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Degree Kirchhoff Index of Bicyclic Graphs

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Abstract. Let G be a connected graph with vertex set $V(G)$. The degree Kirchhoff index of G is defined as $S'(G) = \sum_{\{u,v\}\subseteq V(G)} d(u)d(v)R(u,v)$, where $d(u)$ is the degree of vertex u, and $R(u,v)$ denotes the resistance distance between vertices u and v. In this paper, we characterize the graphs having maximum and minimum degree Kirchhoff index among all n-vertex bicyclic graphs with exactly two cycles.

1 Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph with n vertices and m edges. In this paper, all graphs considered are assumed to be connected. The distance $d(v,u)$ between the vertices v and u of the graph G is defined as the length of a shortest path between v and u. The Wiener index is defined as the sum of distances between all unordered pairs of vertices

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v).$$

It is one of the most used topological indices, well correlated with many physical and chemical properties of a variety of classes of chemical compounds.

A multiplicative weighted version of the Wiener index is the Schultz index of the second kind $S(G)$ defined as

$$S(G) = \sum_{\{u,v\}\subseteq V(G)} d(u)d(v)d(u,v).$$

It was introduced by Gutman in [6], where it is also pointed out that the relation

$$S(G) = 4W(G) - (2n-1)(n-1)$$

holds for trees.

The Kirchhoff index (or resistance index) is defined, in analogy to the Wiener index [1], by

$$Kf(G) = \sum_{\{u,v\}\subseteq V(G)} R(u,v) = \frac{1}{2}\sum_{v\in V(G)} R(v).$$

where $R(u,v)$ denotes the resistance distance (see [5]) between vertices u and v and $R(v)$ stands for the sum of resistance distances between the vertex v and all other

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vertices of G:

$$R(v) = R(v|G) = \sum_{u \in V(G)} R(u,v).$$

Recently, a new index called the *degree Kirchhoff index* was introduced in [2] and further studied in [3,4,7]. It is defined as

$$S'(G) = \sum_{\{u,v\} \subseteq V(G)} d(u)d(v)R(u,v) = \frac{1}{2} \sum_{v \in V(G)} d(v)S'(v),$$

where

$$S'(v) = S'(v|G) = \sum_{u \in V(G)} d(u)R(u,v). \tag{1.1}$$

By the definitions of $S(G)$, $S'(G)$, and $R(u,v) \le d(u,v)$, we have $S'(G) \le S(G)$.

For a tree, $R(u,v) = d(u,v)$, we have $Kf(G) = W(G)$ and $S'(G) = S(G)$. In [3], the authors determined the first two maximum and the first two minimum degree Kirchhoff indices of unicyclic graphs. Reference [4] found the fully loaded unicyclic graphs with maximum and minimum degree Kirchhoff index and the extremal cacti with minimum degree Kirchhoff index. Specifically, they gave the minimum degree Kirchhoff index of bicyclic graphs with two edge disjoint cycles. In this paper, we further characterize the graphs having maximum and minimum degree Kirchhoff index among all n-vertex bicyclic graphs with exactly two cycles.

2 Preliminary

Lemma 2.1 ([5]) *Let x be a cut vertex of a graph G, and let a and b be vertices occurring in different components that arise upon deletion of x. Then $R(a,b) = R(a,x) + R(x,b)$.*

Lemma 2.2 ([3]) *Let G_1 and G_2 be connected graphs with disjoint vertex sets and with m_1 and m_2 edges, respectively. Let $u_1 \in V(G_1)$, $u_2 \in V(G_2)$. Construct the graph G by identifying the vertices u_1 and u_2, and denote the so-obtained vertex by u. Then*

$$S'(G) = S'(G_1) + S'(G_2) + 2m_1 S'(u_2|G_2) + 2m_2 S'(u_1|G_1). \tag{2.1}$$

Denote by $H_{n,3}$ the graph obtained by adding $n-3$ pendant vertices to a vertex of C_3 and by U_3^n the graph obtained by adding a path with $n-3$ vertices to a vertex of C_3.

Lemma 2.3 ([3]) *Let G be an unicyclic graph with n vertices. Then $S'(U_3^n) \ge S'(G) \ge S'(H_{n,3})$.*

Lemma 2.4 *Let G_0 be a connected graph with m_0 edges and $u, w \in V(G_0)$ such that $S'(u|G_0) \le S'(v|G_0) \le S'(w|G_0)$ for all $v \in V(G_0)$. If F, G, H are graphs obtained from G_0 by attaching a pendant vertex v_1 at u, v, w, respectively. Then $S'(u|F) \le S'(x|G) \le S'(v_1|H)$ for all $x \in V(G)$.*

By equation (1.1), we have

$$\begin{aligned} S'(v_1|G) &= \sum_{t\in V(G)} d_G(t)R(t,v_1) = \sum_{t\in V(G)} d_G(t)(R(t,v)+1) \\ &= \sum_{t\in V(G_0)-\{v\}} d_{G_0}(t)(R(t,v)+1) + d_{G_0}(v)+1 \\ &= \sum_{t\in V(G_0)} d_{G_0}(t)R(t,v) + \sum_{t\in V(G_0)} d_{G_0}(t)+1 \\ &= S'(v|G_0)+2m_0+1, \end{aligned}$$

and for $x \in V(G_0)$, we have

$$\begin{aligned} S'(x|G) &= \sum_{t\in V(G)} d_G(t)R(t,x) \\ &= \sum_{t\in V(G_0)-\{v\}} d_{G_0}(t)R(t,x) + (d_{G_0}(v)+1)R(v,x) + R(v_1,x) \\ &= \sum_{t\in V(G_0)} d_{G_0}(t)R(t,x) + 2R(v,x)+1 = S'(x|G_0)+2R(v,x)+1. \end{aligned}$$

Note that $0 \le R(v,x) \le m_0$ for $x \in V(G_0)$, we and have

$$S'(x|G_0)+1 \le S'(x|G) \le S'(x|G_0)+2m_0+1.$$

On the other hand,

$$\begin{aligned} S'(u|F) &= \sum_{t\in V(F)} d_F(t)R(t,u) \\ &= \sum_{t\in V(G_0)-\{u\}} d_{G_0}(t)R(t,u) + (d_{G_0}(u)+1)R(u,u) + R(v_1,u) \\ &= \sum_{t\in V(G_0)} d_{G_0}(t)R(t,u) + R(v_1,u) = S'(u|G_0)+1, \\ S'(v_1|H) &= \sum_{t\in V(H)} d_H(t)R(t,v_1) \\ &= \sum_{t\in V(G_0)-\{w\}} d_{G_0}(t)(R(t,w)+1) + (d_{G_0}(w)+1)R(w,v_1) \\ &= \sum_{t\in V(G_0)} d_{G_0}(t)R(t,w) + \sum_{t\in V(G_0)} d_{G_0}(t)+1 = S'(w|G_0)+2m_0+1. \end{aligned}$$

Since $S'(u|G_0) \le S'(v|G_0) \le S'(w|G_0)$ for all $v \in V(G_0)$, we have

$$S'(u|F) \le S'(x|G) \le S'(v_1|H) \quad \forall x \in V(G).$$

Theorem 2.5 *Let G_0 be a connected graph with n vertices and let m_0 edges and $\mathcal{G}_{n,k} = \mathcal{G}_{n,k}(G_0)$ be the set of all connected graphs of order $n+k$ that have G_0 as an induced subgraph such that the components of $G - E(G_0)$ are trees $T_1, T_2, \ldots, T_n$, and each T_i intersects G_0 at a single point. Choose vertices u and w such that $S'(u|G_0) \le S'(v|G_0) \le S'(w|G_0)$ for all $v \in V(G_0)$, and construct a graph $G_k = G_0 + wv_1 + P_k$ from G_0 by attaching a path $P_k = v_1v_2\cdots v_k$ at w, and a graph $G'_k = G_0 + \{uv_1, uv_2, \ldots, uv_k\}$ from G_0 by attaching k pendant vertices $v_1, \ldots, v_k$ at u. Then*

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(i) *G_k has the maximal degree Kirchhoff index and G'_k has the minimum degree Kirchhoff index among $\mathcal{G}_{n,k}$;*

(ii) *$S'(u|G'_k) \le S'(x|H) \le S'(v_k|G_k)$ for all $H \in \mathcal{G}_{n,k}$ and $x \in V(H)$.*

Proof We will prove the theorem by induction on k.

If $k = 1$, let $G \in \mathcal{G}_{n,1}$, then G can be obtained by attaching a pendant vertex v_1 at a vertex v of G_0. By Lemma 2.2, we have

$$S'(G) = S'(G_0) + 2S'(v|G_0) + 2m_0 + 1.$$

So $S'(G_0) + 2S'(u|G_0) + 2m_0 + 1 \le S'(G) \le S'(G_0) + 2S'(w|G_0) + 2m_0 + 1$, *i.e.*, $S'(G'_1) \le S'(G) \le S'(G_1)$. And $S'(u|G'_1) \le S'(x|H) \le S'(v_1|G_1)$ for all $H \in \mathcal{G}_{n,1}$ and $x \in V(H)$ from Lemma 2.4. Results (i) and (ii) hold for $k = 1$.

Suppose results (i) and (ii) hold for $k = t$. If $H \in \mathcal{G}_{n,t+1}$, then H can be obtained from a graph $H' \in \mathcal{G}_{n,t}$ by attaching a pendant vertex v_{t+1} at a vertex $v \in V(H')$. By Lemma 2.2, we have

$$S'(H) = S'(H') + 2S'(v|H') + 2m_t + 1$$

where $m_t = |E(H')| = m_0 + t$. And $S'(G'_t) \le S'(H') \le S'(G_t)$ and $S'(u|G'_t) \le S'(v|H') \le S'(v_t|G_t)$ from the induction hypothesis.

So, $S'(G'_t) + 2S'(u|G'_t) + 2m_t + 1 \le S'(H) \le S'(G_t) + 2S'(v_t|G_t) + 2m_t + 1$, *i.e.*, $S'(G'_{t+1}) \le S'(H) \le S'(G_{t+1})$.

On the other hand, as in the proof of Lemma 2.4, we have

$$S'(x|H) = S'(x|H') + 2R(v,x) + 1 \quad \text{and} \quad 0 \le R(v,x) \le m_t$$

for $x \in V(H')$, and

$$S'(v_{t+1}|H) = S'(v|H') + 2m_t + 1$$
$$S'(u|G'_{t+1}) = S'(u|G'_t) + 1$$
$$S'(v_{t+1}|G_{t+1}) = S'(v_t|G_t) + 2m_t + 1.$$

By the induction hypothesis, $S'(u|G'_t) \le S'(v|H') \le S'(v_t|G_t)$, and so we have $S'(u|G'_{t+1}) \le S'(x|H) \le S'(v_{t+1}|G_{t+1})$ for all $x \in V(H)$.

Therefore, results (i) and (ii) hold for $k = t + 1$. ∎

3 The Main Results

Let G be a bicyclic graph of order n, with exactly two cycles C_{k_1}, C_{k_2}, with skeleton graph either $B_1(k_1,k_2)$ or $B_2(k_1,k_2)$ (see Figure 1).

Denote by $\mathcal{B}_1(k_1,k_2)$ the set of bicyclic graphs of order n with the skeleton graph $B_1(k_1,k_2)$, and by $\mathcal{B}_1$ the set of bicyclic graphs of order n with the skeleton graph $B_1(k_1,k_2)$ for all $k_1 \ge 3$ and $k_2 \ge 3$.

For $x \in V(B_1(k_1,k_2))$, $B_1^x(k_1,k_2,S)$ is the graph obtained by identifying x of $B_1(k_1,k_2)$ with the center v of the star S_{t+1}, and denote the so obtained vertex by x; *i.e.*, $B_1^x(k_1,k_2,S)$ is the graph obtained $B_1(k_1,k_2)$ by attaching t pendant vertices at x, where $t = n - k_1 - k_2 + 1$; $B_1^x(k_1,k_2,P)$ is the graph obtained by identifying x of $B_1(k_1,k_2)$ with an end vertex v of the path P_{t+1}, and denote the so obtained vertex by x; $B_1^x(k_1,k_2,P)$ is the graph obtained from $B_1(k_1,k_2)$ by attaching a path P_t at x.

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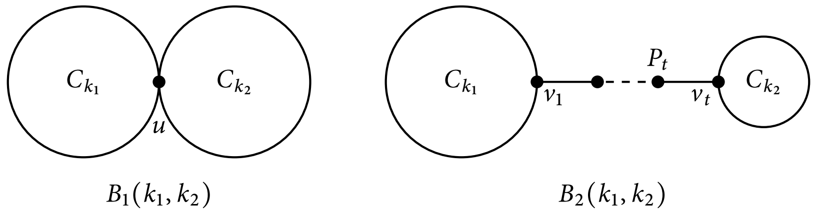

Figure 1: The skeleton graphs of bicyclic graphs ($k_1 \geq k_2$).

Let u be the unique common vertex of C_{k_1} and C_{k_2}, and w is a vertex of C_{k_1} such that $d(u,w) \geq d(v,u)$ for all $v \in V(C_{k_1})$. By computing, we have

$$S'(B_1(k_1,k_2)) = \tfrac{1}{3}(k_1^3 + k_2^3 + 2k_2k_1^2 + 2k_1k_2^2 - 3k_1 - 3k_2);$$
$$S'(u|B_1(k_1,k_2)) = \tfrac{1}{3}(k_1^2 + k_2^2 - 2),$$
$$S'(w|B_1(k_1,k_2)) = \begin{cases} \frac{1}{3}(k_1^2 + k_2^2 - 2) + \frac{1}{2}k_1k_2, & \text{if } k_1 \text{ is even,} \\ \frac{1}{3}(k_1^2 + k_2^2 - 2) + \frac{1}{2k_1}k_2(k_1^2 - 1), & \text{if } k_1 \text{ is odd,} \end{cases}$$
$$S'(S_t) = 2t^2 - 5t + 3, \quad S'(v|S_t) = t - 1,$$
$$S'(P_t) = \tfrac{1}{3}(t-1)(2t^2 - 4t + 3), \quad S'(v|P_t) = (t-1)^2.$$

Theorem 3.1 *If $G \in \mathcal{B}_1(k_1,k_2)$, then*

$$S'\big(B_1^u(k_1,k_2,S)\big) \leq S'(G) \leq S'\big(B_1^w(k_1,k_2,P)\big).$$

Proof For all $x \in V(B_1(k_1,k_2))$, it is computed out that

$$S'\big(u|B_1(k_1,k_2)\big) \leq S'\big(x|B_1(k_1,k_2)\big) \leq S'\big(w|B_1(k_1,k_2)\big).$$

By taking $G_0 = B_1(k_1,k_2)$ in Theorem 2.5, we have $S'(B_1^u(k_1,k_2,S)) \leq S'(G) \leq S'(B_1^w(k_1,k_2,P))$. ■

The result shows that $S'(B_1^w(k_1,k_2,P))$ has the maximal degree Kirchhoff index, and $S'(B_1^u(k_1,k_2,S))$ has the minimum degree Kirchhoff index among $\mathcal{B}_1(k_1,k_2)$.

Lemma 3.2 *If $k_1 > 3$, then $S'(B_1^u(k_1 - 1,k_2,S)) < S'(B_1^u(k_1,k_2,S))$.*

Proof By Lemma 2.2, we have

$$S'\big(B_1^u(k_1,k_2,S)\big) = S'\big(B_1(k_1,k_2)\big) + S'(S_t) + 2(t-1)S'\big(u|B_1(k_1,k_2)\big) + 2(k_1+k_2)S'(v|S_t)$$

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and

$$\begin{aligned} &S'\big(B_1^u(k_1,k_2,S)\big) - S'\big(B_1^u(k_1-1,k_2,S)\big)\\ &\quad= \tfrac{1}{3}(k_1^2+4k_1k_2-9k_1-8k_2+11)+\tfrac{2}{3}t(2k_1-4)\\ &\quad\geq \tfrac{1}{3}\big[(k_1^2-5k_1+11\big]+\tfrac{2}{3}t(2k_1-4)\\ &\quad= \tfrac{1}{3}\Big[\Big(k_1-\tfrac{5}{2}\Big)^2+\tfrac{19}{4}\Big]+\tfrac{2}{3}t(2k_1-4)>0. \end{aligned}$$

So $S'(B_1^u(k_1-1,k_2,S)) < S'(B_1^u(k_1,k_2,S))$. ■

Theorem 3.3 *If $G \in \mathcal{B}_1$, then $S'(G) \geq S'(B_1^u(3,3,S))$.*

Proof For $G \in \mathcal{B}_1$, there are $k_1 \geq k_2 \geq 3$ such that $G \in \mathcal{B}_1(k_1,k_2)$. By Theorem 3.1, we have

$$S'(G) \geq S'\big(B_1^u(k_1,k_2,S)\big).$$

Using Lemma 3.2 repeatedly, we have $S'(G) \geq S'(B_1^u(k_1,k_2,S)) \geq S'(B_1^u(3,3,S))$. ■

Theorem 3.4 *If $G \in \mathcal{B}_1$, then $S'(G) \leq S'(B_1^w(3,3,P))$.*

Proof For $G \in \mathcal{B}_1$, there are $k_1 \geq k_2 \geq 3$ such that $G \in \mathcal{B}_1(k_1,k_2)$. By Theorem 3.1, we have

$$S'(G) \leq S'(B_1^w\big(k_1,k_2,P)\big).$$

Let $U_{k_1}^{n_1}$ be the induced subgraph of $B_1^w(k_1,k_2,P)$ by the cycle C_{k_1} and the path P_t, *i.e.*, $U_{k_1}^{n_1}$ is an unicyclic graph of order n_1 obtained from C_{k-1} by attaching a path P_t at w, where $n_1 = k_1 + t$. By Lemma 2.2, we have

$$S'\big(B_1^w(k_1,k_2,P)\big) = S'(U_{k_1}^{n_1}) + S'(C_{k_2}) + 2k_2S'(u\,|\,U_{k_1}^{n_1}) + 2n_1S'(u\,|\,C_{k_2})$$

and

$$\begin{aligned} &S'\big(B_1^w(3,k_2,P)\big) - S'\big(B_1^w(k_1,k_2,P)\big)\\ &\quad= S'(U_3^{n_1}) - S'(U_{k_1}^{n_1}) + 2k_2\big(S'(w\,|\,U_3^{n_1}) - S'(w\,|\,U_{k_1}^{n_1})\big)\\ &\quad\geq 0 \end{aligned}$$

So we have

$$S'\big(B_1^w(3,k_2,P)\big) \geq S'\big(B_1^w(k_1,k_2,P)\big). \tag{3.1}$$

Now, let w' be a vertex of C_{k_2} such that $d(u,w') \geq d(x,u)$ for all $x \in V(C_{k_2})$. Then

$$S'\big(x\,|\,B_1(3,k_2)\big) \leq S'\big(w'\,|\,B_1(3,k_2)\big)$$

for all $x \in V(B_1(3,k_2))$. By Lemma 2.2, we have

$$S'\big(B_1^w(3,k_2,P)\big) \leq S'\big(B_1^{w'}(3,k_2,P)\big).$$

By equation (3.1), we have

$$S'\big(B_1^{w'}(k_2,3,P)\big) \leq S'\big(B_1^{w'}(3,3,P)\big).$$

So $S'(G) \leq S'(B_1^w(k_1,k_2,P)) \leq S'(B_1^w(3,3,P))$. ■

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For the skeleton graph $G = B_2(k_1, k_2)$ (see Figure 1), we have

$$x \in V(C_{k_1} - v_1), S'(x|G) =$$
$$\tfrac{1}{3}(k_1^2 + k_2^2 - 2) + (t-1)^2 + 2R(x, u_1)(k_2 + t - 1) + 2k_2(t-1),$$
$$x \in V(C_{k_2} - v_t), S'(x|G) =$$
$$\tfrac{1}{3}(k_1^2 + k_2^2 - 2) + (t-1)^2 + 2R(x, u_t)(k_1 + t - 1) + 2k_1(t-1),$$
$$v_l \in V(P_t), S'(v_l|G) =$$
$$\tfrac{1}{3}(k_1^2 + k_2^2 - 2) + (l-1)^2 + (t-l-1)^2 + 2k_1(l-1) + 2k_2(t-l-1).$$

From above, we can get the following results by direct computation.

(i) If w is a vertex of $B_2(k_1, k_2)$ such that $S'(w|B_2(k_1, k_2)) \geq S'(x|B_2(k_1, k_2))$ for all $x \in V(B_2(k_1, k_2))$, then either w is a vertex of C_{k_1} such that $d(w, v_1) \geq d(x, v_1)$ for all $x \in V(C_{k_1})$, or w is a vertex of C_{k_2} such that $d(w, v_t) \geq d(x, v_t)$ for all $x \in V(C_{k_2})$.

(ii) If v is a vertex of $B_2(k_1, k_2)$ such that $S'(v|B_2(k_1, k_2)) \leq S'(x|B_2(k_1, k_2))$ for all $x \in V(B_2(k_1, k_2))$, then $v = v_l \in P_t$, where $l = \lfloor \frac{t+1}{2} \rfloor$.

Denote by $\mathcal{B}_2(k_1, k_2)$ the set of bicyclic graphs of order n whose skeleton graph is $B_2(k_1, k_2)$, and by $\mathcal{B}_2$ the set of bicyclic graphs of order n whose skeleton graph is $B_2(k_1, k_2)$ for all $k_1 \geq 3$ and $k_2 \geq 3$.

For $x \in V(B_2(k_1, k_2))$, $B_2^x(k_1, k_2, S)$ is the bicyclic graph obtained from $B_2(k_1, k_2)$ by attaching k pendant vertices at x; $B_2^x(k_1, k_2, P)$ is the bicyclic graph obtained from $B_2(k_1, k_2)$ by attaching a path P_k at x, where $k = n - k_1 - k_2 - t + 2$.

Using Theorem 2.5, we can get the following result.

Lemma 3.5 *If $G \in \mathcal{B}_2(k_1, k_2)$, then*

$$S'\big(B_2^{v_l}(k_1, k_2, S)\big) \leq S'(G) \leq S'\big(B_2^{w}(k_1, k_2, P)\big),$$

where w is a vertex of $B_2(k_1, k_2)$ such that $S'(w|B_2(k_1, k_2)) \geq S'(x|B_2(k_1, k_2))$ and v is a vertex of $B_2(k_1, k_2)$ such that $S'(v|B_2(k_1, k_2)) \leq S'(x|B_2(k_1, k_2))$ for all $x \in V(B_2(k_1, k_2))$.

Theorem 3.6 *If $G \in \mathcal{B}_2$, then $S'(G) \leq S'(B_2(3, 3))$.*

Proof For $G \in \mathcal{B}_2$, there are $k_1 \geq k_2 \geq 3$ such that $G \in \mathcal{B}_2(k_1, k_2)$. By Lemma 3.5, we have $S'(G) \leq S'(B_1^w(k_1, k_2, P))$.

Now, using equation (2.1) and Theorem 2.5, we have

$$S'\big(B_2^w(k_1, k_2, P)\big) \leq S'\big(B_2(k_1, k_2)\big).$$

Let $U_{k_1}^{n_1}$ be the induced subgraph of $B_2(k_1, k_2)$ by its cycle C_{k_1} and path P_t, where $n = k_1 + k_2 + t - 2$ and $n_1 = k_1 + t - 1$. Then $U_{k_1}^{n_1}$ is an unicyclic graph of order n_1, and $u = v_t$ is the vertex of degree 1 in $U_{k_1}^{n_1}$. By Lemma 2.2, we have

$$S'\big(B_2(k_1, k_2)\big) = S'(U_{k_1}^{n_1}) + S'(C_{k_2}) + 2k_2 S'(u_t|U_{k_1}^{n_1}) + 2n_1 S'(u|C_{k_2}).$$
$$S'\big(B_2(3, k_2)\big) = S'(U_3^{n_1}) + S'(C_{k_2}) + 2k_2 S'(u_t|U_3^{n_1}) + 2n_1 S'(u|C_{k_2}).$$

Since $S'(U_{k_1}^{n_1}) \leq S'(U_3^{n_1})$ from Lemma 2.3 and $S'(u_t|U_{k_1}^{n_1}) \leq S'(u_t|U_3^{n_1})$, we have

$$S'\big(B_2(k_1, k_2)\big) \leq S'\big(B_2(3, k_2)\big).$$

Similarly,

$$S'\big(B_2(3,k_2)\big) \le S'\big(B_2(3,3)\big).$$

So $S'(G) \le S'(B_2(3,3))$. ∎

Theorem 3.7 *If $G \in \mathcal{B}_2$, then $S'(G) \ge S'(B_1^u(3,3,S))$.*

Proof For $G \in \mathcal{B}_2$, there are $k_1 \ge k_2 \ge 3$ such that $G \in \mathcal{B}_2(k_1,k_2)$. By Lemma 3.5, we have $S'(G) \ge S'(B_2^{v_l}(k_1,k_2,S))$.

Let U be the unicyclic subgraph from $B_2^{v_l}(k_1,k_2,S)$ by deleting all vertices in $C_{k_2} - \{v_t\}$. Also H_{n_1,k_1} is the unicyclic graph obtained by attaching $n-k_1-k_2+1$ pendant vertices to a vertex of C_{k_1}, where $n_1 = n-k_2+1$.

By Lemma 2.2, we have

$$S'\big(B_2^{v_l}(k_1,k_2,S)\big) = S'(U) + S'(C_{k_2}) + 2k_2 S'(v_t|U) + 2n_1 S'(v_t|C_{k_2}),$$
$$S'\big(B_1^u(k_1,k_2,S)\big) = S'(H_{n_1,k_1}) + S'(C_{k_2}) + 2k_2 S'(u|H_{n_1,k_1}) + 2n_1 S'(u|C_{k_2}).$$

Since $S'(U) \ge S'(H_{n_1,k_1})$ from Lemma 2.3, and $S'(v_t|U) \ge S'(u|H_{n_1,k_1})$, where $u = v_1$, we have

$$S'(B_1^u(k_1,k_2,S)) \le S'(B_2^{v_l}(k_1,k_2,S)).$$

By Theorem 3.3, we have $S'(G) \ge S'(B_1^u(k_1,k_2,S)) \ge S'(B_1^u(3,3,S))$. ∎

Theorem 3.8 *If G is a bicyclic graph of order n with exactly two cycles, then*

$$S'(B_1^u(3,3,S)) \le S'(G) \le S'(B_2(3,3)).$$

Proof By Theorems 3.3, 3.4, 3.6, and 3.7, we only need to prove that $S'(B_1^w(3,3,P)) \le S'(B_2(3,3))$.

Let U_3^{n-2} be the unicyclic graph of order $n-2$ obtained by attaching a path P_{n-4} to a vertex of C_3. Then

$$S'(u|U_3^{n-2}) \le S'(v|U_3^{n-2}),$$

where u is a vertex of C_3 with degree 2 in U_3^{n-2}, and v is the vertex with degree 1 in U_3^{n-2}. By Lemma 2.2, we have

$$S'\big(B_1^u(3,3,P)\big) = S'(C_3) + S'(U_3^{n-2}) + 6S'(u|U_3^{n-2}) + 2(n-2)S'(u|C_3),$$
$$S'\big(B_2^u(3,3)\big) = S'(C_3) + S'(U_3^{n-2}) + 6S'(v|U_3^{n-2}) + 2(n-2)S'(v|C_3)$$

So $S'(B_1^u(3,3,P)) \le S'(B_2(3,3))$. ∎

Note that the minimum degree Kirchhoff index of bicyclic graphs with exactly two cycles is also obtained in [4] by using other method.

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