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On s-semipermutable or s-quasinormally Embedded Subgroups of Finite Groups

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Abstract. Suppose that G is a finite group and H is a subgroup of G. H is said to be s-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$; H is said to be s-quasinormally embedded in G if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G. In every non-cyclic Sylow subgroup P of G we fix some subgroup D satisfying $1 < |D| < |P|$ and study the structure of G under the assumption that every subgroup H of P with $|H| = |D|$ is either s-semipermutable or s-quasinormally embedded in G. Some recent results are generalized and unified.

1 Introduction

All groups considered in this paper are finite. We use conventional notions and notation. G always means a group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$ and G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a *formation*, provided that

(i) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and
(ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for any normal subgroups M, N of G.

A formation $\mathcal{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation.

A subgroup H of G is called *s-quasinormal* (or *s-permutable*, *π-quasinormal*) in G provided H permutes with all Sylow subgroups of G, *i.e.*, $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by Kegel in [6] and has been studied extensively by Deskins [2] and Schmidt [11]. More recently, Zhang and Wang [15] generalized s-quasinormal subgroups to s-semipermutable subgroups. A subgroup H is said to be *s-semipermutable* in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$. Clearly, every s-quasinormal subgroup of G is an s-semipermutable subgroup of G, but the converse does not hold. Many authors consider minimal or maximal subgroups of a Sylow subgroup of a group when investigating the structure of G, such as in [1, 2] and [5–15], *etc.* For example, in [5] Han proves the following result.

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Theorem 1.1 (Han) *Let p be a prime dividing the order of a group G satisfying $(|G|, p-1) = 1$ and P a Sylow p-subgroup of G. Suppose there exists a nontrivial subgroup D of P such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-semipermutable in G. Then G is p-nilpotent.*

As another generalization of the s-quasinormality, Ballester-Bolinches *et al.* [1] introduce the following concept: a subgroup H of G is said to be *s-quasinormally embedded* in G if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G. In [14], Wei and Guo provide a result as follows.

Theorem 1.2 (Wei and Guo) *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if there is a subgroup D of P such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-quasinormally embedded in G.*

The aim of this article is to unify and improve the above theorems using s-semipermutable and s-quasinormally embedded subgroups. Our main theorem is as follows.

Theorem 3.5 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or s-quasinormally embedded in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathcal{F}$.*

2 Basic Definitions and Preliminary Results

In this section, we collect some known results that are useful later.

Lemma 2.1 *Suppose that H is an s-semipermutable subgroup of G. Then the following assertions hold.*

(i) *If $H \leq K \leq G$, then H is s-semipermutable in K.*
(ii) *Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N.*
(iii) *If $H \leq O_p(G)$, then H is s-permutable in G.*

Proof (i) is [15, Property 1], (ii) is [15, Property 2], and (iii) is [15, Lemma 3]. ■

Lemma 2.2 ([1]) *Suppose that U is s-quasinormally embedded in a group G, and let $H \leq G$ and $K \trianglelefteq G$. Then the following assertions hold.*

(i) *If $U \leq H$, then U is s-quasinormally embedded in H.*

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(ii) *UK is s-quasinormally embedded in G and UK/K is s-quasinormally embedded in G/K.*
(iii) *If $K \leq H$ and H/K is s-quasinormally embedded in G/K, then H is s-quasinormally embedded in G.*

Lemma 2.3 ([13]) *Let G be a group, K an s-quasinormal subgroup of G and P a Sylow p-subgroup of K, where p is a prime. If either $P \leq O_p(G)$ or $K_G = 1$, then P is s-quasinormal in G.*

Lemma 2.4 ([11]) *If P is an s-quasinormal p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.5 ([13]) *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.*
(i) *If N is normal in G of order p, then $N \leq Z(G)$.*
(ii) *If G has cyclic Sylow p-subgroup, then G is p-nilpotent.*
(iii) *If $M \leq G$ and $[G : M] = p$, then $M \trianglelefteq G$.*

Lemma 2.6 ([12]) *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every maximal subgroup of P is s-semipermutable in G, then G is p-nilpotent.*

Lemma 2.7 ([3, III, 5.2, and IV, 5.4]) *Suppose that p is a prime and G is a minimal non-p-nilpotent group,* i.e., *G is not a p-nilpotent group but whose proper subgroups are all p-nilpotent.*
(i) *G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*
(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*
(iii) *The exponent of P is p or 4.*

Lemma 2.8 ([7]) *Let H be a nilpotent subgroup of a group G. Then the following statements are equivalent:*
(i) *H is s-quasinormal in G;*
(ii) *$H \leq F(G)$ and H is s-quasinormally embedded in G.*

Lemma 2.9 *Let N be an elementary abelian normal p-subgroup of a group G. If there exists a subgroup D in N such that $1 < |D| < |N|$ and every subgroup H of N with $|H| = |D|$ is s-semipermutable in G, then there exists a maximal subgroup M of N such that M is normal in G.*

Lemma 2.10 ([3, VI, 4.10]) *Assume that A and B are two subgroups of a group G and $G \neq AB$. If $AB^g = B^g A$ holds for any $g \in G$, then either A or B is contained in a nontrivial normal subgroup of G.*

The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Its definition and important properties can be found in [4, X, 13]. We would like to give the following basic facts we will use in our proof.

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Lemma 2.11 ([4, X, 13]) *Let G be a group and M a subgroup of G.*

(i) *If M is normal in G, then $F^*(M) \leq F^*(G)$.*

(ii) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \operatorname{Soc}\big(F(G)C_G(F(G))/F(G)\big)$;*

(iii) *$F^*\big(F^*(G)\big) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

Lemma 2.12 ([10]) *Let $\mathcal{F}$ be a saturated formation containing $\mathfrak{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathcal{F}$.*

3 Main Results

In this section, we will prove our main results.

Theorem 3.1 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is either s-semipermutable or s-quasinormally embedded in G, then G is p-nilpotent.*

Proof Assume that the theorem is not true and let G be a counterexample of minimal order. We derive a contradiction in several steps.

By Lemmas 2.1 and 2.2, the following two steps are obvious.

Step 1. $O_{p'}(G) = 1$.

Step 2. G has a unique minimal normal subgroup N and G/N is p-nilpotent. Moreover, $\Phi(G) = 1$.

Step 3. $O_p(G) = 1$: If $O_p(G) \neq 1$, then step 2 yields $N \leq O_p(G)$ and $\Phi\big(O_p(G)\big) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, we conclude that $O_p(G) \cap M$ is normal in G. The uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Furthermore, $P \cap M < P$, and, thus there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Hence, $P = NP_1$. By hypothesis, P_1 is s-semipermutable or s-quasinormally embedded in G. Suppose first P_1 is s-semipermutable in G. Then $P_1 M_q$ is a group for $q \neq p$. Hence

$$P_1 \langle M_p, M_q | q \in \pi(M), q \neq p \rangle = P_1 M$$

is a group. Then $P_1 M = M$ or G by maximality of M. If $P_1 M = G$, then

$$P = P \cap P_1 M = P_1(P \cap M) = P_1,$$

a contradiction. If $P_1 M = M$, then $P_1 \leq M$. Therefore, $P_1 \cap N = 1$ and N is of prime order. Then the p-nilpotency of G/N implies the p-nilpotency of G, a contradiction. Therefore, we may assume that P_1 is s-quasinormally embedded in G. Then there is an s-quasinormal subgroup K of G such that $P_1 \in \operatorname{Syl}_p(K)$. If $K_G \neq 1$, then $N \leq K$. Since N is a normal p-subgroup of K and $P_1 \in \operatorname{Syl}_p(K)$, we have that $N \leq P_1$, a contradiction. Hence $K_G = 1$, and so by Lemma 2.3 P_1 is s-quasinormal in G. By

Lemma 2.4, $O^p(G) \le N_G(P_1)$, $P_1 \trianglelefteq G$. Then $N \cap P_1 = 1$ and $|N| = p$. By Lemma 2.5, $N \le Z(G)$ and hence G is p-nilpotent, a contradiction.

By Step 1 and Step 3, we have the following.

Step 4. There is no p-nilpotent minimal normal subgroup of G.

Step 5. The final contradiction: If $N \cap P \le \Phi(P)$, then N is p-nilpotent by Tate's theorem [3, Satz 4.7, p. 431], contrary to Step 4. Consequently, there is a maximal subgroup P_1 of P such that $P = (N \cap P)P_1$. By the hypothesis, if P_1 is s-quasinormally embedded in G, there is an s-quasinormal subgroup K of G such that $P_1 \in \mathrm{Syl}_p(K)$. If $K_G \ne 1$, then $N \le K$ and $P_1 \cap N \in \mathrm{Syl}_p(N)$. Clearly, $P \cap N \in \mathrm{Syl}_p(N)$. Thus $P \cap N \le P_1 \cap N \le P_1$, contrary to the choice of P_1. Therefore, $K_G = 1$, P_1 is s-quasinormal in G by Lemma 2.3, then $P_1 \trianglelefteq G$. This leads to $P_1 = 1$ and $|P| = p$, G is p-nilpotent by Lemma 2.5(ii), a contradiction. Now we can assume that all maximal subgroups of P are s-semipermutable in G. Then G is p-nilpotent by Lemma 2.6, a contradiction. ■

Theorem 3.2 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or s-quasinormally embedded in G, then G is p-nilpotent.*

Proof Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

Step 1. $O_{p'}(G) = 1$: If $O_{p'}(G) \ne 1$, Lemmas 2.1(ii) and 2.1(iii) guarantee that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

Step 2. $|D| > p$: Suppose that $|D| = p$. Since G is not p-nilpotent, G has a minimal non-p-nilpotent subgroup G_1. By Lemma 2.7(i), $G_1 = [P_1]Q$, where $P_1 \in \mathrm{Syl}_p(G_1)$ and $Q \in \mathrm{Syl}_q(G_1)$, $p \ne q$. Let $X/\Phi(P_1)$ be a subgroup of $P_1/\Phi(P_1)$ of order p, $x \in X \backslash \Phi(P_1)$ and $L = \langle x \rangle$. Then L is of order p or 4 by Lemma 2.7(iii). By the hypotheses, L is either s-semipermutable or s-quasinormally embedded in G, thus in G_1 by Lemmas 2.1(i) and 2.2(i). First, suppose that $L = \langle x \rangle$ is s-quasinormally embedded in G_1 for every element $x \in P_1$, then by Lemma 2.8 $\langle x \rangle$ is s-quasinormal in G_1. Thus $LQ \le G_1$. Therefore, $LQ = L \times Q$. Then $G_1 = P_1 \times Q$, a contradiction. Therefore, $L = \langle x \rangle$ is s-semipermutable in G_1 for every element $x \in P_1$. Thus $LQ \le G_1$. Therefore, $LQ = L \times Q$. Then $G_1 = P_1 \times Q$, a contradiction.

Step 3. $|P : D| > p$: This follows from Theorem 3.1.

Step 4. P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-semipermutable in G: Assume that $H \le P$ such that $|H| = |D|$ and H is s-quasinormally embedded in G. Then there exists a normal subgroup M such that $|G : M| = p$ and $G = MH$. Since $|P : D| > p$ by Step 3, M satisfies the hypotheses of the theorem. The choice of G yields that M is p-nilpotent. It is easy to see that G is p-nilpotent, contrary to the choice of G.

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Step 5. If $N \leq P$ and N is minimal normal in G, then $|N| \leq |D|$: Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, N is elementary abelian. By Lemma 2.9, N has a maximal subgroup which is normal in G, contrary to the minimality of N.

Step 6. Suppose that $N \leq P$ and N is minimal normal in G. Then G/N is p-nilpotent: If $|N| < |D|$, G/N satisfies the hypotheses of the theorem by Lemma 2.1(ii). Thus G/N is p-nilpotent by the minimal choice of G. So we may suppose that $|N| = |D|$ by Step 5. We will show that every cyclic subgroup of P/N of order p or order 4 (when P/N is a non-abelian 2-group) is s-semipermutable in G/N. Let $K \leq P$ and $|K/N| = p$. By Step 2, N is non-cyclic, so are all subgroups containing N. Hence there is a maximal subgroup $L \neq N$ of K such that $K = NL$. Of course, $|N| = |D| = |L|$. Since L is s-semipermutable in G by the hypotheses, $K/N = LN/N$ is s-semipermutable in G/N by Lemma 2.1(ii). If $p = 2$ and P/N is non-abelian, take a cyclic subgroup X/N of P/N of order 4. Let K/N be maximal in X/N. Then K is maximal in X and $|K/N|=2$. Since X is non-cyclic and X/N is cyclic, there is a maximal subgroup L of X such that N is not contained in L. Thus $X = LN$ and $|L| = |K| = 2|D|$. By the hypotheses, L is s-semipermutable in G. By Lemma 2.1(ii), $X/N = LN/N$ is s-semipermutable in G/N. Hence G/N satisfies the hypotheses. By the minimal choice of G, G/N is p-nilpotent.

Step 7. $O_p(G) = 1$: Suppose that $O_p(G) \neq 1$. Take a minimal normal subgroup N of G contained in $O_p(G)$. By Step 6, G/N is p-nilpotent. It is easy to see that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$. Hence $O_p(G)$ is an elementary abelian p-group. On the other hand, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that $O_p(G) \cap M = 1$, $N = O_p(G)$ and $M \cong G/N$ is p-nilpotent. Then G can be written as $G = N(M \cap P)M_{p'}$, where $M_{p'}$ is the normal p-complement of M. Pick a maximal subgroup S of $M_p = P \cap M$. Then $NSM_{p'}$ is a subgroup of G with index p. Since p is the minimal prime in $\pi(G)$, we know that $NSM_{p'}$ is normal in G. Now by Step 3 and the induction, we have $NSM_{p'}$ is p-nilpotent. Therefore, G is p-nilpotent, a contradiction.

Step 8. The minimal normal subgroup L of G is not p-nilpotent: If L is p-nilpotent, then it follows from the fact that $L_{p'} \operatorname{char} L \lhd G$ that $L_{p'} \leq O_{p'}(G) = 1$. Thus L is a p-group. Whence $L \leq O_p(G) = 1$ by Step 7, a contradiction.

Step 9. G is a non-abelian simple group: Suppose that G is not a simple group. Take a minimal normal subgroup L of G. Then $L < G$. If $|L|_p > |D|$, then L is p-nilpotent by the minimal choice of G, contrary to Step 8. Hence $|L|_p \leq |D|$. Take $P_* \geq L \cap P$ such that $|P_*| = p|D|$. Hence P_* is a Sylow p-subgroup of P_*L. Since every maximal subgroup of P_* is of order $|D|$, every maximal subgroup of P_* is s-semipermutable in G by hypotheses, thus in P_*L by Lemma 2.1(i). Now applying Theorem 3.1, we get P_*L is p-nilpotent. Therefore, L is p-nilpotent, contrary to Step 8.

Step 10. The final contradiction: Suppose that H is a subgroup of P with $|H| = |D|$ and Q is a Sylow q-subgroup with $q \neq p$. Then $HQ^g = Q^g H$ for any $g \in G$ by the hypotheses that H is s-semipermutable in G. Since G is simple by Step 9, $G = HQ$ from Lemma 2.10, the final contradiction. ∎

The following corollary is immediate from Theorem 3.2.

Corollary 3.3 *Suppose that G is a group. If every non-cyclic Sylow subgroup of G has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or s-quasinormally embedded in G, then G has a Sylow tower of supersolvable type.*

Theorem 3.4 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups, and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or s-quasinormally embedded in G. Then $G \in \mathcal{F}$.*

Proof Suppose that P is a non-cyclic Sylow p-subgroup of E, $\forall p \in \pi(E)$. Since P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or s-quasinormally embedded in G by hypotheses, thus in E by Lemmas 2.1(i) and 2.2(i). Applying Corollary 3.3, we conclude that E has a Sylow tower of supersolvable type. Let q be the maximal prime divisor of $|E|$ and $Q \in \mathrm{Syl}_q(E)$. Then $Q \trianglelefteq G$. Since $(G/Q, E/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathcal{F}$. For any subgroup H of Q with $|H| = |D|$, since $Q \leq O_q(G)$, H is s-permutable in G by Lemmas 2.1(iii) and 2.8. Since s-permutable implies weakly s-permutable and $F^*(Q) = Q$ by Lemma 2.11, we get $G \in \mathcal{F}$ by applying Lemma 2.12. ∎

Theorem 3.5 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups, and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or s-quasinormally embedded in G. Then $G \in \mathcal{F}$.*

Proof We distinguish two cases:

Case 1. $\mathcal{F} = \mathcal{U}$.

Let G be a minimal counter-example.

Step 1. Every proper normal subgroup N of G containing $F^*(E)$ (if it exists) is supersolvable: If N is a proper normal subgroup of G containing $F^*(E)$, then $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 2.11(iii), $F^*(E) = F^*\big(F^*(E)\big) \leq F^*(E \cap N) \leq F^*(E)$, so $F^*(E \cap N) = F^*(E)$. For any Sylow subgroup P of $F^*(E \cap N) = F^*(E)$, P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or s-quasinormally embedded in G by hypotheses, thus in N

by Lemmas 2.1(i) and 2.2 (i). So N and $N \cap H$ satisfy the hypotheses of the theorem, the minimal choice of G implies that N is supersolvable.

Step 2. $E = G$: If $E < G$, then $E \in \mathfrak{U}$ by Step 1. Hence $F^*(E) = F(E)$ by Lemma 2.11. It follows that every Sylow subgroup of $F^*(E)$ is normal in G. By Lemmas 2.1(iii) and 2.8, every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-permutable in G. Applying Lemma 2.12 for the special case $\mathfrak{F} = \mathfrak{U}$, $G \in \mathfrak{U}$, a contradiction.

Step 3. $F^*(G) = F(G) < G$: If $F^*(G) = G$, then $G \in \mathfrak{F}$ by Theorem 3.4, contrary to the choice of G. So $F^*(G) < G$. By Step 1, $F^*(G) \in \mathfrak{U}$ and $F^*(G) = F(G)$ by Lemma 2.11.

Step 4. The final contradiction: Since $F^*(G) = F(G)$, each non-cyclic Sylow subgroup P of $F^*(G)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-permutable in G by Lemmas 2.1(iii) and 2.8. Applying Lemma 2.12, $G \in \mathfrak{U}$, a contradiction.

Case 2. $\mathfrak{F} \neq \mathfrak{U}$.

By hypotheses, every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or s-quasinormally embedded in G, thus in E by Lemmas 2.1(i) and 2.2(i). Applying Case 1, $E \in \mathfrak{U}$. Then $F^*(E) = F(E)$ by Lemma 2.11. It follows that each Sylow subgroup of $F^*(E)$ is normal in G. By Lemmas 2.1 (iii) and 2.8, each non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-permutable in G. Applying Lemma 2.12, $G \in \mathfrak{F}$. These complete the proof of the theorem. ■

The following corollaries are immediate from Theorem 3.5.

Corollary 3.6 *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Then $G \in \mathfrak{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is s-semipermutable in G.*

Corollary 3.7 *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Then $G \in \mathfrak{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is s-semipermutable in G.*

Corollary 3.8 *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Then $G \in \mathfrak{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is s-quasinormally embedded in G.*

Corollary 3.9 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is s-quasinormally embedded in G.*

Corollary 3.10 ([8, Theorem 3.4]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is s-quasinormal in G.*

Corollary 3.11 ([9, Theorem 3.3]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is s-quasinormal in G.*

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