This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I.G. Connell, Department of Mathematics, McGill University, Montreal, P.Q.

RELLICH'S EMBEDDING THEOREM FOR A "SPINY URCHIN"

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From the plane R^2 we remove the union of the sets S_k (k = 1, 2, ...) defined as follows (using the notation $z = x + iy$):

$$S_k = \{z : \arg z = n\pi 2^{-k} \text{ for some integer } n; |z| \geq k\} .$$

The remaining connected open set Ω we call the spiny urchin.

The eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u \qquad \text{on } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

has a discrete spectrum! This is a consequence of the theorem to follow, which is Rellich's theorem [3] for Ω. Note that the spiny urchin is too irregular (e.g. $\bar{\Omega} = R^2$) to satisfy the hypotheses of our previous extension of Rellich's theorem [1]. We do not know if Rellich's theorem can fail for some quasi-bounded region [Ω is quasibounded if $\text{dist}(x, \partial\Omega) \to 0$ as

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$x(\epsilon\Omega) \to \infty]$; we guess that it may fail for example in the case that $\partial\Omega$ consists only of isolated points.

THEOREM. Let Ω be the spiny urchin. Then the embedding map $H_0^1(\Omega) \to L_2(\Omega)$ is compact.

Proof. We refer to [1] for definitions, and also for part of the proof.

Introduce the sets

$\Omega_{jk} = \{z : j\pi 2^{-k} \le \arg z \le (j+1)\pi 2^{-k};\ k \le |z| < k+1\}$ for $j, k = 0, 1, 2, \ldots$. Then

$$R^n = \bigcup_{k=0}^{\infty} \bigcup_{j=0}^{2^k} \Omega_{jk} .$$

Let $\varphi \in C_0^\infty(\Omega)$ and let z be given in Ω, say $z \in \Omega_{jk}$. For simplicity suppose $j = 0$. Then, with $z = re^{i\theta}$,

$$|\varphi(z)|^2 = \left|\int_0^{\arg z} \frac{\partial\varphi}{\partial\theta}(r,\theta)\, d\theta\right|^2$$

$$\le \arg z \int_0^{\arg z} \left|\frac{\partial\varphi}{\partial\theta}(r,\theta)\right|^2 d\theta$$

$$\le 2^{1-k}\pi \int_0^{2^{-k}\pi} r^2 |\nabla\varphi|^2\, d\theta$$

$$\le 2^{1-k}\pi(k+1)^2 \int_0^{2^{-k}\pi} |\nabla\varphi|^2\, d\theta .$$

By integration over Ω_{0k} we have

$$\int_{\Omega_{0k}} |\varphi(z)|^2 dz \le 2^{1-k}\pi(k+1)^2 \int_0^{2^{-k}\pi} \int_k^{k+1} \int_0^{2^{-k}\pi} |\nabla\varphi(r,\theta)|^2 d\theta\, r\, dr\, d\theta_1$$

$$= 2^{1-2k}\pi^2(k+1)^2 \int_{\Omega_{0k}} |\nabla\varphi(z)|^2\, dz .$$

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The same inequality holds for any Ω_{jk}.

Now if Ω_R denotes $\Omega \cap [|z| \geq R]$, then for positive integers R, Ω_R is contained in the union of Ω_{jk}'s with $k \geq R$. By summing the above inequality over all such Ω_{jk}, we obtain

$$\int_{\Omega_R} |\varphi(z)|^2 dz \leq 2\pi^2 \Sigma_{k=R}^{\infty} 2^{-2k}(k+1)^2 \Sigma_{j=0}^{2^k} \int_{\Omega_{jk}} |\nabla\varphi(z)|^2 dz$$

$$\leq \{2\pi^2 \Sigma_R^{\infty} 2^{-2k}(k+1)^2\} \int_{\Omega_R} |\nabla \varphi(z)|^2 dz.$$

Since the expression in brackets approaches zero as $R \to \infty$, the proof may be completed as in Theorem 3 of [1], by using a well-known criterion for compactness of a linear operator.

Note that the proof applies to a wide class of open sets found by removing radial line segments from the plane. Similarly, one could remove parallel line segments. Also, it is possible to construct such sets in higher dimensions. In all cases it is essential to be able to partition the space R^n so that an inequality of the form

$$\int_{\Omega_R} |\varphi|^2 dx \leq C(R) \int_{\Omega_R} |\nabla \varphi|^2 dx, \quad \varphi \in C_o^{\infty}(\Omega)$$

$$C(R) \to 0 \quad \text{as} \quad R \to \infty$$

can be obtained by a summation.

The theorem also holds for the spaces $W_o^{m,p}(\Omega)$; see [2] for related results.

Added in proof: In the paper "Compact Sobolev embeddings for unbounded domains," submitted to the Pacific Journal of Mathematics, R. Adams has generalised the theorem above; he has also verified the conjecture that Rellich's theorem can fail if $\partial\Omega$ consists of isolated points.

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