SUBLATTICES OF A FREE LATTICE

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Introduction. Professor R. A. Dean has proved (**1**, Theorem 3) that a completely free lattice generated by a countable partially ordered set is isomorphic to a sublattice of a free lattice. In particular, it follows that a free product of countably many countable chains can be isomorphically embedded in a free lattice. Generalizing this we show (2.1) that the class of all lattices that can be isomorphically embedded in free lattices is closed under the operation of forming free lattice-products with arbitrarily many factors. We also prove (2.4) that this class is closed under the operation of forming simply ordered sums with denumerably many summands. Finally we show (2.7) that every finite dimensional sublattice of a free lattice is finite.

The first theorem mentioned above is based on a result (1.3) of a rather general nature concerning free products of algebraic systems. It is perhaps worth noting that the *amalgamation property* considered there has played a role in the investigations of other embedding problems in universal algebra. Compare in this connection Fraissé (**2**) and Jónsson (**3**).

1. A theorem in universal algebra. We consider a class **K** of algebras, or algebraic systems, $\mathfrak{A} = \langle A, F_0, F_1, \ldots, F_\xi, \ldots\rangle_{\xi<\alpha}$, where A is a non-empty set, α is a finite or infinite ordinal and, for each $\xi < \alpha$, F_ξ is an operation of some finite rank μ_ξ over A, that is, F_ξ is a function on A^{μ_ξ} to A. The ordinal α and the natural numbers μ_ξ are assumed to be the same for all members of **K**. Actually we shall identify the algebra $\mathfrak{A}$ with its underlying set A. Assuming the notions of isomorphism, homomorphism, and subalgebra to be known, we recall the definitions of a free **K**-algebra and of a free **K**-product.

We say that A is a *free* **K**-*algebra generated by* X if and only if $A \in$ **K**, $X \subseteq A$, A is generated by X, and for any mapping f of X into an algebra $B \in$ **K** there exists a homomorphism g of A into B such that $g(x) = f(x)$ for all $x \in X$.

We say that A is a *free* **K**-*algebra* if and only if there exists a set X such that A is a free **K**-algebra generated by X.

We say that A is a *free* **K**-*product of* A_i, $i \in I$, if and only if $A \in$ **K**, I is a non-empty set, $A_i \in$ **K** and A_i is a subalgebra of A for all $i \in I$, A is generated by the set $\bigcup_{i\in I} A_i$, and for any homomorphisms f_i of the algebras A_i into an algebra $B \in$ **K** there exists a homomorphism g of A into B such that $g(x) = f_i(x)$ whenever $i \in I$ and $x \in A_i$.

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(The free **K**-product just defined might be properly called an inner free **K**-product. An outer free **K**-product of algebras $B_i \in \mathbf{K}$, $i \in I$, would consist of an algebra $A \in \mathbf{K}$ together with isomorphisms of the algebras B_i into A, such that A is an inner free **K**-product of the images A_i of the algebras B_i.)

To insure the existence of free **K**-algebras and of free **K**-products we shall assume that **K** is a non-trivial equational class in the wider sense, that is, **K** is the class of all models of some finite or infinite set of equations, and there exists an algebra $A \in \mathbf{K}$ having at least two elements. We also assume that **K** has the following property:

Definition 1.1. A class **K** of algebraic systems is said to have the *embedding property* if and only if for any $A, B \in \mathbf{K}$ there exists $C \in \mathbf{K}$ such that A and B are isomorphic to subsystems of C.

These assumptions are somewhat stronger than is necessary, but they are sufficiently general for the present purpose.

Under these assumptions concerning **K** we have:

For any non-empty set X there exists an algebra A such that A is a free **K**-algebra generated by X.

Suppose A is a free **K**-algebra generated by X, and f is a one-to-one mapping of X onto a subset Y of an algebra B. Then B is a free **K**-algebra generated by Y if and only if there exists an isomorphism g of A onto B such that $g(x) = f(x)$ for all $x \in X$.

If I is a non-empty set and if $B_i \in \mathbf{K}$ for each $i \in I$, then there exist an algebra A and isomorphisms of all the algebras B_i onto subalgebras A_i of A such that A is a free **K**-product of A_i, $i \in I$.

Suppose A is a free **K**-product of $A_i, i \in I$, and for each $i \in I$ suppose f_i is an isomorphism of A_i onto a subalgebra B_i of an algebra B. Then B is a free **K**-product of B_i, $i \in I$, if and only if there exists an isomorphism g of A onto B such that $g(x) = f_i(x)$ whenever $i \in I$ and $x \in A_i$.

Suppose with the elements i of a non-empty set I there are associated pairwise disjoint, non-empty subsets X_i of an algebra A, let $X = \bigcup_{i \in I} X_i$, and for each $i \in I$ let A_i be the subalgebra of A, which is generated by X_i. Then A is a free **K**-algebra generated by X if and only if A is a free **K**-product of A_i, $i \in I$, and for each $i \in I$, A_i is a free **K**-algebra generated by X_i.

We now introduce the amalgamation property mentioned in the introduction.

Definition 1.2. A class **K** of algebraic systems is said to have the *amalgamation property* if and only if the following conditions are satisfied:

If $A, B_0, B_1 \in \mathbf{K}$ and if f_0 and f_1 are isomorphisms of A into B_0 and into B_1, respectively, then there exist $C \in \mathbf{K}$ and isomorphisms g_0 and g_1 of B_0 and of B_1, respectively, into C, such that $g_0 f_0(x) = g_1 f_1(x)$ whenever $x \in A$.

THEOREM 1.3. *Suppose the class* **K** *of algebraic systems is non-trivial and equational in the wider sense, and assume that* **K** *has the embedding property* *and the amalgamation property. If A is a free* **K**-*product of A_i, $i \in I$, if B_i is a subalgebra of A_i for each $i \in I$, and if B is the subalgebra of A that is generated by the set $\bigcup_{i\in I} B_i$, then B is a free* **K**-*product of B_i, $i \in I$.*

Proof. For convenience we assume that I is the set of all ordinals $\xi < \alpha$, where α is some fixed ordinal. There exist $C \in \mathbf{K}$ and isomorphisms f_ξ of B_ξ onto subalgebras C_ξ of C for all $\xi < \alpha$, such that C is a free **K**-product of C_ξ, $\xi < \alpha$. Consequently there exists a homomorphism h of C into B such that $h(x) = f_\xi^{-1}(x)$ whenever $\xi < \alpha$ and $x \in C_\xi$. Since B is generated by the union of the algebras B_ξ, h must actually map C onto B. The proof will therefore be complete if we prove that h is one-to-one.

We shall show that there exists an increasing sequence of algebras $D_0 = C$, $D_1, D_2, \ldots, D_\alpha$ in **K** and a sequence of functions $g_0, g_1, \ldots, g_\xi, \ldots, \xi < \alpha$, such that the following conditions hold for each $\xi < \alpha$:

(1) g_ξ maps A_ξ isomorphically into $D_{\xi+1}$.

(2) $g_\xi(y) = f_\xi(y)$ whenever $y \in B_\xi$.

In fact, suppose $0 < \lambda \leqslant \alpha$, and suppose D_ξ has been defined for all $\xi < \lambda$, and g_ξ has been defined for all ξ with $\xi + 1 < \lambda$ in such a way that (1) and (2) hold whenever $\xi + 1 < \lambda$. If λ is a limit ordinal, then we let D_λ be the union of the algebras D_μ with $\mu < \lambda$. Thus D_ξ is defined for all $\xi < \lambda + 1$. Since the conditions $\xi + 1 < \lambda + 1$ and $\xi + 1 < \lambda$ are equivalent, we see that g_ξ is defined and that (1) and (2) hold whenever $\xi + 1 < \lambda + 1$. If λ is not a limit ordinal, say $\lambda = \mu + 1$, then f_μ maps B_μ isomorphically into D_μ (because C is a subalgebra of D_μ), and the identity automorphism of B_μ maps B_μ isomorphically into A_μ. By the amalgamation property this implies that there exist $D_\lambda \in \mathbf{K}$, an isomorphism g_μ of A_μ into D_μ, and an isomorphism k_μ of D_μ into D_λ such that $g_\mu(x) = k_\mu f_\mu(x)$ for all $x \in B_\mu$. We may assume that D_λ is an extension of D_μ, and that k_μ is the identity automorphism of D_μ, so that $g_\mu(x) = f_\mu(x)$ for all $x \in B_\mu$. Thus D_ξ has been selected for all $\xi < \lambda + 1$ and g_ξ has been selected for all ξ with $\xi + 1 < \lambda + 1$, and the conditions (1) and (2) hold whenever $\xi + 1 < \lambda + 1$. An easy induction now establishes the existence of all the required algebras D_ξ and functions g_ξ.

Each of the algebras $D_{\xi+1}$ with $\xi < \alpha$ is a subalgebra of D_α, and therefore g_ξ maps A_ξ isomorphically into D_α. Since A is a free **K**-product of A_ξ, $\xi < \alpha$, it follows that there exists a homomorphism g of A into D_α such that $g(y) = g_\xi(y)$ whenever $\xi < \alpha$ and $y \in A_\xi$. In particular, if $x \in C_\xi$, then $y = f_\xi^{-1}(x) = h(x)$ belongs to B_ξ, and therefore $g(y) = g_\xi(y) = f_\xi(y) = x$. Thus $gh(x) = x$ whenever x belongs to one of the algebras C_ξ, whence it follows that $gh(x) = x$ for all $x \in C$. This shows that h is one-to-one, and the proof is complete.

COROLLARY 1.4. *Suppose the class* **K** *of algebraic systems is non-trivial and is equational in the wider sense, and assume that* **K** *has the embedding property and the amalgamation property. If A is a free* **K**-*product of A_i, $i \in I$, and if for*

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each $i \in I$, A_i is isomorphic to a subalgebra of a free **K**-*algebra with m_i generators, then A is isomorphic to a subalgebra of a free* **K**-*algebra with*

$$\sum_{i\in I} m_i$$

generators.

Proof. Let

$$m = \sum_{i\in I} m_i,$$

and let F be a free **K**-algebra generated by a set X with m elements. Then

$$X = \bigcup_{i\in I} X_i$$

where the sets X_i are pairwise disjoint and X_i has m_i elements. If for each $i \in I$, F_i is the subalgebra of F that is generated by X_i, then F is a free **K**-product of F_i, $i \in I$. Furthermore, F_i is a free **K**-algebra generated by X_i, whence it follows that A_i is isomorphic to a subalgebra B_i of F_i. If B is the subalgebra of F that is generated by the set

$$\bigcup_{i\in I} B_i,$$

then we infer by (1.3) that B is a free **K**-product of B_i, $i \in I$. Consequently A is isomorphic to B.

An example shows that the amalgamation property cannot be dropped from the hypothesis of (1.3) and (1.4). Let **K** be the class of all systems consisting of a group G together with a homomorphism α of G into its centre. That is, in addition to the group axioms the systems in **K** satisfy the conditions

$$\alpha(xy) = \alpha(x)\alpha(y), \qquad \alpha(x)y = y\alpha(x).$$

Regarded as a group, a free **K**-system F generated by a set X is a direct product of a free group F_0 generated by X and a free Abelian group F_1 generated by the set of all elements of the form $\alpha^k(x)$ with $x \in X$ and $k = 1, 2, 3, \ldots$. Since two elements of a free group commute if and only if they are powers of the same element, it follows that two elements a and b of F commute if and only if $a = u^p v$ and $b = u^q w$ where $u \in F_0$, $v, w \in F_1$, and p and q are integers. From this we infer that every free **K**-system F has the following property: If $a, b \in F$, and $ab = ba$, then there exist integers p and q, not both zero, such that $a^q b^{-p}$ is in the centre of F. It obviously follows that every subsystem of a free system also has this property.

Let G be a free **K**-product of G_0 and G_1, where G_1 is a free **K**-system generated by a one-element set $\{x\}$ and G_1 is a free Abelian group generated by an infinite set $\{y_0, z_0, y_1, z_1, \ldots\}$ together with the endomorphism α that takes y_i into y_{i+1} and z_i into z_{i+1}. Letting A_0, B_0, A_1, and B_1 be the subgroups of G generated by the sets $\{x\}$, $\{\alpha^k(x) \mid k = 1, 2, \ldots\}$, $\{y_0, z_0\}$, and $\{y_1, z_1, y_2, z_2, \ldots\}$, respectively, and using $*$ and $\times$ to denote free group-

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products and direct products, we find that $G_0 = A_0 \times B_0$ and $G_1 = A_1 \times B_1$, and therefore $G = (A_0 * A_1) \times B_0 \times B_1$. It follows that no element of A_1 except the identity belongs to the centre of G, because no other element commutes with x. Consequently $y_0{}^q z_0{}^{-p}$ does not belong to the centre of G unless $p = q = 0$, and since $y_0 z_0 = z_0 y_0$ this shows that G is not isomorphic to a subsystem of a free **K**-system. Inasmuch as G_0 is a free **K**-system and G_1 is isomorphic to the centre of a free **K**-system, it follows that the conclusion of (1.4) fails for the class **K**.

Observing that in the proof of (1.4) the only use made of the amalgamation property was through (1.3), we infer that the above class **K** must also violate the conclusion of (1.3). An even simpler example of a non-trivial equational class, having the embedding property but not satisfying the conclusion of (1.3), is the class **K** of all groups G such that $a^2b^2 = b^2a^2$ for all $a, b \in G$. Since there exist non-Abelian groups having this property (for example, the group of all permutations of a three-element set), a free **K**-algebra with two generators or, equivalently, a free **K**-product of two infinite cyclic groups, cannot be Abelian. If A is a free **K**-algebra generated by a two-element set $\{a_0, a_1\}$, and therefore the free **K**-product of the infinite cyclic groups A_0 and A_1 generated by the sets $\{a_0\}$ and $\{a_1\}$, then the subgroup B of A generated by $\{a_0{}^2, a_1{}^2\}$ is Abelian, and is therefore not a free **K**-product of the subgroups generated by $\{a_0{}^2\}$ and $\{a_1{}^2\}$.

2. Sublattices of a free lattice. We begin by applying the results of the preceding section to lattices. The class **K** of all lattices is non-trivial and equational. The direct product of two lattices is therefore a lattice, and since every lattice has a one-element sublattice it follows that **K** has the embedding property. In Jónsson **(3)**, in the proof of Theorem 3.5, it is shown that **K** has an amalgamation property that is stronger than the one considered here. Using (1.4) we therefore obtain:

THEOREM 2.1. *If A is a free lattice-product of A_i, $i \in I$, and if for each $i \in I$, A_i is isomorphic to a sublattice of a free lattice with m_i generators, then A is isomorphic to a sublattice of a free lattice with*

$$\sum_{i \in I} m_i$$

generators.

COROLLARY 2.2. *If A is a free lattice-product of A_i, $i \in I$, and if each A_i is a denumerable chain, then A is isomorphic to a sublattice of a free lattice with m generators, where m is the cardinal of I in case I is non-denumerable, and $m = 3$ in case I is denumerable.*

LEMMA 2.3. *Suppose m is an infinite cardinal and F is a free lattice with m generators. If $a, b \in F$ and $b < a$, then the lattice quotient a/b contains, as a sublattice, a free lattice with m generators.*

Proof. If X is the set of generators of F, then there exists a finite subset Y of X such that a and b belong to the sublattice F' of X, which is generated by Y. The sublattice F'' of F, which is generated by the set $Z = X - Y$, can be mapped homomorphically into a/b by a function f such that $f(x) = b + ax$ whenever $x \in Z$. We shall show that f is an isomorphism.

Since F is a free lattice-product of F' and F'', it follows from 1.3 that the lattice D generated by a, b, and Z is a free lattice-product of F'' and of the two-element lattice $E = \{a, b\}$. Letting $\bar{F}$ be a lattice obtained by adjoining a zero element 0 and a unit element 1 to F'' we map E and F'' into $\bar{F}$ by mapping a into 1, b into 0, and each element of F'' into itself. These isomorphisms have a common extension g which is a homomorphism of D into $\bar{F}$. Since f is a homomorphism of F'' into D, gf is a homomorphism of F'' into $\bar{F}$. Furthermore, $gf(x) = g(b + ax) = 0 + 1x = x$ for all $x \in Z$, and therefore $gf(x) = x$ for all $x \in F''$. Consequently f is an isomorphism of F'' into a/b, and the proof is complete.

Given two non-empty subsets B and C of a partially ordered set A, we shall write $B \leqslant C$ if and only if either $B = C$ or else $b < c$ for all $b \in B$ and $c \in C$. It is obvious that the non-empty subsets of a partially ordered set form, under this relation, another partially ordered set.

THEOREM 2.4. *If $m \geqslant 3$, and if the lattice A is the union of a denumerable chain $\mathscr{A}$ of sublattices each of which is isomorphic to a sublattice of a free lattice with m generators, then A is isomorphic to a sublattice of a free lattice with m generators.*

Proof. Since a free lattice with three generators contains as a sublattice a free lattice with infinitely many generators, we may assume that m is infinite.

Let F be a free lattice with m generators and let $\mathscr{B}$ be the family of all quotients a/b with $a, b \in F$ and $b < a$. Then $\mathscr{B}$ is a partially ordered set. Furthermore, for any two quotients a/b and c/d in F, if $a/b < c/d$, that is, if $b < a < d < c$, then it follows by 2.3 that there exist $x, y \in F$ such that $a < y < x < d$ and therefore

$$a/b < x/y < c/d.$$

Consequently, if $\mathscr{C}$ is a maximal chain in $\mathscr{B}$, then $\mathscr{C}$ is dense-in-itself, and $\mathscr{A}$ is therefore order-isomorphic to a subchain $\mathscr{C}'$ of $\mathscr{C}$. By (2.3) and the hypothesis, each of the lattices $B \in \mathscr{A}$ is isomorphic to a sublattice B' of the corresponding quotient in $\mathscr{C}'$, and we conclude that the union A' of these lattices B' is a sublattice of F, and that A is isomorphic to A'.

THEOREM 2.5. *Suppose A is a lattice with a zero element* 0 *and a unit element* 1, *and assume that B and C are sublattices of A such that*

$$B \cap C = \phi \text{ and } B \cup C = A - \{0, 1\},$$
$$b + c = 1 \text{ and } bc = 0 \text{ whenever } b \in B \text{ and } c \in C.$$

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If B and C are isomorphic to sublattices of a free lattice with m generators, where $m \geqslant 3$, then so is A.

Proof. We may assume that m is infinite. If F is a free lattice with m generators, then F is not modular, and hence there exist $a, b, c \in F$ such that

$$(1) \qquad ac < b < a < b + c.$$

We can further assume that a is additively irreducible. In fact, by (2.3) the lattice quotient a/b contains as a sublattice a free lattice F' with m generators, and F' contains an element a' that is multiplicatively reducible (in F' and therefore also in F), and consequently a' is additively irreducible. Furthermore, $b < a' < a$ and hence $a'c \leqslant ac < b$. Thus (1) holds with a replaced by a'. We henceforth assume that a is additively irreducible.

Again using (2.3), we select an additively irreducible element $d \in F$ with $c < d < b + c$, and we show that

$$(2) \qquad ad < b + ad < a < b + c, \qquad ad < c + ad < d < b + c.$$

Since $c < d < b + c$, it follows that $b \nleq d$ and therefore $ad < b + ad$. Also $b + ad \leqslant a$, and an equality would imply that $a = ad$ (because a is additively irreducible and $b < a$). From this we could infer that $a \leqslant d$, hence $b + c \leqslant a + c \leqslant d$, contrary to our choice of d. Thus $b + ad < a$. The inequality $a < b + c$ is part of our hypothesis (1). Since $b < a < b + c$, we have $c \nleq a$, hence $ad < c + ad$. Furthermore $c + ad \leqslant d$, and equality is excluded because it would imply that $d = ad$, hence $c < d \leqslant a$. The last inequality in (2) holds because of our choice of d.

By (2.3) the lattice quotients $a/(b + ad)$ and $d/(c + ad)$ contain free lattices F_1 and F_2 with m generators, and by hypothesis it follows that there exist functions f and g mapping B and C isomorphically into F_1 and F_2, respectively. Observing that $x + y = b + c$ and $xy = ad$ whenever $x \in F_1$ and $y \in F_2$, we obtain the desired isomorphism h of A into F by letting $h(x) = f(x)$ for all $x \in B$, $h(x) = g(x)$ for all $x \in C$, $h(0) = ad$, and $h(1) = b + c$.

In proving our last result we need the observation that a free lattice, and hence every sublattice of a free lattice, satisfies a special case of the distributive law.

LEMMA 2.6. *Suppose F is a free lattice and $u, a, b, c \in F$.*

(i) *If $u = ab = ac$, then $u = a(b + c)$.*

(ii) *If $u = a + b = a + c$, then $u = a + bc$.*

Proof. By Whitman (**4**, Theorem 2, Corollary 2), the canonical representation of u,

$$u = \prod_{i<n} u_i$$

has the property that if

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$$u = \prod_{j<m} v_j,$$

then each of the elements u_i contains one of the elements v_j. Under the hypothesis of (i) it follows that each of the elements u_i either contains a or else contains both b and c, and in either case we therefore have $a(b + c) \leqslant u_i$. Consequently $a(b + c) \leqslant u$. The opposite inclusion is obvious, and (ii) follows by duality.

THEOREM 2.7. *Every finite dimensional sublattice of a free lattice is finite.*

Proof. We shall actually prove the stronger statement that every finite dimensional lattice A which satisfies the condition (i) of (2.6) is finite. Assuming that this holds for all lower dimensional cases, consider the case when the dimension of A is n.

Let M be the set of all the atoms of A, choose $a \in M$, and let $N = M - \{a\}$. Then $ab = 0$ for all $b \in N$, and letting

$$c = \sum_{b \in N} b$$

we infer from (2.6 i) together with the finiteness of the dimension of A that $ac = 0$. Therefore $c \neq 1$, and by the inductive hypothesis the quotient $c/0$ must be finite. In particular this shows that N is finite, and therefore M is finite. Since, by the inductive hypothesis, all the quotients $1/b$ with $b \in M$ are finite, and since every member of A except the element 0 belongs to at least one of these quotients, we conclude that A must be finite.

Analysing the proof of the last theorem we can actually find an upper bound for the number of elements in an n dimensional sublattice A of a free lattice. We first prove by induction that A has at most n atoms. We simply observe that, in the notation used above, the atoms of the lattice quotient $c/0$ are precisely the elements of N, and infer by the inductive hypothesis that N has at most $n - 1$ elements. A second induction proves that A has at most $2 \cdot (n!)$ elements. For each element of A, except the element 0, is contained in one of the quotients $1/b$ with $b \in M$, and the fact that 0 belongs to none of these quotients is more than made up for since 1 belongs to all the quotients. Actually these estimates can be considerably improved. For instance, if $n = 3$, then A has at most 8 elements, and if $n > 3$, then A has at most $n - 1$ atoms.

Finally, Professor R. Dilworth has observed that by a slight modification of our proof it can be shown that if a sublattice A of a free lattice satisfies the double chain condition, then A is finite. The set of all lattice quotients a/b of A, ordered by set-inclusion, satisfies the minimal condition, and one need therefore only consider the case in which A has the additional property that every quotient properly contained in A is finite. Under this assumption the finiteness of A follows as in the proof of (2.7).

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