ON THE CAUCHY PROBLEM FOR THE n-DIMENSIONAL WAVE EQUATION

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CONSIDER the n-dimensional wave equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial t^2}. \tag{1}$$

Then the Cauchy, or initial value problem requires a solution $u(P, t)$ of (1) at the point P subject to the initial conditions:

$$\begin{aligned} u(P, 0) &= f(P), \\ \left[\frac{\partial}{\partial t} u(P, t)\right]_{t=0} &= g(P), \end{aligned} \tag{2}$$

where $f(P)$, $g(P)$ are defined everywhere. In our case we suppose $g(P) = 0$. Then the appropriate solution of the Cauchy problem is given by Courant and Hilbert (**1**) as

$$u(P, t) = \frac{t\sqrt{\pi}}{\Gamma(n/2)}\left(\frac{\partial}{\partial t^2}\right)^{(n-1)/2}(t^{n-2}M(P, t)), \tag{3}$$

when n is odd, and as

$$u(P, t) = \frac{2t}{\Gamma(n/2)}\left(\frac{\partial}{\partial t^2}\right)^{n/2}\int_0^t r^{n-2}M(P, r)\frac{rdr}{\sqrt{t^2-r^2}}, \tag{4}$$

when n is even (although the factor $2t$ is inadvertently omitted in (1)). Here $r = r(P, Q)$ is the Euclidean distance between the points P and Q, and $M(P, r)$ is defined as the mean value of the function $u(Q, 0)$ taken over the surface $\Sigma(r)$ of the n-sphere $B(r)$ with centre P and radius r. Thus

$$M(P, r) \equiv \frac{1}{\omega_n}\int_{\Sigma(r)} u(Q, 0)d\omega, \tag{5}$$

where $d\omega$ is the element of area and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the n-dimensional unit sphere.

For (3) and (4) to be formal solutions of (1) Courant and Hilbert require $u(P, 0)$ to be continuously differentiable in all n arguments, $(n+3)/2$ times for n odd, and $(n+4)/2$ times for n even. In cases of physical interest however, where solutions of the Cauchy problem are known to exist, then (3), (4) *are* these solutions and hence $u(Q, 0)$ need only be sufficiently differentiable to make the expressions meaningful, and satisfy the wave equation. Thus in the rest of this paper we will omit a rigorous discussion of the conditions sufficient for the establishment of the formulae, and assume that necessary and sufficient conditions can be determined individually for each properly posed Cauchy problem.

While (3) and (4) are perfectly general, in some cases of physical interest the function $u(P, 0)$ may be difficult to derive or the integrals not easily handled. It seems of interest to find formulae equivalent to (3) and (4) involving $\nabla^2 u(P, 0)$ rather than $u(P, 0)$. We shall see also that the new forms demonstrate more clearly the nature of the wave.

We begin with the Green's function for the potential equation $\nabla^2 u = 0$ in the case of the n-sphere of radius r_0:

$$G(P, Q;\ r_0) = \frac{1}{2\pi}\log\frac{r_0}{r}, \quad (n=2);$$

$$= \frac{1}{(n-2)\omega_n}(r^{2-n} - r_0^{2-n}). \quad (n>2) \tag{6}$$

From the equations

$$\nabla^2 G(P, Q;\ r_0) = -\delta(P, Q),$$
$$\nabla^2 u(Q, 0) = F(Q), \tag{7}$$

(δ is the Dirac delta function) we arrive at the identities

$$M(P, r_0) = u(P, 0) + \frac{1}{2\pi}\int_{S(r_0)} \log\frac{r_0}{r}\, F(Q)dS_Q, \quad (n=2) \tag{8a}$$

$$M(P, r_0) = u(P, 0) + \frac{1}{(n-2)\omega_n}\int_{B(r_0)} (r^{2-n} - r_0^{2-n})F(Q)dV_Q, \quad (n>2) \tag{8b}$$

where in (8b) dV_Q is the element of volume at the variable point Q, while in (8a) $S(r_0)$ is the circle (2-sphere) centre P, radius r_0, and dS_Q is the element of area.

From (8b) and (3), setting $r = t$, it follows for n odd that

$$u(P, t) = \frac{\sqrt{\pi}}{\Gamma(n/2)}\, t\left(\frac{\partial}{\partial t^2}\right)^{(n-1)/2}\left[t^{n-2}(u(P, 0) + \frac{1}{(n-2)\omega_n}\int_{B(t)} (r^{2-n} - t^{2-n})FdV_Q)\right]$$

$$= u(P, 0) + \frac{1}{(n-2)\omega_n}\int_{B(t)} Fr^{2-n}dV_Q. \tag{9}$$

For $n = 3$ $\omega_2 = 4\pi$ and so

$$u(P, t) = u(P, 0) + \frac{1}{4\pi}\int_{B(t)} r^{-1}FdV_Q, \tag{10}$$

a form of solution already used by Randall (2).

After substituting (8b) in (4) and changing the order of integration we obtain for n even and $\neq 2$

$$u(P,t) = u(P,0) + \frac{t}{\pi^{n/2}}\left(\frac{\partial}{\partial t^2}\right)^{n/2} \int_{B(t)} r^{2-n} F dV_Q \int_r^t v\sqrt{t^2-v^2}\,.\,v^{n-4}dv,$$

$$= u(P,0) + \frac{t}{4\pi^{n/2}}\left(\frac{\partial}{\partial t^2}\right)^{(n-4)/2} \int_{B(t)} F/\sqrt{t^2-r^2}$$

$$\sum_{k=1}^{n/2} a_{n,k} r^{n-2k-2}(t^2-r^2)^{k-1} dV_Q, \quad (11)$$

where the $a_{n,k}$ are constants determined from the recurrence relationship:

$$\frac{a_{n,k+1}}{a_{n,k}} = \frac{n-2k-2}{2k-1}, \quad a_{n,1} = 1. \quad (11a)$$

In (11), for the case $n = 4$ the differentiation has been performed explicitly; but for higher values of n, such differentiation is not possible because of the singularity at $r = t$.

The same process, employing (8a) and (4), yields for $n = 2$.

$$u(P,t) = u(P,0) - \frac{1}{2\pi}\int_{S(t)} \operatorname{sech}^{-1}\frac{t}{r}\,.\,F dS_Q. \quad (12)$$

Equations (9), (11), (12) show explicitly how the time-dependent potential is the sum of a static potential and a time-dependent wave; the formulae also demonstrate clearly that if initially $\nabla^2 u = 0$ everywhere, then the potential remains at its initial value for all time. Also it may be noted that in (3) and (4) differentiation with respect to t cannot be performed completely, while in (9), (11), (12) this differentiation has already been carried out for odd values of n and for $n = 2, 4$.

If use is made of the generalised Helmholtz theorem:

$$u(P) = \frac{-1}{(n-2)\omega_n}\int_\infty G(P,Q)F(Q)dV_Q, \quad (13)$$

where $G(P, Q)$ is the Green's function for the infinite domain and the integration is over all space, then $u(P, t)$ may be expressed entirely in terms of the Laplacian $\nabla^2 u(Q, 0)$, that is, the divergence of the static field grad u. In physical applications where $\nabla^2 u$ but not grad u has been obtained this may prove a more appropriate formula.

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REFERENCES

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