sigma(x) = y$. Put $z = [x, y]$, and set $u = x + y$ and $v = x - y$. Let $L_1 = Fu$ (the 1-eigenspace of σ) and $L_{-1} = Fv \oplus Fz$ (the -1-eigenspace of σ). Then, $[u, v] = [x + y, x - y] = -2[x, y] = -2z$. We write $[z, x] = ax + by + cz$. Then we obtain $[z, y] = -\sigma([z, x]) = -\sigma(ax + by + cz) = -bx - ay + cz$ and so the Jacobi identity implies $[bx + ay - cz, x] + [ax + by + cz, y] = 0$. Therefore,

$$\{-az - c(ax + by + cz)\} + \{az + c(-bx - ay + cz)\} = 0$$

and $c(a + b)x + c(a + b)y = 0$, which implies $c = 0$ or $a + b = 0$.

Case 1 $c = 0$.

In this case, we have $[x, y] = z$, $[z, x] = ax + by$, $[z, y] = -bx - ay$. We write the matrix of $\mathrm{ad}(z)$ restricted to the space $Fx \oplus Fy$ as

$$\mathrm{ad}\, z = \begin{pmatrix} a & -b \\ b & -a \end{pmatrix}.$$

Then, its characteristic polynomial is $t^2 - a^2 + b^2$. Hence, $\mathrm{ad}\, z\,|_{Fx \oplus Fy}$ is similar to one of

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where $\lambda = (a^2 - b^2)^{1/2}$. If $\lambda \neq 0$, that is $a^2 - b^2 \neq 0$, then we have certain elements $x', y' \in Fx \oplus Fy$ such that $[z', x'] = \lambda' x'$, $[z', y'] = -\lambda' y'$ with $z' = [x', y'] \neq 0$ and $\lambda' \neq 0$. This means that $L \simeq \mathrm{sl}_2$. If $a = b = 0$, then we see $L \simeq \mathfrak{H}$. Now we suppose $a = b \neq 0$. Thus, we have $[z, u] = 0$, $[z, v] = 2au$, $[u, v] = -2z$. Put $\mu = (-a)^{1/4}$, and set $z' = z/\mu$, $u' = \mu u$, $v' = v/(2\mu^2)$. Then,

$$[v', u'] = [v, u]/(2\mu) = z/\mu = z',$$

$$[v', z'] = (-a)u/(\mu^3) = u',$$

$$[u', z'] = 0.$$

As is easy to check, again from Jacobson's book, [6], on page 12 under heading (d) one finds our algebra here is just the one with $\alpha = 1$ and we have

$$\sigma(v') = -v', \quad \sigma(u') = u', \quad \sigma(z') = -z'.$$

We denote this algebra by $L(\alpha = 1)$.

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Next we suppose $a = -b \neq 0$. Thus, we have $[z, v] = 0$, $[z, u] = 2av$, $[u, v] = -2z$. Put $\nu = a^{1/4}$, and set $z' = z/\nu$, $u' = -u/(2\nu^2)$, $v' = \nu v$. Then,

$$[u', v'] = -[u, v]/(2\nu) = z/\nu = z',$$
$$[u', z'] = -[u, z]/(2\nu^3) = \nu v = v',$$
$$[v', z'] = 0.$$

Here we have once again that our algebra is just $L(\alpha = 1)$ and

$$\sigma(u') = u', \quad \sigma(v') = -v', \quad \sigma(z') = -z'.$$

Case 2 $c \neq 0, a + b = 0$.

In this case, we have $[x, y] = z$, $[z, x] = [z, y] = ax - ay + cz$, and hence $[u, v] = -2z$, $[z, u] = 2av + 2cz4$, $[z, v] = 0$. Set $u' = -u/(2c)$, $v' = cv$, $z' = z$. Then,

$$[u', v'] = -[u, v]/2 = z = z',$$
$$[u', z'] = -[u, z]/(2c) = av/c + z = av'/(c^2) + z,$$
$$[v', z'] = 0.$$

This time we get that our algebra is the one from [6] page 12 having $\beta = a/c^2$, which we denote as $L(\beta = a/c^2)$, and so have

$$\sigma(u') = u', \quad \sigma(v') = -v', \quad \sigma(z') = -z'$$

With the notation developed above we see that the preceding arguments, together with the results in [6] about isomorphisms between these algebras, establish the following result.

Theorem 2.5 *Let F be an algebraically closed field of characteristic* 0. *Let L be a three dimensional Lie algebra with a regular plus-minus pair. Then L is isomorphic to one of* sl_2, $\mathfrak{H}$, $L(\alpha = 1)$ *or* $L(\beta = r)$ *for any r in F. Moreover no two distinct algebras in this list are isomorphic.*

Remark Finally we want to point out that the Lie algebra $K(u, 1)$ with $u \in F$ is isomorphic to $L(\alpha = 1)$ if $u = -1$, and is isomorphic to $L(\beta = -\frac{u}{(u+1)^2})$ if $u \neq -1$. As a consequence of our work we see that a three dimensional three graded Lie algebra has a regular plus-minus pair or no plus-minus pair at all.

3 Plus-Minus Pairs in Some General Classes of Lie Algebras

In this section we will show that Borcherds Lie algebras have plus-minus pairs. Since these generalize the well-known Kac-Moody Lie algebras our results apply to these

https://doi.org/10.4153/CMB-2002-053-4 Published online by Cambridge University Press

as well. We next go on to investigate $\mathbf{Z}^n$-graded Lie algebras and see that with certain other assumptions these also have plus-minus pairs. Our method is to establish slightly more general factorization results in the universal enveloping algebras of these Lie algebras and then show how this gives rise to plus-minus pairs. The techniques are quite general and no doubt apply to other situations as well.

Let $\mathfrak{g}$ be a rank l Borcherds Lie algebra over F with the standard Cartan subalgebra $\mathfrak{h}$ and Chevalley generators $\{e_1, \dots, e_l, f_1, \dots, f_l\}$, and let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ the derived subalgebra of $\mathfrak{g}$. Put $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$, and take a complement $\mathfrak{h}''$ of $\mathfrak{h}'$ in $\mathfrak{h}$ with $\mathfrak{h} = \mathfrak{h}'' \oplus \mathfrak{h}'$. Then, $\mathfrak{g} = \mathfrak{h}'' \oplus \mathfrak{g}'$. Let $\mathfrak{g}_+$ be the subalgebra of $\mathfrak{g}$ generated by $e_1, \dots, e_l$, and $\mathfrak{g}_-$ the subalgebra of $\mathfrak{g}$ generated by $f_1, \dots, f_l$ (*cf.* [4], [7], [8], [9], [10], [12]).

Proposition 3.1 *Let $\mathfrak{g}$ be a rank l Borcherds Lie algebra, and let $I \cup J = \{1, 2, \dots, l\}$ be a partition of $\{1, 2, \dots, l\}$ into disjoint subsets. Then,*

$$U(\mathfrak{g}) = \Big(\prod_{i \in I} U(Fe_i)\Big) U(\mathfrak{g}_-) U(\mathfrak{h}'') U(\mathfrak{g}_+) \Big(\prod_{j \in J} U(Ff_j)\Big).$$

Proof Let $\mathfrak{g}_+^i$ be the standard homogeneous complementary subalgebra of Fe_i in $\mathfrak{g}_+$, and $\mathfrak{g}_-^i$ the standard homogeneous complementary subalgebra of Ff_i in $\mathfrak{g}_-$. For each $k = 1, \dots, l$ we put $h_k = [e_k, f_k]$ and $\mathfrak{h}_k = Fh_{k+1} \oplus \cdots \oplus Fh_l$, and we set $I_k = I \cap \{1, \dots, k\}$ and $J_k = J \cap \{1, \dots, k\}$. We make free use of the PBW Theorem as well as the fact that $Fe_i \oplus Ff_i \oplus Fh_i$ is either sl_2 or $\mathfrak{H}$ so has a regular plus-minus pair.

If $1 \in I$, then

$$\begin{aligned} U(\mathfrak{g}) &= U(\mathfrak{g}_-)\, U(\mathfrak{h})\, U(\mathfrak{g}_+) \\ &= U(\mathfrak{g}_-^1)\, U(Ff_1)\, U(\mathfrak{h}'')\, U(\mathfrak{h}_1)\, U(Fh_1) U(Fe_1) U(\mathfrak{g}_+^1) \\ &= U(\mathfrak{g}_-^1)\, U(\mathfrak{h}'')\, U(\mathfrak{h}_1)\, U(Ff_1)\, U(Fh_1) U(Fe_1) U(\mathfrak{g}_+^1) \\ &= U(\mathfrak{g}_-^1)\, U(\mathfrak{h}'')\, U(\mathfrak{h}_1)\, U(Fe_1)\, U(Ff_1) U(Fe_1) U(\mathfrak{g}_+^1) \\ &= U(Fe_1) U(\mathfrak{g}_-^1)\, U(\mathfrak{h}'')\, U(\mathfrak{h}_1)\, U(Ff_1) U(Fe_1) U(\mathfrak{g}_+^1) \\ &= U(Fe_1) U(\mathfrak{g}_-) U(\mathfrak{h}'')\, U(\mathfrak{h}_1) U(\mathfrak{g}_+). \end{aligned}$$

In the other case when $1 \in J$ by using the same type of argument we have

$$U(\mathfrak{g}) = U(\mathfrak{g}_-)\, U(\mathfrak{h}'')\, U(\mathfrak{h}_1) U(\mathfrak{g}_+)\, U(Ff_1).$$

If we began with

$$U(\mathfrak{g}) = \Big(\prod_{i \in I_k} U(Fe_i)\Big)\, U(\mathfrak{g}_-) U(\mathfrak{h}'')\, U(\mathfrak{h}_k)\, U(\mathfrak{g}_+) \Big(\prod_{j \in J_k} U(Ff_j)\Big),$$

then, again using the same method, we can obtain

$$U(\mathfrak{g}) = \Big(\prod_{i \in I_{k+1}} U(Fe_i)\Big)\, U(\mathfrak{g}_-) U(\mathfrak{h}'')\, U(\mathfrak{h}_{k+1})\, U(\mathfrak{g}_+) \Big(\prod_{j \in J_{k+1}} U(Ff_j)\Big).$$

Thus after several applications of this process we reach the stated result. ■

We next see that this gives the desired plus-minus pair.

Corollary 3.2 *Let $\mathfrak{g}$ be a Borcherds Lie algebra of finite rank. Then,*

$$U(\mathfrak{g}) = U(\mathfrak{g}_{\pm})U(\mathfrak{g}_{\mp})U(\mathfrak{h}'')U(\mathfrak{g}_{\pm}).$$

Hence, Borcherds Lie algebras have plus-minus pairs. In particular, perfect Kac-Moody Lie algebras or, more generally, perfect Borcherds Lie algebras have regular plus-minus pairs.

Proof We just take one of I and J to be empty. This leads to the result. Then, for example, let $P = \mathfrak{h}'' \oplus \mathfrak{g}_+$ and $M = \mathfrak{g}_-$. This gives a plus-minus pair. ■

We next generalize the previous discussion by considering $\mathbf{Z}^n$-graded Lie algebras. Thus, let $Q = \bigoplus_{i=1}^n \mathbf{Z}\alpha_i$ be a free abelian group of rank n generated by $\alpha_1, \dots, \alpha_n$, and let $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$ be a Lie algebra graded by Q. Put $\Delta = \{\alpha \in Q \mid \mathfrak{g}_\alpha \neq 0\}$. We also assume that $\mathbf{Z}\alpha_1 \cap \Delta = \{0, \pm\alpha_1\}$, and that $L = \mathfrak{g}_{\alpha_1} \oplus [\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}] \oplus \mathfrak{g}_{-\alpha_1}$ is a subalgebra with a plus-minus pair $(\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1})$ in L. (Thus, if L is three dimensional Proposition 2.4 implies L is isomorphic to either sl_2 or $\mathfrak{H}$.) We also suppose that there exists a complementary subalgebra $\mathfrak{g}_0'$ of $[\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}]$ in $\mathfrak{g}_0$ with $\mathfrak{g}_0 = [\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}] \oplus \mathfrak{g}_0'$. An element $\alpha = \sum_{i=1}^n c_i\alpha_i \in Q$ is called *positive* (resp. *negative*), that is $\alpha > 0$ (resp. $\alpha < 0$), if there is an index i satisfying $c_i > 0$ (resp. $c_i < 0$) and $c_{i+1} = c_{i+2} = \cdots = c_n = 0$. Put $\Delta_+ = \{\alpha \in \Delta \mid \alpha > 0\}$ and $\Delta_- = \{\alpha \in \Delta \mid \alpha < 0\}$. Let $\mathfrak{g}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$, and $\mathfrak{g}_\pm' = \bigoplus_{\alpha \in \Delta_\pm \setminus \{\alpha_1\}} \mathfrak{g}_\alpha$. Then, $\mathfrak{g}_\pm = \mathfrak{g}_{\pm\alpha_1} \oplus \mathfrak{g}_\pm'$, and we see that $\mathfrak{g}_{\pm\alpha_1} \oplus \mathfrak{g}_\mp'$ are subalgebras. In this situation we have the following result.

Proposition 3.3 *Let $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$ be a graded Lie algebra with the extra conditions as above. Then,*

$$U(\mathfrak{g}) = U(\mathfrak{g}_{\alpha_1})\, U(\mathfrak{g}_-)U(\mathfrak{g}_0')U(\mathfrak{g}_+).$$

Moreover, letting $P = \mathfrak{g}_+ \oplus \mathfrak{g}_0'$ and $M = \mathfrak{g}_-$ gives a plus-minus pair for $\mathfrak{g}$.

Proof Using our assumptions we see that

$$\begin{aligned}
U(\mathfrak{g}) &= U(\mathfrak{g}_-)U(\mathfrak{g}_0)\,U(\mathfrak{g}_+)\\
&= U(\mathfrak{g}_-')U(\mathfrak{g}_{-\alpha_1})U(\mathfrak{g}_0')U([\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}])U(\mathfrak{g}_{\alpha_1})\,U(\mathfrak{g}_+')\\
&= U(\mathfrak{g}_-')U(\mathfrak{g}_0')\,U(\mathfrak{g}_{-\alpha_1})U([\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}])\,U(\mathfrak{g}_{\alpha_1})\,U(\mathfrak{g}_+')\\
&= U(\mathfrak{g}_-')U(\mathfrak{g}_0')\,U(\mathfrak{g}_{\alpha_1})U(\mathfrak{g}_{-\alpha_1})\,U(\mathfrak{g}_{\alpha_1})\,U(\mathfrak{g}_+')\\
&= U(\mathfrak{g}_{\alpha_1})U(\mathfrak{g}_-')\,U(\mathfrak{g}_0')U(\mathfrak{g}_{-\alpha_1})\,U(\mathfrak{g}_{\alpha_1})\,U(\mathfrak{g}_+')\\
&= U(\mathfrak{g}_{\alpha_1})U(\mathfrak{g}_-')\,U(\mathfrak{g}_{-\alpha_1})U(\mathfrak{g}_0')U(\mathfrak{g}_{\alpha_1})\,U(\mathfrak{g}_+')\\
&= U(\mathfrak{g}_{\alpha_1})U(\mathfrak{g}_-)\,U(\mathfrak{g}_0')U(\mathfrak{g}_+).
\end{aligned}$$
■

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Remark It can be seen that many EALA's and some of the root-graded Lie algebras (*cf.* [1], [2], [3], [13]) satisfy the hypothesis of Proposition 3.3.

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