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Homotopy Classification of Projections in the Corona Algebra of a Non-simple C^*-algebra

Lawrence G. Brown and Hyun Ho Lee

Abstract. We study projections in the corona algebra of $C(X) \otimes K$, where K is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space and $X = [0,1], [0,\infty), (-\infty,\infty)$, or $[0,1]/\{0,1\}$. Using BDF's essential codimension, we determine conditions for a projection in the corona algebra to be liftable to a projection in the multiplier algebra. We also determine the conditions for two projections to be equal in K_0, Murray–von Neumann equivalent, unitarily equivalent, or homotopic. In light of these characterizations, we construct examples showing that the equivalence notions above are all distinct.

1 Introduction

Our goal is to study projections in the corona algebra of a non-simple stable rank one C^*-algebra. The work presented here originated from a lifting problem: Let A be a C^*-algebra and D a closed ideal of A. We are interested in whether every unitary in A/D is liftable to a partial isometry in A. It happens whenever D has an approximate identity of projections, a weaker condition than real rank zero. We are concerned with the case where D has stable rank one. It might not be possible in general, but constructing an explicit counter-example is not trivial. One solution is to find a stable, projectionless, stable rank one algebra D such that $K_0(D)$ is non-trivial, and then consider an extension of D by $C(\mathbb{T})$ that comes from a unitary $\mathbf{u}$ in the corona algebra of D with non-trivial K_1-class. *i.e.,*

$$0 \longrightarrow D \longrightarrow A \longrightarrow C(\mathbb{T}) \longrightarrow 0$$

We can observe that $\mathbf{u}$ cannot be lifted to a unitary (if so, $[\mathbf{u}] = 0$, which is a contradiction) nor can it be lifted to a partial isometry, because there are no any non-zero projections available to be the defect projections of a partial isometry.

If we denote the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space by K, the cone and suspension of K are (stably) projectionless, but their K_0-groups are trivial. Let I be the cone or suspension of K and suppose that we have a projection $\mathbf{p}$ in the corona algebra of I, denoted by $\mathcal{C}(I)$, which does

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not lift (stably), but its K_0-class does lift. If a is a self adjoint element that lifts $\mathbf{p}$ in the multiplier algebra $M(I)$, we take D_0 to be the C^*-algebra generated by a and I so that the quotient D_0/I is isomorphic to $\mathbb{C}$. Then the Busby invariant is determined by sending $\mathbf{1}$ to $\mathbf{p}$, and we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{j} & D_0 & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \mathbf{p} & & \\ 0 & \longrightarrow & I & \longrightarrow & M(I) & \longrightarrow & \mathcal{C}(I) & \longrightarrow & 0. \end{array}$$

By the long exact sequence, we have

$$\begin{array}{ccccccc} K_0(I) & \longrightarrow & K_0(D_0) & \longrightarrow & K_0(\mathbb{C}) & \xrightarrow{\partial_0} & K_1(I) \\ & & \downarrow & & \downarrow \mathbf{p} & & \| \\ K_0(I) & \longrightarrow & K_0(M(I)) & \longrightarrow & K_0(\mathcal{C}(I)) & \xrightarrow{\partial_0} & K_1(I). \end{array}$$

Since $\partial_0([\mathbf{p}]_0) = 0$, $\partial_0 \colon K_0(\mathbb{C}) \to K_1(I)$ becomes trivial. Thus $K_0(D_0) \cong K_0(\mathbb{C})$. In particular, $K_0(D_0)$ is non-trivial. The facts that p and also any finite sum $p \oplus \cdots \oplus p$ do not lift imply that D_0 is stably projectionless. This leads us to the question: when does such a projection $\mathbf{p}$ in the corona algebra of I exist?

As a corollary we can obtain a stably projectionless, stable rank one C^*-algebra D from D_0 such that its K_0-group is non-trivial, which is also of independent interest. Note that D is not simple from our construction. There is a rather substantial literature on simple, stably projectionless C^*-algebras with K-groups equal to $K_i(\mathbb{C})$ or trivial K-groups [9, 10, 14, 16]. In Elliott's classification program these algebras have played important roles since the Jiang-Su algebra appeared. But interest is growing in non-simple C^*-algebras that absorb a stably projectionless, (strongly) self-absorbing C^*-algebra. We hope that our construction is worthwhile to those interested in classification.

Let I be of the form $C(X) \otimes K$ where $X = [0,1], [0,\infty), (-\infty,\infty)$ or $[0,1]/\{0,1\}$. In what follows we represent an element $\mathbf{f}$ in the corona algebra of I by finite partitions and operator valued functions on the subintervals that agree modulo compacts at partition points. This approach is proven to be useful when $\mathbf{f}$ is a projection since the functions on the subintervals can be taken to be projection-valued. Thus a projection in the corona algebra of I is locally liftable in the above sense and their transitions are described by pairs of projections in $B(H)$ whose difference is compact. Further, we can quantify the transitions using the essential codimension. This quantification allows us not only to solve lifting problems but also to give conditions for homotopy equivalence, unitary equivalence, Murray–von Neumann equivalence, and K_0-equivalence of two projections in the corona algebra.

This paper is arranged as follows. In Section 2 we review the notion of essential codimension of two projections in $B(H)$ and derive some facts that will be needed later. (In fact, the definition of essential codimension was given in [3], and some properties were provided without proofs in [1]. Here we review the definition and provide complete proofs of its properties.) In Section 3 we give a necessary and sufficient condition for the liftability of a projection in the corona algebra of I. In Section 4, we give criteria for homotopy equivalence $\sim_h$, unitary equivalence $\sim_u$, and Murray–von Neumann equivalence $\sim$ of two projections $\mathbf{p}, \mathbf{q}$ in $\mathcal{C}(I) \otimes M_n$ for some n. In addition, we determine the condition for the liftability of the K_0-class of a projection; *i.e.,* we clarify when it becomes trivial in K_0. Then we construct a projection that does not lift but whose K_0-class does lift if, applicable. Also, we construct examples such that $[p]_0 = [q]_0$ in K_0 but $p \nsim q$, $p \sim q$ but $p \nsim_u q$, and $p \sim_u q$ but $p \nsim_h q$. Section A is an appendix that reviews some rudiments of continuous fields of Hilbert spaces and proves some results that are crucial ingredients for the above.

2 Essential Codimension

Throughout the article H denotes a separable infinite dimensional Hilbert space, K the C^*-algebra of compact operators on H, and $B(H)$ the C^*-algebra of bounded operators on H.

Definition 2.1 ([3]) When p, q are projections in $B(H)$ such that $p - q \in K$, we define the *essential codimension* of p and q, which will be denoted as $[p:q]$. If p and q have infinite rank, let V, W be isometries such that $VV^* = q, WW^* = p$. Then

$$[p:q] = \begin{cases} \operatorname{Ind}(V^*W) & \text{if } p, q \text{ have infinite rank,} \\ \operatorname{rank}(p) - \operatorname{rank}(q) & \text{if } p, q \text{ have finite rank.} \end{cases}$$

Note that $[p:q]$ is independent of the choice of V and W. In fact, if we have isometries V_1, V_2 such that $V_1V_1^* = q$ and $V_2V_2^* = q$, then $U = V_2^*V_1$ is a unitary and $UV_1^*W = V_2^*W$. It follows that $\operatorname{Ind}(V_1^*W) = \operatorname{Ind}(V_2^*W)$. The other case is proved similarly.

Proposition 2.2 $[\,:\,]$ *has the following properties:*

(i) *if $p_2 \leq p_1$, then $[p_1:p_2]$ is the usual codimension of p_2 in p_1, which is* $\operatorname{rank}(p_1 - p_2)$*;*
(ii) $[p_1:p_2] = -[p_2:p_1]$*;*
(iii) $[p_1:p_3] = [p_1:p_2] + [p_2:p_3]$*;*
(iv) $[p_1 + p_1':p_2 + p_2'] = [p_1:p_2] + [p_1':p_2']$ *when sensible.*

Proof For (i), let V_i be the isometries corresponding to p_i for $i = 1, 2$. Then $V_2^*V_1$ is a co-isometry, because $p_2p_1 = p_2$. Hence,

$$\begin{aligned}\operatorname{Ind}(V_2^*V_1) = \dim\ker(V_2^*V_1) &= \operatorname{rank}(1 - (V_2^*V_1)^*V_2^*V_1) = \operatorname{Tr}(V_1^*(p_1 - p_2)V_1) \\ &= \operatorname{Tr}(p_1 - p_2) = \operatorname{rank}(p_1 - p_2)\end{aligned}$$

if p and q are infinite rank, where $\operatorname{Tr}\colon L^1(H) \to \mathbb{C}$ is the usual trace map.

(ii) is evident from the definition.

For (iii), if the p_i's have finite rank, it is easy. If the p_i's have infinite rank, we choose isometries such that $V_iV_i^* = p_i$. Then $V_3^*V_2V_2^*V_1 - V_3^*V_1 \in K$, and therefore $\operatorname{Ind}(V_3^*V_1) = \operatorname{Ind}(V_3^*V_2V_2^*V_1) = \operatorname{Ind}(V_3^*V_2) + \operatorname{Ind}(V_2^*V_1)$.

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Finally, note that $p_i + p_i'$ is a projection if and only if p_i and p_i' have orthogonal ranges or $p_i p_i' = 0$. If both p_i and p_i' have finite rank, then

$$\operatorname{rank}(p_i + p_i') = \operatorname{rank}(p_i) + \operatorname{rank}(p_i') \quad \text{for} \quad i = 1, 2,$$

and (iv) is obvious. If p_i has infinite rank and p_i' has finite rank, then $p_i + p_i' - p_i \in K$ for $i = 1, 2$:

$$\begin{aligned}[p_1 + p_1' : p_2 + p_2'] &= [p_1 + p_1' : p_1] + [p_1 : p_2] + [p_2 : p_2 + p_2'] \\ &= \operatorname{rank}(p_1') + [p_1 : p_2] - \operatorname{rank}(p_2') \\ &= [p_1 : p_2] + [p_1' : p_2'].\end{aligned}$$

If both p_i and p_i' have infinite rank, note that $p_1' p_2, p_2' p_1 \in K$. If $U : H \to H \oplus H$ is a unitary, then $V = \begin{bmatrix} V_1 & V_1' \end{bmatrix} U$ and $W = \begin{bmatrix} V_2 & V_2' \end{bmatrix} U$ are suitable isometries for $p_1 + p_1'$ and $p_2 + p_2'$, where V_i and V_i' are isometries such that $V_i V_i^* = p_i$ and $V_i' V_i'^* = p_i'$ for $i = 1, 2$. So

$$\begin{aligned}[p_1 + p_1' : p_2 + p_2'] &= \operatorname{Ind}(V^* W) = \operatorname{Ind}(U^*(V_1^* V_2 \oplus (V_1')^* V_2')U) \\ &= \operatorname{Ind}(V_1^* V_2 \oplus (V_1')^* V_2') = \operatorname{Ind}(V_1^* V_2) + \operatorname{Ind}((V_1')^* V_2') \\ &= [p_1 : p_2] + [p_1' : p_2'].\end{aligned}$$

■

Lemma 2.3 *Let p and q be projections in $B(H)$ such that $p - q \in K$. If there is a unitary $U \in 1 + K$ such that $UpU^* = q$, then $[p : q] = 0$. In particular, if $\|p - q\| < 1$, then $[p : q] = 0$.*

Proof If the ranges of p and q are finite dimensional, $\operatorname{rank}(p) = \operatorname{rank}(UpU^*) = \operatorname{rank}(q)$. Now assume that p and q are infinite dimensional and let W be an isometry such that $WW^* = p$. Then $V = UW$ is an isometry such that $VV^* = q$.

$$[p : q] = \operatorname{Ind}(V^* W) = \operatorname{Ind}(W^* U^* W) = \operatorname{Ind}(W^* W + \text{compact}) = \operatorname{Ind}(I) = 0.$$

Now if $\|p - q\| < 1$, we can take $a = (1 - q)(1 - p) + qp \in 1 + K$. Since $aa^* = a^* a = 1 - (p - q)^2 \in 1 + K$,

$$\|a^* a - 1\| = \|p - q\|^2 < 1, \quad \|aa^* - 1\| = \|p - q\|^2 < 1.$$

Moreover, $ap = qp = qa$. Hence, a is an invertible element and $U = a(a^* a)^{-\frac{1}{2}} \in 1 + K$ is a unitary such that $UpU^* = q$. ■

If a pair of projections p, q such that $p - q \in K$ is given, we can diagonalize $p - q$ by the spectral theorem. Let $E_\lambda(T)$ be the eigenspace of an operator T corresponding to the eigenvalue λ. Then we can characterize $[p : q]$ in terms of $E_1(p - q)$ and $E_{-1}(p - q)$. Though this could be done by a direct method, here we take a more complicated approach based on the classification of pairs of projections that was first given by Dixmier [6], and Kreĭn, Kranosel′skiĭ, and Mil′man [11], independently. The virtue of this approach is that we can also obtain information about other eigenvalues that will be crucial for proving Corollary 2.6.

Proposition 2.4 $[p:q] = \dim E_1(p-q) - \dim E_{-1}(p-q)$.

Proof Let M and N be the ranges of p and q respectively, and let $H_{11} = M \cap N$, $H_{10} = M \cap N^\perp$, $H_{01} = M^\perp \cap N$, $H_{00} = M^\perp \cap N^\perp$, and $H_0 = (H_{00} \oplus H_{10} \oplus H_{01} \oplus H_{11})^\perp$. It is possible to identify both $H_0 \cap M$ and $H_0 \cap M^\perp$ with $L^2(X)$ for some measure space X in such a way that $p|_{H_0}$, which is denoted by p_0, and $q|_{H_0}$, which is denoted by q_0, are given by

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q_0 = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix},$$

where ϕ is a measurable function on X such that $0 < \phi(x) < \frac{\pi}{2}$ for $x \in X$. Here p_0 and q_0 operate on $L^2(X) \oplus L^2(X)$, and the matrices are operator matrices, whose entries are multiplication operators.

If we denote $p|_{H_{11}}$ by p_{11} and $p|_{H_{10}}$ by p_{10}, then $p = p_{11} + p_{10} + p_0$. Similarly, $q = q_{11} + q_{01} + q_0$. Now $p - q = p_{10} - q_{01} + p_0 - q_0 \in K$ implies that $p_0 - q_0 \in K(H_0)$ and $p_{10} - q_{01} \in K(H_{10} \oplus H_{01})$. Then since

$$p_0 - q_0 = \begin{pmatrix} \sin^2\phi & -\cos\phi\sin\phi \\ -\cos\phi\sin\phi & -\sin^2\phi \end{pmatrix} \in K(H_0),$$

X is a discrete space $\{x_n\}$ and $\phi(x_n) \to 0$ as $n \to \infty$. Therefore

$$U = \bigoplus \begin{pmatrix} \cos\phi(x_n) & \sin\phi(x_n) \\ -\sin\phi(x_n) & \cos\phi(x_n) \end{pmatrix} \in 1 + K(H_0).$$

Since U is unitary and $U^* p_0 U = q_0$, $[p_0 : q_0] = 0$ by Lemma 2.3. On the other hand, $p_{10} - q_{01} \in K(H_{10} \oplus H_{01})$ means $\operatorname{rank}(p_{10})$ and $\operatorname{rank}(q_{01})$ are finite, and

$$[p_{10} : q_{01}] = \operatorname{rank}(p_{10}) - \operatorname{rank}(q_{01}) = \dim(E_1(p-q)) - \dim(E_{-1}(p-q)).$$

Since $[p:q] = [p_0:q_0] + [p_{10}:q_{01}] + [p_{11}:q_{11}]$ by Proposition 2.2, $[p:q] = [p_{10}:q_{01}]$, and the result follows. ■

Remark 2.5 (i) $[p:q]$ is the number of $+1$'s minus the number of -1's in the diagonalization of $(p-q)$.

(ii) The other non-zero points in the spectrum of $p - q$ come from $p_0 - q_0$. They are $\sin\phi(x_n)$ and $-\sin\phi(x_n)$. Note that this part of the spectrum is symmetric about 0 (even considering multiplicity); *i.e.*, $\dim E_\lambda(p-q) = \dim E_{-\lambda}(p-q)$ for $0 < \lambda < 1$.

Corollary 2.6 *Suppose projections $p_t, q_t \in B(H)$ are defined for each t in an interval. Then $[p_t : q_t]$ is constant if $p_t - q_t$ is norm continuous in K*

Proof Fix a point t; we are going to show that there is a $\delta > 0$ such that whenever $|t - s| < \delta$, we have $[p_s : q_s] = [p_t : q_t]$.

Let $A_1 = p_t - q_t$. Since A_1 is compact, its spectrum $\sigma(A_1)$ is discrete. So we can take a neighborhood $U = U_{-1} \cup U_0 \cup U_1$ containing $\sigma(A_1)$, where the U_i's are disjoint open disks centered at i such that the distance between them is positive. Choose a positive number R such that $\gamma_{\pm 1}$, the circles of radius R centered at ± 1, are contained in $U_{\pm 1}$ and $\sigma(A_1) \cap U_{\pm 1} \subset V_{\pm 1}$, the interior of $\gamma_{\pm 1}$. By the semicontinuity of spectrum, there is a $\delta > 0$ such that $\sigma(p_s - q_s) \subset V_{-1} \cup U_0 \cup V_1$ for $|s - t| < \delta$. Now set $A_2 = p_s - q_s$. By the Riesz functional calculus we have projections

$$r_i = \frac{1}{2\pi i} \int_{\gamma_1} (z - A_i)^{-1} dz \quad \text{and} \quad s_i = \frac{1}{2\pi i} \int_{\gamma_{-1}} (z - A_i)^{-1} dz$$

for $i = 1, 2$. Moreover, if we take δ small enough, we also have $\|r_1 - r_2\| < 1$ and $\|s_1 - s_2\| < 1$. Then $\operatorname{rank}(r_1) = \operatorname{rank}(r_2)$ and $\operatorname{rank}(s_1) = \operatorname{rank}(s_2)$ by Lemma 2.3.

Note that

$$\operatorname{rank}(s_i) = \sum_{1-R<\lambda<1} \dim E_{-\lambda}(A_i) + \dim E_{-1}(A_i).$$

Similarly,

$$\operatorname{rank}(r_i) = \sum_{1-R<\lambda<1} \dim E_{\lambda}(A_i) + \dim E_{1}(A_i).$$

So Remark 2.5(ii) implies that

$$\begin{aligned} [p_1 : q_1] &= \dim E_1(A_1) - \dim E_{-1}(A_1) \\ &= \operatorname{rank}(r_1) - \operatorname{rank}(s_1) = \operatorname{rank}(r_2) - \operatorname{rank}(s_2) \\ &= \dim E_1(A_2) - \dim E_{-1}(A_2) = [p_2 : q_2]. \end{aligned}$$ ■

Now we want to prove the most important property of the essential codimension.

Theorem 2.7 *Let p and q be projections in $B(H)$ such that $p - q \in K$. There is a unitary U in $1 + K$ such that $UpU^* = q$ if and only if $[p : q] = 0$.*

Proof One direction is proved by Lemma 2.3.

For the other direction, suppose that $[p : q] = 0$. Using the notation in the the proof of Proposition 2.4, we know that p_0 and q_0 are unitarily equivalent by a unitary U_0. In addition, from $[p : q] = \operatorname{rank}(p_{10}) - \operatorname{rank}(q_{01}) = 0$, there is a unitary W in $B(H_{10} \oplus H_{01})$ such that $W p_{10} W^* = q_{01}$. Now we have a unitary $U = U_0 + W + 1_{H_{11}} + 1_{H_{00}} \in 1 + K$ such that $UpU^* = q$. ■

Remark 2.8 Let E be a Hilbert B-module. If $\pi, \sigma \colon A \to \mathcal{L}(E)$ are representations, we say that π and σ are properly asymptotically unitarily equivalent and write $\pi \approxeq \sigma$ if there is a continuous path of unitaries $u \colon [0, \infty) \to \mathcal{U}(K(E) + \mathbb{C}1_E)$, $u = (u_t)_{t \in [0,\infty)}$ such that

- $\lim_{t\to\infty} \|u_t \pi(a) u_t^* - \sigma(a)\| = 0$ for each $a \in A$;
- $u_t \pi(a) u_t^* - \sigma(a) \in K(E)$ for each $t \in [0, \infty)$ and $a \in A$.

The use of the word "proper" reflects the crucial fact that all implementing unitaries are of the form "identity + compact" [5]. A result of Dadarlat and Eilers shows that if $\phi, \psi \colon A \to M(B \otimes K)$ is a Cuntz pair, then $[\phi, \psi]$ vanishes in $KK(A, B)$ if and only if $\phi \oplus \gamma \approxeq \psi \oplus \gamma$ for some representation $\gamma \colon A \to M(B \otimes K)$. As a corollary, they have shown that if $\phi, \psi \colon A \to B(H)$ is a Cuntz pair of admissible representations (faithful, non-degenerate, and its image does not contain any non-trivial compacts), then $[\phi, \psi]$ vanishes in $KK(A, \mathbb{C})$ if and only if $\phi \approxeq \psi$. Now we apply this to $A = \mathbb{C}$. Without loss of generality we can assume p and q, which come from a Cuntz pair in $KK(\mathbb{C}, \mathbb{C})$, are very close if $[p : q] = 0$. Then $z = pq + (1 - p)(1 - q) \in 1 + K$ is invertible and $pz = zq$. If we consider the polar decomposition of z as $z = u|z|$, it is easy to check that $u \in 1 + K$ and $upu^* = q$. Thus Theorem 2.7 can be obtained from a KK-theoretic result.

In the next section, we need the following facts.

Proposition 2.9 *Let p, q be projections such that $p - q \in K$.*

(i) *If q has finite rank, then $[p : q] \geq -\operatorname{rank}(q)$.*
(ii) *If $1 - q$ has finite rank, then $[p : q] \leq \operatorname{rank}(1 - q)$.*

Proof Straightforward. ■

3 Lifting Projections

Let X be $[0, 1]$, $[0, \infty)$, $(-\infty, \infty)$ or $\mathbb{T} = [0, 1]/\{0, 1\}$ and let $I = C(X) \otimes K$, which is the C^*-algebra of (norm continuous) functions from X to K, vanishing at infinity where applicable. Then $M(I)$ is $C_b(X, B(H))$ the set of bounded functions from X to $B(H)$ where $B(H)$ is given the double-strong topology. Let $\mathcal{C}(I) = M(I)/I$ be the corona algebra of I and let $\pi \colon M(I) \to \mathcal{C}(I)$ be the natural quotient map.

In general, an element $\mathbf{f}$ of the corona algebra $\mathcal{C}(I)$ can be represented as follows: Consider a finite partition of X, or $X \setminus \{0, 1\}$ when $X = \mathbb{T}$, given by partition points $x_1 < x_2 < \cdots < x_n$, all of which are in the interior of X, and divide X into $n + 1$ (closed) subintervals $X_0, X_1, \ldots, X_n$. Take $f_i \in C_b(X_i, B(H))$ such that $f_i(x_i) - f_{i-1}(x_i) \in K$ for $i = 1, 2, \ldots, n$ and $f_0(x_0) - f_n(x_0) \in K$, where $x_0 = 0 = 1$ if X is $\mathbb{T}$. We call $(f_0, f_1, \ldots, f_n)$ *a local representation* of $\mathbf{f}$. For example, if $\mathbf{f} = \pi(f)$ for some $f \in M(I)$, then we can take $f_i = f|_{X_i}$. However, the point of this local representation is that, by Theorem 3.1, when $\mathbf{f}$ is a projection we can find $(f_0, f_1, \ldots, f_n)$ such that each f_i is a projection-valued function.

Note that the coset in $\mathcal{C}(I)$ represented by $(f_0, \ldots, f_n)$ consists of functions f in $M(I)$ such that $f - f_i \in C(X_i, K)$ for every i, and $f - f_i$ vanishes (in norm) at any infinite endpoint of X_i. Thus $(f_0, \ldots, f_n)$ and $(g_0, \ldots, g_n)$ define the same element of $\mathcal{C}(I)$ if and only if $f_i - g_i \in C_b(X_i, K)$ for $i = 0, \ldots, n$ and $f_i - g_i$ vanishes at any infinite endpoint of X_i.

J. Calkin [4] showed that in this setting projection can always be lifted when X is a single point. In our situation we have the following theorem.

Theorem 3.1 *If $\mathbf{f}$ is a projection in $\mathcal{C}(I)$, we can find an $(f_0, \ldots, f_n)$ as above such that each f_i is projection-valued.*

Proof Let f be an element of $M(I)$ such that $\pi(f) = \mathbf{f}$. Without loss of generality, we can assume f is a self-adjoint element such that $0 \leq f \leq 1$.

(i) Suppose X does not contain any infinite point. Choose a point t_0 in X. Since $f(t_0) - f^2(t_0)$ is compact, there are a, b with $0 < a < b < 1/4$ such that $\sigma(f(t_0) - f^2(t_0)) \subset U_{ab} = [0, a) \cup (b, 1/4]$. Since $f - f^2$ is norm continuous, there is an open set O_{t_0} containing t_0 such that $\sigma(f(t) - f(t)^2) \subset U_{ab}$ for all t in O_{t_0}. Let α and β be the points in $(0, 1/2)$ such that $\alpha - \alpha^2 = a$ and $\beta - \beta^2 = b$. Then $\sigma(f(t))$ omits (α, β) for t in O_{t_0}. If ϕ_{t_0} is a continuous function such that $\phi_{t_0} = 0$ on $[0, \alpha]$ and $\phi_{t_0} = 1$ on $[\beta, 1]$, then $\phi_{t_0}(f) - f$ is in I and $\phi_{t_0}(f(t))$ is a projection for t in O_{t_0}.

Since X is compact and O_{t_0}'s give an open cover, there is a partition and corresponding values $t_0, t_1, \dots, t_n$ such that $X_i \subset O_{t_i}$. Then let $f_i = \phi_{t_i}|_{X_i}$.

(ii) Let X be $[0, \infty)$. Since $f^2 - f$ vanishes at ∞, there is $M > 0$ such that $\|f^2(t) - f(t)\| < 1/3(1 - 1/3)$ if $t \geq M$. Then $\sigma(f(t))$ omits $(1/3, 2/3)$ for such t. If ϕ is a continuous fucntion such that $\phi = 0$ on $[0, 1/3]$ and $\phi = 1$ on $[2/3, 1]$, then $\phi(f) - f$ is in I and $\phi(f(t))$ is a projection for $t \geq M$. As in part (i) find a local representation $(f_0, \dots, f_{n-1})$ for $[0, M]$. Then take $X_n = [M, \infty)$ and $f_n = \phi(f)|_{X_n}$

(iii) The case $X = (-\infty, \infty)$ is similar to (ii). ∎

Given a (projection-valued) representation $(f_0, \dots, f_n)$ of the projection $\mathbf{f}$ in $\mathcal{C}(I)$, we can associate integers $k_i = [f_i(x_i) : f_{i-1}(x_i)]$ for $i = 1, \dots, n$ and $k_0 = [f_0(x_0) : f_n(x_0)]$ in the circle case.

Proposition 3.2 *If all k_i's are equal to 0, then* $\mathbf{f}$ *is liftable to a projection in $M(I)$.*

Proof First we consider the case $X = [0, 1]$, $[0, \infty)$, or $(-\infty, \infty)$. We construct perturbations $f_1', \dots, f_n'$ of $f_1, \dots, f_n$ so that $f_i'(x_i) = f_{i-1}'(x_i)$ for $i = 1 \cdots n$ and f_n' agrees with f_n in the some neighborhood of ∞ if the right end point of X is ∞.

Observe that if $k_i = [f_i(x_i) : f_{i-1}(x_i)] = 0$ for some i, there is a unitary U in $1 + K$ such that $U f_i(x_i) U^* = f_{i-1}(x_i)$ by Theorem 2.7. Since the set of such unitaries is path connected, there is a norm continuous path $U_i : X_i \to \mathcal{U}(H) \cap (1 + K)$ such that $U_i(x_i) f_i(x_i) U_i(x_i)^* = f_{i-1}(x_i)$ and $U_i = 1$ in a left neighborhood of x_{i+1}. If $i = n$ and the right endpoint of X_i is ∞, the last condition is replaced by $U_i(t) = 1$ for $t \geq x_n + 1$. Then let $f_i' = U_i f_i U_i^*$.

If $X = \mathbb{T}$, construct $f_1', \dots, f_n'$ as above, and also construct a similar perturbation of f_0. ∎

If $\mathbf{f}$, which is represented by $(f_0, f_1, \dots, f_n)$, is liftable to a projection g in $M(I)$, we also represent $\mathbf{f}$ by $(g_0, \dots, g_n)$ where $g_i = g|_{X_i}$. Then, for each i, $[g_i(x) : f_i(x)]$ is defined for all x. From Corollary 2.6 this function must be constant on X_i. Let $l_i = [g_i(x) : f_i(x)]$. Since $g_i(x_i) = g_{i-1}(x_i)$, we have

$$\big[g_i(x_i) : f_i(x_i)\big] + \big[f_i(x_i) : f_{i-1}(x_i)\big] = \big[g_{i-1}(x_i) : f_{i-1}(x_i)\big]$$

by Proposition 2.2(iii). Thus,

$$(3.1) \qquad l_i - l_{i-1} = -k_i \quad \text{for} \quad i > 0 \quad \text{and} \quad l_0 - l_n = -k_0 \quad \text{in the circle case.}$$

Moreover, by Proposition 2.9 and Lemma 2.3 we have the following restrictions on l_i.

(3.2) If for some x in X_i, $f_i(x)$ has finite rank, then $l_i \geq -\operatorname{rank}(f_i(x))$.

(3.3) If for some x in X_i, $1 - f_i(x)$ has finite rank, then $l_i \leq \operatorname{rank}(1 - f_i(x))$.

(3.4) If either end point of X_i is infinite, then $l_i = 0$.

The above provides necessary conditions for a projection $\mathbf{f}$ in $\mathcal{C}(I)$ to be liftable to a projection in $M(I)$. We claim that these necessary conditions are also sufficient. To show this, we need a well-known identification of a strongly continuous projection-valued function on a topological space with a continuous field of Hilbert spaces [8, 252–253]. Given a separable Hilbert space H, there is a one-to-one correspondence between complemented subfields $\mathcal{H} = ((H_x)_{x\in X}, \Gamma)$ of the constant field defined by H over a paracompact space T and strongly continuous projection-valued functions $p\colon X \mapsto B(H)$, where H_x is $p(x)H$. Thus two continuous fields of Hilbert spaces defined by p and p' are isomorphic if and only if there is a double strongly continuous valued function u on T such that $uu^* = p'$ and $u^*u = p$.

The following lemma, for which we claim no originality, plays an important role in the proof of Theorem 3.4. However, its proof is rather long, so we include the proof in an Appendix (see Corollary A.5).

Lemma 3.3 *If X is a separable metric space such that $\dim(X) \leq 1$ and $\mathcal{H}$ is a continuous field of Hilbert spaces over X such that $\dim(H_x) \geq n$ for every x in X, then $\mathcal{H}$ has a trivial subfield of rank n. Equivalently, if p is a strongly continuous projection-valued function on X such that $\operatorname{rank}(p(x)) \geq n$ for every x in X, then there is a norm continuous projection-valued function q such that $q \leq p$ and $\operatorname{rank}(q(x)) = n$ for every x in X.*

Theorem 3.4 *A projection $\mathbf{f}$ in $\mathcal{C}(I)$ represented by $(f_0, \dots, f_n)$ is liftable to a projection in $M(I)$ if and only if there exist $l_0, \dots, l_n$ satisfying* (3.1)–(3.4).

Proof Given l_i's satisfying (3.2), (3.3), and (3.4), we will show there exists a perturbation $(g_0, \dots, g_n)$ of $(f_0, \dots, f_n)$ such that $l_i = [g_i(x_i) : f_i(x_i)]$. It follows from this and (3.1) that $[g_i(x_i) : g_{i-1}(x_i)] = 0$ for $i > 0$ and $[g_0(x_0) : g_n(x_0)] = 0$ in the circle case.

$l_i = 0$: Take $g_i = f_i$. Note that by (3.4) this implies the necessary condition on the perturbation at any infinite endpoints.
$l_i > 0$: By Lemma 3.3 and (3.3) the continuous field determined by $1 - f_i$ has a rank l_i trivial subfield that induces a norm continuous projection-valued function $q \leq 1 - f_i$. So we take $g_i = f_i + q$.
$l_i < 0$: Similarly, by Lemma 3.3 and (3.2) the continuous field determined by f_i has a rank $-l_i$ trivial subfield which induces a projection-valued function $q' \leq f_i$. So we take $g_i = f_i - q'$. ■

Corollary 3.5 *If X has an infinite endpoint, then $\mathbf{p} \oplus \mathbf{0}$ and $\mathbf{p} \oplus \mathbf{1}$ both liftable implies that $\mathbf{p}$ is liftable.*

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Proof There can be only one choice of the l_i's satisfying (3.1) and (3.4), and thus there is a single choice of l_i's that works for both $\mathbf{p} \oplus \mathbf{0}$ and $\mathbf{p} \oplus \mathbf{1}$. Then (3.2) for $\mathbf{p} \oplus \mathbf{0}$ implies (3.2) for $\mathbf{p}$, and (3.3) for $\mathbf{p} \oplus \mathbf{1}$ implies (3.3) for $\mathbf{p}$. ■

It will be shown in Example 4.14 that this implication does not hold if X has no infinite endpoint.

4 Homotopy Classification of Projections

Recall that $K_i(\mathcal{C}(I)) = K_{i+1}(I)$ (i taken mod 2). Thus

$$K_0(\mathcal{C}(I)) = \begin{cases} \mathbb{Z}, & \text{if } X = (-\infty, \infty) \text{ or } \mathbb{T}(= [0,1]/\{0,1\}) \\ 0, & \text{otherwise.} \end{cases}$$

We want to analyze these isomorphisms in a concrete way to compute K_0-classes of projections in $\mathcal{C}(I)$, where $I = C(X) \otimes K$ and $X = (-\infty, \infty)$ or $X = \mathbb{T} = [0,1]/\{0,1\}$, in terms of the representations of Section 3. Since $M_n(I) \simeq I$, we can represent a projection $\mathbf{f}$ in $M_n(\mathcal{C}(I))$ by $(f_1, f_2, \dots, f_n)$, where f_i's are in $M_n(M(I)) = M(M_n(I))$ and each of them is projection-valued: $f_i(x)$ is in $M_n(B(H)) \simeq B(H^n)) \simeq B(H)$. Thus it will suffice to compute K_0-classes of projections in $\mathcal{C}(I)$.

Given two nontrivial projections $\mathbf{p}, \mathbf{q}$ in $\mathcal{C}(I)$, let us assume that there is a partial isometry $\pi(u)$ in $\mathcal{C}(I)$ such that $\pi(u)^*\pi(u) = \mathbf{p}$ and $\pi(u)\pi(u)^* = \mathbf{q}$ for some u in $M(I)$. If $(p_0, \dots, p_n)$ and $(q_0, \dots, q_n)$ are local liftings of $\mathbf{p}$ and $\mathbf{q}$ respectively, using the same partition, we can represent $\pi(u)$ by $(u_0, \dots, u_n)$ where $u_i = q_i u|_{X_i} p_i$. Thus the following holds:

$$q_i - u_i u_i^*, \quad p_i - u_i^* u_i \in C(X_i) \otimes K.$$

This implies that for x in X_i we can view $u_i(x)$ as a Fredholm operator from $p_i(x)H$ to $q_i(x)H$, and thus we can define the Fredholm index for $u_i(x)$. As we shall see, this index plays a key role.

Proposition 4.1 $\operatorname{Ind}(u_i(x))$ *is constant on* X_i*. In particular, if* X_i *has an infinite endpoint, then* $\operatorname{Ind}(u_i(x)) = 0$.

Proof For x_0 in X_i we have

$$\ker\big(u_i(x_0)\big) = \big\{h \in p_i(x_0)H \mid u_i(x_0)^* u_i(x_0) h = 0\big\} = E_1\big(p_i(x_0) - u_i(x_0)^* u_i(x_0)\big)$$

(recall that $E_\lambda(T)$ is the eigenspace of T corresponding to the eigenvalue λ for T in $B(H)$). Similarly, $\ker(u_i(x_0)^*) = E_1(q_i(x_0) - u_i(x_0)u_i(x_0)^*)$.

Note that the spectrum of $p_i(x_0) - u_i(x_0)^* u_i(x_0)$ omits an interval $(1 - \epsilon, 1)$ for some ϵ in $(0, 1)$. Then there is a $\delta > 0$ such that $\sigma(p_i(x_0) - u_i(x_0)^* u_i(x_0))$ omits the interval $\big(1 - \frac{2\epsilon}{3}, 1 - \frac{\epsilon}{3}\big)$ for $|x - x_0| < \delta$.

By the Riesz functional calculus (see the proof of Corollary 2.6),

$$\dim\big(E_1\big(p_i(x_0) - u_i(x_0)^* u_i(x_0)\big)\big) = \sum_{0 \le \lambda \le \epsilon/3} \dim\big(E_{1-\lambda}\big(p_i(x) - u_i(x)^* u_i(x)\big)\big)$$

for $|x - x_0| < \delta$. Similarly,

$$\dim\big(E_1\big(q_i(x_0) - u_i(x_0)u_i(x_0)^*\big)\big) = \sum_{0\le\lambda\le\epsilon/3} \dim\big(E_{1-\lambda}\big(q_i(x) - u_i(x)u_i(x)^*\big)\big)$$

for $|x - x_0| < \delta$. Since

$$\begin{aligned}\dim E_{1-\lambda}\big(p_i(x) - u_i(x)^*u(x)\big) &= \dim E_\lambda\big(u_i(x)^*u_i(x)\big) = \dim E_\lambda\big(u_i(x)u_i(x)^*\big)\\ &= \dim E_{1-\lambda}\big(q_i(x) - u_i(x)u(x)^*\big)\end{aligned}$$

for $\lambda > 0$,

$$\sum_{0<\lambda\le\epsilon/3} \dim\big(E_{1-\lambda}\big(p_i(x) - u_i(x)^*u_i(x)\big)\big) = \sum_{0<\lambda\le\epsilon/3} \dim\big(E_{1-\lambda}\big(q_i(x) - u_i(x)u_i(x)^*\big)\big).$$

Thus,

$$\begin{aligned}&\operatorname{Ind}\big(u_i(x_0)\big)\\ &\quad= \dim\big(E_1\big(p_i(x_0) - u_i(x_0)^*u_i(x_0)\big)\big) - \dim\big(E_1\big(q_i(x_0) - u_i(x_0)u_i(x_0)^*\big)\big)\\ &\quad= \dim\big(E_1\big(p_i(x) - u_i(x)^*u_i(x)\big)\big) - \dim\big(E_1\big(q_i(x) - u_i(x)u_i(x)^*\big)\big)\\ &\quad= \operatorname{Ind}\big(u_i(x)\big) \quad \text{for} \quad |x - x_0| < \delta.\end{aligned}$$

The claim follows from the fact that each X_i is connected. ■

We will denote the index of u_i on X_i by t_i. If $u_i(x_i) = v_i|u_i(x_i)|$ is a polar decomposition of $u_i(x_i)$ in $B(H)$ and $u_{i-1}(x_i) = v_{i-1}|u_{i-1}(x_i)|$ is a polar decomposition of $u_{i-1}(x_i)$, then

$$\begin{aligned}t_i &= \big[p_i(x_i) : v_i^* v_i\big] - \big[q_i(x_i) : v_i v_i^*\big],\\ t_{i-1} &= \big[p_{i-1}(x_i) : v_{i-1}^* v_{i-1}\big] - \big[q_{i-1}(x_i) : v_{i-1} v_{i-1}^*\big].\end{aligned} \tag{4.1}$$

The facts that $p_i(x_i) - p_{i-1}(x_i), q_i(x_i) - q_{i-1}(x_i) \in K$ imply that $u_i(x_i) - u_{i-1}(x_i) \in K$. Then it is easily shown that

$$v_i - v_{i-1} \in K,\ \ v_i^* v_i - v_{i-1}^* v_{i-1}, v_i v_i^* - v_{i-1} v_{i-1}^* \in K.$$

By Proposition 2.2,

$$\begin{aligned}\big[p_i(x_i) : p_{i-1}(x_i)\big] &= \big[p_i(x_i) : v_i^* v_i\big] + \big[v_i^* v_i : v_{i-1}^* v_{i-1}\big] + \big[v_{i-1}^* v_{i-1} : p_{i-1}(x_i)\big],\\ \big[q_i(x_i) : q_{i-1}(x_i)\big] &= \big[q_i(x_i) : v_i v_i^*\big] + \big[v_i v_i^* : v_{i-1} v_{i-1}^*\big] + \big[v_{i-1} v_{i-1}^* : q_{i-1}(x_i)\big].\end{aligned}$$

By subtracting, we have

$$\left[p_i(x_i):p_{i-1}(x_i)\right]-\left[q_i(x_i):q_{i-1}(x_i)\right]=\\ t_i-t_{i-1}+\left[v_i^*v_i:v_{i-1}^*v_{i-1}\right]-\left[v_iv_i^*:v_{i-1}v_{i-1}^*\right].$$

Let W,V be isometries such that $WW^*=v_i^*v_i, VV^*=v_{i-1}^*v_{i-1}$. Then $V'=v_{i-1}V, W'=v_iW$ are isometries such that $V'V'^*=v_{i-1}v_{i-1}{}^*, W'W'^*=v_iv_i^*$. Then

$$\begin{aligned}[v_iv_i^*:v_{i-1}v_{i-1}^*]&=\operatorname{Ind}(V'^*W')=\operatorname{Ind}(V^*v_{i-1}^*v_iW)\\&=\operatorname{Ind}(V^*v_i^*v_iW)\quad(v_{i-1}-v_i\in K)\\&=\operatorname{Ind}(V^*WW^*W)=\operatorname{Ind}(V^*W)\\&=[v_i^*v_i:v_{i-1}^*v_{i-1}].\end{aligned}$$

Thus, if we let $k_i=[p_i(x_i):p_{i-1}(x_i)]$, $l_i=[q_i(x_i):q_{i-1}(x_i)]$, then we have

$$t_i-t_{i-1}=k_i-l_i. \tag{4.2}$$

Proposition 4.2 *Suppose there is a partial isometry $\pi(u)$ in $\mathcal{C}(I)$, where $I=C(-\infty,\infty)\otimes K$ or $C(\mathbb{T})\otimes K$, such that $\pi(u)^*\pi(u)=\mathbf{p}$ and $\pi(u)\pi(u)^*=\mathbf{q}$ for some u in $M(I)$. If $(p_0,\dots,p_n)$ and $(q_0,\dots,q_n)$ are local liftings of $\mathbf{p}$ and $\mathbf{q}$ respectively, then*

$$\sum_{i=1}^{n}\left[p_i(x_i):p_{i-1}(x_i)\right]=\sum_{i=1}^{n}\left[q_i(x_i):q_{i-1}(x_i)\right]$$

or, in the circle case,

$$\sum_{i=1}^{n}\left[p_i(x_i):p_{i-1}(x_i)\right]+\left[p_0(x_0):p_n(x_0)\right]\\=\sum_{i=1}^{n}\left[q_i(x_i):q_{i-1}(x_i)\right]+\left[q_0(x_0):q_n(x_0)\right].$$

Proof By taking a sum in both sides of (4.2), we have $\sum k_i-\sum l_i=t_n-t_0$. Using the second part of Proposition 4.1 we have $t_n=t_0=0$.

In the circle case, add (4.2) for $i=1,\dots,n+1$ (i taken modulo $n+1$). ∎

Lemma 4.3 *If a projection $\mathbf{f}$ in $\mathcal{C}(I)$, where $I=C(-\infty,\infty)\otimes K$ or $C(\mathbb{T})\otimes K$, has two different local liftings $(f_1,\dots,f_n)$ and $(g_1,\dots,g_n)$, then $\sum[f_i(x_i):f_{i-1}(x_i)]=\sum[g_i(x_i):g_{i-1}(x_i)]$.*

Proof Note that

$$\left[g_i(x_i):f_i(x_i)\right]+\left[f_i(x_i):f_{i-1}(x_i)\right]+\left[f_{i-1}(x_i):g_{i-1}(x_i)\right]=\left[g_i(x_i):g_{i-1}(x_i)\right].$$

Equivalently,

$$\big[f_i(x_i) : f_{i-1}(x_i)\big] - \big[g_i(x_i) : g_{i-1}(x_i)\big] = \big[f_i(x_i) : g_i(x_i)\big] - \big[f_{i-1}(x_i) : g_{i-1}(x_i)\big]\,.$$

Hence, by Corollary 2.6, in the $(-\infty, \infty)$-case

$$\sum\big[f_i(x_i) : f_{i-1}(x_i)\big] - \sum\big[g_i(x_i) : g_{i-1}(x_i)\big] = \big[f_n(x_n) : g_n(x_n)\big] - \big[f_0(x_1) : g_0(x_1)\big]\,.$$

Since $f_n - g_n \in C_0[x_n, \infty) \otimes K$ and $f_0 - g_0 \in C_0(-\infty, x_1] \otimes K$, $[f_n(x_n) : g_n(x_n)] = [f_0(x_1) : g_0(x_1)] = 0$ by Corollary 2.6, and the conclusion follows.

In the circle case the conclusion follows similarly, by adding $n+1$ equations. ■

Now we are ready to define a map $\chi\colon K_0(\mathcal{C}(I)) \to \mathbb{Z}$ as follows: Let $\alpha = [\mathbf{p}] - [\mathbf{q}]$ be an element of $K_0(\mathcal{C}(I))$, and let $(p_0, \dots, p_n)$ and $(q_0, \dots, q_n)$ be local liftings of $\mathbf{p}$ and $\mathbf{q}$ respectively. Then

$$\chi(\alpha) = \sum_{i=1}^{n}\big[p_i(x_i) : p_{i-1}(x_i)\big] - \sum_{i=1}^{n}\big[q_i(x_i) : q_{i-1}(x_i)\big]$$

or

$$\sum_{i=1}^{n+1}\big[p_i(x_i) : p_{i-1}(x_i)\big] - \sum_{i=1}^{n+1}\big[q_i(x_i) : q_{i-1}(x_i)\big]$$

with $x_{n+1} = x_0$ in the case $X = [0,1]/\{0,1\}$. Note that χ is well defined by Proposition 4.2 and Lemma 4.3. The next goal is to show that χ is an isomorphism.

Lemma 4.4 *Let $(p_0, \dots, p_n)$ and $(q_0, \dots, q_n)$ be local liftings of $\mathbf{p}$ and $\mathbf{q}$ such that* $\operatorname{rank}(1 - q_i(x)) = \operatorname{rank}(q_i(x)) = \infty$ *for each x in X_i.*

If

$$\sum_{i=1}^{n}[p_i(x_i) : p_{i-1}(x_i)] = \sum_{i=1}^{n}[q_i(x_i) : q_{i-1}(x_i)],$$

or, in the circle case,

$$\sum_{i=1}^{n}\big[p_i(x_i) : p_{i-1}(x_i)\big] + \big[p_0(x_0) : p_n(x_0)\big] = \sum_{i=1}^{n}\big[q_i(x_i) : q_{i-1}(x_i)\big] + \big[q_0(x_0) : q_n(x_0)\big]\,,$$

then we can find a perturbation $(q_0', \dots, q_n')$ of $\mathbf{q}$ such that $[p_i(x_i) : p_{i-1}(x_i)] = [q_i'(x_i) : q_{i-1}'(x_i)]$ for $i = 1, \dots, n$ (in the circle case, $i = 1, \dots, n+1$ with i modulo $n+1$).

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Proof Let $[p_i(x_i):p_{i-1}(x_i)] = k_i$, $[q_i(x_i):q_{i-1}(x_i)] = l_i$, and $d_i = k_i - l_i$. Then note that $\sum d_i = 0$ by assumption.

Let $q'_0 = q_0$. Suppose that we have $q'_0, \dots, q'_i$ such that $[p_j(x_j):p_{j-1}(x_j)] = [q'_j(x_j):q'_{j-1}(x_j)] = k_j$ for $j = 1, \dots, i$. Then $[q_{i+1}(x_{i+1}):q'_i(x_{i+1})] = l_{i+1} - \sum_{k=1}^{i} d_k$ by Lemma 4.3.

If $d_{i+1} + \sum_{k=1}^{i} d_k \geq 0$, let q be a projection-valued (norm continuous) function such that $q \leq 1 - q_{i+1}$ and $\operatorname{rank}(q(x)) = d_{i+1} + \sum_{k=1}^{i} d_k$, using Lemma 3.3.

Take $q'_{i+1} = q + q_{i+1}$. Then

$$\begin{aligned}
[q'_{i+1}(x_{i+1}):q'_i(x_{i+1})] &= [q_{i+1}(x_{i+1}):q'_i(x_{i+1})] + [q(x_{i+1}):0] \\
&= l_{i+1} - \sum_{k=1}^{i} d_k + d_{i+1} + \sum_{k=1}^{i} d_k \\
&= l_{i+1} + k_{i+1} - l_{i+1} = k_{i+1}
\end{aligned}$$

If $d_{i+1} + \sum_{k=1}^{i} d_k < 0$, let q be a projection-valued (norm continuous) function such that $q \leq q_{i+1}$ and $\operatorname{rank}(q(x)) = -(d_{i+1} + \sum_{k=1}^{i} d_k)$, take $q'_{i+1} = q_{i+1} - q$.

Note that

$$\big[q'_{i+1}(x_{i+1}):q'_i(x_{i+1})\big] + \big[q(x_{i+1}):0\big] = \big[q_{i+1}(x_{i+1}):q'_i(x_{i+1})\big]$$

Thus

$$\begin{aligned}
\big[q'_{i+1}(x_{i+1}):q'_i(x_{i+1})\big] &= l_{i+1} - \sum_{k=1}^{i} d_k + \Big(d_{i+1} + \sum_{k=1}^{i} d_k\Big) \\
&= l_{i+1} + d_{i+1} = k_{i+1}
\end{aligned}$$

In the $(-\infty,\infty)$-case we continue this recursion through step $n-1$ and take $q'_n = q_n$. In the circle case we continue the recursion through step n. In both cases we have directly obtained all but one of the equations $[q_i(x_i):q_{i-1}(x_i)] = k_i$, and the last follows from the hypothesis and Lemma 4.3. ■

Next we have an analogous result that is more symmetrical.

Lemma 4.5 *Let $(p_0, \dots, p_n)$ and $(q_0, \dots, q_n)$ be local liftings of $\mathbf{p}$ and $\mathbf{q}$ such that* $\operatorname{rank}(p_i(x)) = \operatorname{rank}(q_i(x)) = \infty$ *for each x in X_i.*

If

$$\sum \big[p_i(x_i):p_{i-1}(x_i)\big] = \sum \big[q_i(x_i):q_{i-1}(x_i)\big]$$

or

$$\sum \big[p_i(x_i):p_{i-1}(x_i)\big] = \sum \big[q'_i(x_i):q'_{i-1}(x_i)\big]$$

for $i = 1, \dots, n+1$ modulo $n+1$, then we can find perturbations $(q'_0, \dots, q'_n)$ of $\mathbf{q}$ and $(p'_0, \dots, p'_n)$ of $\mathbf{p}$ such that $[p'_i(x_i):p'_{i-1}(x_i)] = [q'_i(x_i):q'_{i-1}(x_i)]$ for all i.

Proof The proof proceeds as above with one exception: If $d_{i+1} + \sum_{k=1}^{i} d_k \geq 0$, we make $p'_{i+1} \leq p_i$ rather than making $q'_{i+1} \geq q_i$. ■

Theorem 4.6 *The map $\chi\colon K_0(\mathcal{C}(I)) \to \mathbb{Z}$ is an isomorphism. Thus $[\mathbf{p}] = [\mathbf{q}]$ if and only if $\sum k_i = \sum l_i$ where $(p_0, \dots, p_n)$ and $(q_0, \dots, q_n)$ are local liftings of two projections $\mathbf{p}$ and $\mathbf{q}$ respectively, and $k_i = [p_i(x_i) : p_{i-1}(x_i)]$, $l_i = [q_i(x_i) : q_{i-1}(x_i)]$.*

Proof Suppose that $\alpha = [\mathbf{p}] - [\mathbf{q}]$ is an element of $K_0(\mathcal{C}(I))$ such that $\chi(\alpha) = 0$.

Replace $\mathbf{p}$ with $\mathbf{p} \oplus \mathbf{1} \oplus \mathbf{0}$ and $\mathbf{q}$ with $\mathbf{q} \oplus \mathbf{1} \oplus \mathbf{0}$, and choose local liftings $(p_0, \dots, p_n)$ and $(q_0, \dots, q_n)$.

Then by Lemma 4.4 we arrange $k_i = l_i$ for each i. Since $\operatorname{rank}(p_i(x)) = \operatorname{rank}(q_i(x)) = \infty$, there is a (double) strongly continuous function u_i on each X_i such that $u_i{}^* u_i = p_i, u_i u_i{}^* = q_i$ (see Theorem A.1). Note that $u_{i-1}(x)$ is a unitary from $p_{i-1}(x)H$ onto $q_{i-1}(x)H$ so that $\operatorname{Ind}(u_{i-1}(x_i)) = 0$. Then $k_i = l_i$ implies that

$$\operatorname{Ind}\big(q_i(x_i)u_{i-1}(x_i)p_i(x_i)\big) = -l_i + \operatorname{Ind}\big(u_{i-1}(x_i)\big) + k_i = 0,$$

where the first index is for maps from $p_i(x_i)H$ to $q_i(x_i)H$, and, for example, the index of $p_{i-1}(x_i)p_i(x_i)$ as a map from $p_i(x_i)H$ to $p_{i-1}(x_i)H$ is k_i. Also

$$q_i(x_i)u_{i-1}(x_i)p_i(x_i) - u_{i-1}(x_i) \in K.$$

There is a compact perturbation v_i of $q_i(x_i)u_{i-1}(x_i)p_i(x_i)$ such that $v_i^* v_i = p_i(x_i)$, $v_i v_i^* = q_i(x_i)$, and $v_i - u_{i-1}(x_i) \in K$. By the triviality of the continuous field of Hilbert spaces determined by p_i and the path connectedness of the unitary group of $B(H)$, there is a path $\{v(t) : t \in [x_i, x]\}$ such that $v(t)^* v(t) = v(t)v(t)^* = p_i(t)$, $v(x_i) = u_i(x_i)^* v_i$, and $v(x) = p_i(x)$ for some x in X_i. Then we let $w_i = u_i v$ on $[x_i, x]$ so that

$$\begin{aligned} w_i(x_i) - u_{i-1}(x_i) &= v_i - u_{i-1}(x_i) \in K, \\ w_i^* w_i &= v^* u_i{}^* u_i v = v^* p_i v = p_i, \\ w_i w_i^* &= u_i v v^* u_i{}^* = u_i p_i u_i{}^* = q_i. \end{aligned}$$

Finally, we define

$$u'_i = \begin{cases} w_i, & \text{on } [x_i, x], \\ u_i, & \text{on } [x, x_{i+1}]. \end{cases}$$

In the $(-\infty, \infty)$-case we do the above for $i = 1, \dots, n$ and let $u'_0 = u_0$. In the circle case we do it for $i = 0, \dots, n$. This shows that $\mathbf{p} \sim \mathbf{q}$ and completes the proof of the injectivity of χ. The surjectivity of χ is fairly obvious and will follow from examples presented later in this section. ■

Corollary 4.7 *Suppose that $(p_0, \dots, p_n)$ and $(q_0, \dots, q_n)$ are local liftings of two projections in $\mathcal{C}(I)$. If $\operatorname{rank}(p_i(x)) = \operatorname{rank}(q_i(x)) = \infty$ for some i, then there exists a strongly continuous operator valued function u_i such that $u_i{}^* u_i = p_i$, $u_i u_i{}^* = q_i$, and $u_i(x_i) - u_{i-1}(x_i) \in K$ if and only if $k_i = l_i$, where $k_i = [p_i(x_i) : p_{i-1}(x_i)]$ and $l_i = [q_i(x_i) : q_{i-1}(x_i)]$.*

Corollary 4.8 *$[\mathbf{p}]_0$ is liftable if and only if $\mathbf{p} \oplus \mathbf{1} \oplus \mathbf{0}$ is liftable.*

Proof Since $K_0(M(I)) = 0$, we know that $[\mathbf{p}]_0$ is liftable if and only if $[\mathbf{p}]_0 = 0$. The latter is equivalent to $\sum k_i = 0$. Thus, if $[\mathbf{p}]_0$ is liftable, then $\mathbf{p} \oplus \mathbf{1} \oplus \mathbf{0}$ is liftable to a projection by Theorem 3.4.
Conversely, since the theorem implies that $[\mathbf{p} \oplus \mathbf{1} \oplus \mathbf{0}] = [\mathbf{p}]$, the liftability of $\mathbf{p} \oplus \mathbf{1} \oplus \mathbf{0}$ implies the liftability of $[\mathbf{p}]$. ■

Remark 4.9 In the cases $X = [0, 1]$ and $X = [0, \infty)$, which correspond to $K_0(\mathcal{C}(I)) = 0$, $\mathbf{p}$ is always liftable when there is a local representation of $\mathbf{p}$ such that all the ranks and coranks are infinite.

In the following we further apply the tools involving Fredholm indices and essential codimensions to characterize other equivalence relations as well. From now on X can be any of $[0, 1]$, $[0, \infty)$, $(-\infty, \infty)$, or $[0, 1]/\{0, 1\}$.

Proposition 4.10 *Suppose that two projections $\mathbf{p}, \mathbf{q} \in \mathcal{C}(I)$ are represented by local liftings $(p_0, \dots, p_n)$, $(q_0, \dots, q_n)$ respectively. Let $k_i = [p_i(x_i) : p_{i-1}(x_i)]$ and $l_i = [q_i(x_i) : q_{i-1}(x_i)]$ for $i = 1, \dots, n$. Then $\mathbf{p} \sim \mathbf{q}$ in $\mathcal{C}(I)$ if and only if there exist (finite) t_i's such that*

(i) $\operatorname{rank}(p_i(x)) = t_i + \operatorname{rank}(q_i(x))$, $\forall x \in X_i$,
(ii) $t_i - t_{i-1} = k_i - l_i$,
(iii) $t_i = 0$ *if X_i has an infinite endpoint.*

Proof Assume that $\mathbf{p} \sim \mathbf{q}$ in $\mathcal{C}(I)$. As we already observed, we then have a $(u_0, \dots, u_n)$ such that $q_i - u_i u_i^*, p_i - u_i^* u_i \in C(X_i) \otimes K$. We derived (ii) and (iii) in (4.2) and Proposition 4.1, where t_i denotes the Fredholm index of $u_i(x)$ as an operator from $p_i(x)H$ to $q_i(x)H$. If $\operatorname{rank}(p_i(x))$ is finite, the formula (4.1) implies that $\operatorname{rank}(q_i(x))$ is also finite and $\operatorname{Ind}(u_i(x)) = t_i = \operatorname{rank}(p_i(x)) - \operatorname{rank}(q_i(x))$. Similarly, if $\operatorname{rank}(p_i(x))$ is infinite, $\operatorname{rank}(q_i(x))$ is also. Thus $\operatorname{rank}(p_i(x)) = \operatorname{rank}(q_i(x)) + t_i$ holds. Thus (i) is proved.

Conversely if $\operatorname{rank}(p_i(x)) = \operatorname{rank}(q_i(x)) + t_i$ and $t_i \geq 0$ for a particular i, then $\operatorname{rank}(p_i(x)) \geq t_i$, thus there is a norm continuous projection-valued function $p \leq p_i$ such that $\operatorname{rank}(p(x)) = t_i$ by Lemma 3.3. Let $p_i' = p_i - p$, $q_i' = q_i$. And if $t_i < 0$, then $-t_i \leq \operatorname{rank}(q_i(x))$, and we let $q_i' = q_i - q$, $\operatorname{rank} q = -t_i$, and $p_i' = p_i$. Then $(p_0', \dots, p_n')$ and $(q_0', \dots, q_n')$ are local liftings for $\mathbf{p}$ and $\mathbf{q}$, and we have reduced to the case $t_i = 0$.

So there are continuous functions u_i such that $u_i{}^* u_i = p_i'$, $u_i u_i{}^* = q_i'$ by Proposition A.4. It follows that

$$\begin{aligned}&\operatorname{Ind}\big(q_i'(x_i) u_{i-1}(x_i) p_i'(x_i)\big)\\&\quad = -\big[q_i'(x_i) : q_{i-1}'(x_i)\big] + \operatorname{Ind}\big(u_{i-1}(x_i)\big) + \big[p_i'(x_i) : p_{i-1}'(x_i)\big] = 0\end{aligned}$$

and

$$q_i'(x_i) u_{i-1}(x_i) p_i'(x_i) - u_{i-1}(x_i) \in K.$$

Then by Lemma A.6 we can perturb u_i to get u_i' such that $u_i'(x_i) - u_{i-1}'(x_i) \in K$ as in the proof of Theorem 4.6. ■

Remark 4.11 Although we changed local representations in the second part of the above proof, it is not hard to see that, in terms of the given local representations, we recover the original t_i's via the process discussed before and after Proposition 4.1.

Since $\mathbf{p} \sim_u \mathbf{q}$ is equivalent to $\mathbf{p} \sim \mathbf{q}$ and $\mathbf{1} - \mathbf{p} \sim \mathbf{1} - \mathbf{q}$, we get the following statement immediately from Proposition 4.10.

Corollary 4.12 *Suppose that two projections $\mathbf{p}, \mathbf{q} \in \mathcal{C}(I)$ are represented by local liftings $(p_0, \dots, p_n)$, $(q_0, \dots, q_n)$ respectively. Let $k_i = [p_i(x_i) : p_{i-1}(x_i)]$ and $l_i = [q_i(x_i) : q_{i-1}(x_i)]$ for $i = 1, \dots, n$. Then $\mathbf{p} \sim_u \mathbf{q}$ in $\mathcal{C}(I)$ if and only if there exist (finite) t_i's and s_i's such that*

(i) $\operatorname{rank}(1 - p_i(x)) = s_i + \operatorname{rank}(1 - q_i(x))$, $\forall x \in X_i$,
(ii) $\operatorname{rank}(p_i(x)) = t_i + \operatorname{rank}(q_i(x))$, $\forall x \in X_i$,
(iii) $t_i - t_{i-1} = k_i - l_i$,
(iv) $s_i + t_i = s_{i-1} + t_{i-1}$,
(v) *t_i and s_i are zero if X_i has an infinite endpoint.*

In general, $\mathbf{p} \sim_h \mathbf{q}$ in $\mathcal{C}(I)$ if and only if $\mathbf{upu}^* = \mathbf{q}$, where $\mathbf{u}$ is connected to 1 in the unitary group of $\mathcal{C}(I)$. It is said that a (non-unital) C^*-algebra A has *good index theory* if whenever A is embedded as an ideal of a unital C^*-algebra B and u is a unitary in B/A such that $\partial_1([u]) = 0$ in $K_0(A)$, there is a unitary in B that lifts u (see [2, 2–3]). It was proved by G. Nagy [13] that any stable rank one C^*-algebra has good index theory. Since I is of the form $C(X) \otimes K$ where $\dim X \leq 1$, I has stable rank one and I has good index theory. Also, recall that the unitary group of the multiplier algebra of a stable C^*-algebra is path connected (even contractible). Thus in our situation $\mathbf{p} \sim_h \mathbf{q}$ if and only if $\mathbf{upu}^* = \mathbf{q}$ where $\mathbf{u}$ has trivial K_1-class.

Corollary 4.13 *Suppose that two projections $\mathbf{p}, \mathbf{q}$ in $\mathcal{C}(I)$ are represented by local liftings $(p_0, \dots, p_n)$, $(q_0, \dots, q_n)$ respectively. Let $k_i = [p_i(x_i) : p_{i-1}(x_i)]$ and $l_i = [q_i(x_i) : q_{i-1}(x_i)]$ for $i = 1, \dots, n$. Then $\mathbf{p} \sim_h \mathbf{q}$ in $\mathcal{C}(I)$ if and only if there exist (finite) t_i's, s_i's such that*

(i) $\operatorname{rank}(1 - p_i(x)) = s_i + \operatorname{rank}(1 - q_i(x))$, $\forall x \in X_i$,
(ii) $\operatorname{rank}(p_i(x)) = t_i + \operatorname{rank}(q_i(x))$, $\forall x \in X_i$,
(iii) $t_i - t_{i-1} = k_i - l_i$,
(iv) $s_i + t_i = s_{i-1} + t_{i-1} = 0$,
(v) *t_i and s_i are zero if X_i has an infinite endpoint.*

Proof Assume that $\mathbf{p} \sim_h \mathbf{q}$; *i.e.*, $\mathbf{upu}^* = \mathbf{q}$ where $\mathbf{u}$ has trivial K_1-class. Then (i),(ii),(iii), the first equality of (iv), and (v) follow from Corollary 4.12. By good index theory there is a unitary u in $M(I)$ such that $up_iu^* - q_i \in C(X_i) \otimes K$ for each i. Hence $q_i(x)u(x)p_i(x)$ and $(1 - q_i(x))u(x)(1 - p_i(x))$ are Fredholm operators from $p_i(x)H$ to $q_i(x)H$ and from $(1 - p_i(x))H$ to $(1 - q_i(x))H$ respectively. If $t_i = \operatorname{Ind}(q_i(x)u(x)p_i(x))$ and $s_i = \operatorname{Ind}((1 - q_i(x))u(x)(1 - p_i(x))$, then the proof of Proposition 4.1 and the related discussion imply that

$$\begin{aligned} t_i + s_i &= \operatorname{Ind}\big(q_i(x)u_i(x)p_i(x)\big) + \operatorname{Ind}\big((1 - q_i(x))u_i(x)(1 - p_i(x))\big) \\ &= 0. \end{aligned}$$

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For the converse direction we may assume X has no infinite endpoints, since otherwise $K_1(\mathcal{C}(I)) = 0$. Then as in the proof of Proposition 4.10, we construct $\mathbf{v}$,$\mathbf{w}$ in $\mathcal{C}(I)$ such that $\mathbf{v}^*\mathbf{v} = \mathbf{p}$, $\mathbf{v}\mathbf{v}^* = \mathbf{q}$, $\mathbf{w}^*\mathbf{w} = \mathbf{1} - \mathbf{p}$, $\mathbf{w}\mathbf{w}^* = \mathbf{1} - \mathbf{q}$. Then, *cf.* Remark 4.11, $\mathbf{v}$ and $\mathbf{w}$ are given by local representations $(v_0, \dots, v_n)$, $(w_0, \dots, w_n)$, where $v_i(x)$ is a Fredholm operator of index t_i from $p_i(x)H$ to $q_i(x)H$ and $w_i(x)$ is a Fredholm operator of index s_i from $(1 - p_i(x))H$ to $(1 - q_i(x))H$. Now if $\mathbf{u} = \mathbf{v} + \mathbf{w}$, we see that $[\mathbf{u}]_1 = \mathrm{Ind}(v_i + w_i) = 0$ via the map $K_1(\mathcal{C}(I)) \to K_0(I) \to K_0(K) \simeq \mathbb{Z}$, which is induced from evaluation at a point. ■

We conclude this section with some examples, in particular including a projection in $\mathcal{C}(I)$ that does not lift (stably) but whose K_0-class does lift.

Example 4.14 Consider a partition $\{x_1, x_2\}$ that divides X into three subintervals X_0, X_1, X_2. Let $\mathbf{p}$ be given by a local representation (p_0, p_1, p_2), where

$$\mathrm{rank}(p_0(x_1)) = \mathrm{rank}(p_2(x_2) = 1,$$

$$\mathrm{rank}(p_1(x_1)) = \mathrm{rank}(p_1(x_2)) = 0,$$

$$p_0(0) = p_2(1) = 1 \text{ if } X = [0,1]/\{0,1\},$$

$$\exists a \in X_1 \text{ such that } p_1(a) = 1,$$

$$\exists b \in X_0 \text{ such that } p_0(b) = 0$$

The last condition may be omitted if X has an infinite endpoint.

Then $k_1 = -1$, $k_2 = 1$ (and $k_0 = 0$ in the circle case). Thus the K_0-class of $\mathbf{p}$ is 0. Condition (3.1) is equivalent to $l_0 = l_2$ and $l_1 = l_0 + 1$. If X has no infinite endpoint, by the last condition either (3.2) or (3.4)) implies $l_0 \geq 0$, and (3.3) implies $l_1 \leq 0$. Thus $\mathbf{p}$ does not lift, and similar reasoning shows that $\mathbf{p} \oplus \cdots \oplus \mathbf{p}$ does not lift. However, $\mathbf{p} \oplus \mathbf{0}$ is liftable, since we may now take $l_0 = l_2 = 0, l_1 = 1$. If X has no infinite endpoint, then $\mathbf{p} \oplus \mathbf{1}$ is also liftable, since we may then take $l_0 = l_2 = -1, l_1 = 0$. But in all cases $\mathbf{1} - \mathbf{p}$ is an example of a non-liftable projection such that $(\mathbf{1} - \mathbf{p}) \oplus \mathbf{1}$ is liftable.

Example 4.15 (a) There are easy examples to show that $[\mathbf{p}] = [\mathbf{q}]$ does not imply $\mathbf{p} \sim \mathbf{q}$ and that $\mathbf{p} \sim \mathbf{q}$ does not imply $\mathbf{p} \sim_u \mathbf{q}$. For the first take $\mathbf{p} = \mathbf{1}$ and $\mathbf{q} = \mathbf{0}$, and for the second take $\mathbf{p} = \mathbf{1} \oplus \mathbf{1}$ and $\mathbf{q} = \mathbf{1} \oplus \mathbf{0}$, noting that $M_2(\mathcal{C}(I)) \simeq \mathcal{C}(I)$.

(b) We can also illustrate these phenomena with $[\mathbf{p}] = [\mathbf{q}] \neq 0$ using a partition into two subintervals:

For the first, in the $(-\infty, \infty)$-case, let $\mathrm{rank}(p_0(x_1)) = a$, $\mathrm{rank}(p_1(x_1)) = b$, $\mathrm{rank}(q_0(x_1)) = c$, and $\mathrm{rank}(q_1(x_1)) = d$, where $a, b, c, d < \infty$, $b - a = d - c$, but $a \neq c$. In the circle case add the condition $p_0(0) = p_1(1) = q_0(0) = q_1(1) = 0$.

For the second, note that if p_i's and q_i's just above have finite rank everywhere, then $\mathbf{1} - \mathbf{p} \sim \mathbf{1} - \mathbf{q}$ though $\mathbf{p} \nsim \mathbf{q}$.

If X has an infinite endpoint, then $K_1(\mathcal{C}(I)) = 0$, and hence $\mathbf{p} \sim_u \mathbf{q}$ implies $\mathbf{p} \sim_h \mathbf{q}$. The next example shows that this implication does not hold in the other two cases.

Example 4.16 Let $\mathbf{p}$ and $\mathbf{q}$ be given by projection-valued functions defined on all $[0,1]$(the partition is irrelevant here). Take p so that

$$\operatorname{rank}\big(p(1/2)\big) = 1,\ p(0) = p(1) = 1,\ \text{and}\ \operatorname{rank}\big(p(x)) = \operatorname{rank}((1-p)(x)\big) = \infty$$

for $x \neq 0, 1/2, 1$. Similarly, $\operatorname{rank}(q(1/2)) = 0$, $q(0) = q(1) = 1$, and $\operatorname{rank}(q(x)) = \operatorname{rank}((1-q)(x)) = \infty$ for $x \neq 0, 1/2, 1$. Then $\mathbf{p} \sim \mathbf{q}$ and all t_i's must be 1, and $\mathbf{1} - \mathbf{p} \sim \mathbf{1} - \mathbf{q}$ and all s_i's must be 0.

A Appendix

In this section, we show some results about continuous fields of Hilbert spaces that were used implicitly or explicitly in this article. We refer the reader to [7, 8] for a complete introduction to this notion. Recall that a continuous field of Hilbert spaces over a topological space X is a family of Hilbert spaces $(H_x)_{x\in X}$ together with a vector space Γ consisting of vector sections ξ in the product space $\prod_{x\in X} H_x$ satisfying the following two conditions:

(1) The norm $x \to \|\xi(x)\|$ is continuous on X for each $\xi \in \Gamma$.
(2) The set $\{\xi(x) \mid \xi \in \Gamma\}$ is norm dense in H_x for each $x \in X$.

If H_x is the same Hilbert space H for every x, and Γ consists of all continuous mappings of X into H, $\mathcal{H}$ is called a constant field. A field (isometrically) isomorphic to a constant field is said to be trivial. If $\mathcal{H}^{'} = ((H_x^{'})_{x\in X}, \Gamma^{'})$ is a continuous filed of Hilbert spaces over T, $H_x^{'}$ is a closed subspace of H_x for each t, and $\Gamma^{'} \subset \Gamma$, then $\mathcal{H}^{'}$ is called a subfield of $\mathcal{H}$. Furthermore, we call $\mathcal{H}^{'} = ((H_x^{'})_{x\in X}, \Gamma^{'})$ a complemented subfield of $\mathcal{H} = ((H_x)_{x\in X}, \Gamma)$ if there is a subfield $\mathcal{H}^{''} = ((H_x^{''})_{x\in X}, \Gamma^{''})$ such that $H_x^{'} \oplus H_x^{''} = H_x$ for every x. Also, we say that $\mathcal{H}$ is separable if Γ has a countable subset Λ such that $\{\xi(x) \mid \xi \in \Lambda\}$ is dense in H_x for each x. As we already mentioned, complemented subfields of the constant field over X are in one-to-one correspondence with strongly continuous projection-valued functions from X to $B(H)$.

Theorem A.1 ([7, Lemma 10.8.7]) *If X is paracompact and of finite dimension, every separable continuous field $\mathcal{H} = ((H_x)_{x\in X}, \Gamma)$ of Hilbert spaces over X such that $\dim(H_x) = \infty$ for every x is trivial. Thus two continuous fields $\mathcal{H}$, $\mathcal{H}'$ of Hilbert spaces over X such that $\dim H_x = \dim H_x' = \infty$ are isomorphic.*

Theorem A.2 ([8]) *Let X be a paracompact space and let $\mathcal{H}$ be a separable continuous field of Hilbert spaces over X. Then it is isomorphic to a complemented subfield of a trivial field, and thus is isomorphic to a continuous field defined by a strongly continuous projection-valued function $p\colon X \mapsto B(H)$.*

Given any continuous field $\mathcal{H} = ((H_x)_{x\in X}, \Gamma)$ and any closed subset A of X, there is a continuous subfield $\mathcal{H}^0$ such that

$$H_x^0 = \begin{cases} H_x, & x \notin A, \\ 0, & x \in A. \end{cases}$$

Lemma A.3 and Proposition A.4 may be known by experts and could be deduced from Robert and Tikuisis [15, Theorem 4.3], but we shall give our proofs of these for the convenience of readers. We denote by dim the covering dimension of a topological space.

Lemma A.3 *If X is a separable metric space such that* $\dim X \leq 1$, *if $\mathcal{H}$ is a continuous field of Hilbert spaces over X such that $H_x \neq 0$ for every $x \in X$, if f is a continuous section of $\mathcal{H}$, and if $\epsilon > 0$, then there is a continuous section g such that $\|g(x) - f(x)\| < \epsilon$ and $g(x) \neq 0$ for every $x \in X$.*

Proof There is a countable open cover $\{U_n\}$ of X such that each $\mathcal{H}|_{U_n}$ has a non-vanishing section. By paracompactness of X there is a closed, locally finite refinement, $\{F_n\}$, for $\{U_n\}$. Then for each n we have

$$\mathcal{H}|_{F_n} = \mathcal{L}_n \oplus \mathcal{K}_n, \tag{A.1}$$

where $\mathcal{L}_n$ is a trivial subfield of rank one. Choose a strictly increasing sequence $\{\epsilon_n\}$ of positive numbers such that $\epsilon_n < \epsilon$. We will recursively construct compatible sections g_n over F_n such that $g_n(x) \neq 0$ and $\|g_n(x) - f(x)\| \leq \epsilon_n$ for every $x \in F_n$. The local finiteness ensures that the resulting global section g is continuous on X.
To construct g_n, let $A = F_n \cap (\bigcup_{k=1}^{n-1} F_k)(A = \varnothing$ if $n = 1)$. We have a non-vanishing section g' on A such that $\|g'(x) - f(x)\| \leq \epsilon_{n-1}$ for every $x \in A$, and we wish to extend g' to F_n. Choose ϵ' and ϵ'' so that $\epsilon_{n-1} < \epsilon' < \epsilon'' < \epsilon_n$, and write $f = f^1 \oplus f^2$, $g' = g^1 \oplus g^2$ relative to the decomposition (A.1).

We first extend g^2 to all of F_n so that $\|g^2(x) - f^2(x)\| \leq \epsilon'$ for every $x \in X$. To do this, let h be an arbitrary extension of g^2 to F_n, which exists by [8, Proposition 7]. Then let $B = \{x \in F_n \mid \|h(x) - f^2\| \geq \epsilon'\}$, and note that $A \cap B = \varnothing$. Let $\phi\colon F_n \mapsto [0,1]$ be a continuous function such that $\phi|_B = 1$ and $\phi|_A = 0$, and take $g^2 = \phi f^2 + (1 - \phi)h$.
Next, we extend g^1 to a section k on all of F_n so that $\|k(x) - f^1(x)\|^2 + \|g^2(x) - f^2(x)\|^2 \leq (\epsilon'')^2$. It will be convenient to identify sections of $\mathcal{L}_n$ with complex valued functions and define

$$\phi(s,z) = \begin{cases} z, & \text{if } |z| \leq s, \\ s\frac{z}{|z|}, & \text{if } |z| > s, \end{cases}$$

for $s > 0$ and $z \in \mathbb{C}$. Thus ϕ is continuous on $(0,\infty) \times \mathbb{C}$. Now extend the function $g^1 - f^1$ to l on F_n and let $k(x) = f^1(x) + \phi(\sigma(x), l(x))$, where

$$\sigma(x) = ((\epsilon'')^2 - \|g^2(x) - f^2(x)\|^2)^{1/2}.$$

Finally, we must modify k to obtain the non-vanishing property without changing $k|_A$. Let

$$C = \{x \in F_n \mid g^2(x) = 0\}, \quad D = \{x \in C \mid k(x) = 0\}, \quad \text{and} \quad \delta = \frac{\epsilon_n - \epsilon''}{2}.$$

Since $D \cap A = \varnothing$, there is an open neighborhood V of D such that $\overline{V} \cap A = \varnothing$. Let $G = \overline{V} \cap C$ and $E = ((\overline{V} \setminus V) \cap C) \cup \{x \in G \mid \|k(x)\| = \delta\}$. By dimension theory, there is a non-vanishing continuous function r on G such that $r|_E = k|_E$. Then define $g^1|_C$ by

$$g^1(x) = \begin{cases} \phi(\delta, r(x)) & \text{if } x \in V \cap C \text{ and } \|k(x)\| < \delta, \\ k(x) & \text{otherwise on } C. \end{cases}$$

Note that if $x_n \in V \cap C$, $\|k(x_n)\| < \delta$ and if $x_n \to x$ for some x not satisfying these conditions, then $r(x) = k(x)$ and $\|k(x)\| \le \delta$. Thus

$$g^1(x_n) \to \phi(\delta, r(x)) = k(x)$$

This implies that g^1 is continuous on C and $\|g^1 - k\| < 2\delta$.

Now g^1 is defined on $C \cup A$, and we extend g^1 to F_n so that $\|g^1(x) - k(x)\| \le 2\delta$ for every x in F_n. This can be done as in the previous paragraph. Then $g_n = g^1 \oplus g^2$ satisfies all required properties. ■

Proposition A.4 *If X is a separable metric space such that* $\dim X \le 1$, *and if $\mathcal{H}$ and $\mathcal{K}$ are separable continuous fields of Hilbert spaces over X such that* $\dim H_x = \dim K_x$ *for every $x \in X$, then $\mathcal{H} \cong \mathcal{K}$.*

Proof For $n = 1, 2, \dots$, let $U_n = \{x \in X \mid \dim H_x \ge n\}$, an open set. Let $\mathcal{L}_n$ be a trivial line bundle over U_n, extended by zero as to be a continuous field over X. Thus continuous sections of $\mathcal{L}_n$ can be identified with continuous complex-valued functions on X that vanish on $X \smallsetminus U_n$. We will show that $\mathcal{H} \cong \bigoplus_1^\infty \mathcal{L}_n$. Since the same argument applies to $\mathcal{K}$, the result follows.

To do this, we construct recursively a sequence $\{e_n\}$ such that e_n is a continuous section of $\mathcal{H}|_{U_n}$, such that

$$\|e_n(x)\| = 1 \quad \text{for every } x \in U_n, \langle e_n(x), e_m(x)\rangle = 0 \quad \text{if } n < m \text{ and } x \in U_m.$$

We will impose additional conditions on the e_n's, but first we point out that any such e_n's give rise to complemented subfields $\mathcal{M}_n$, where $(\mathcal{M}_n)_x = \operatorname{span}(e_n(x))$. Moreover if we write $\mathcal{H} = \mathcal{H}'_n \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$, then $\dim(H'_n)_x = \max(\dim H_x - n, 0)$. It will be enough to consider the case $n = 1$. Note that for any continuous section f of $\mathcal{H}$ the function c_1 defined by

$$c_1(x) = \begin{cases} \langle f(x), e_1(x)\rangle, & x \in U_1 \\ 0, & x \notin U_1 \end{cases}$$

is continuous, since $|\langle f(x), e_1(x)\rangle| \le \|f(x)\|$ and f vanishes on $X \setminus U_1$. Thus $g = f - c_1 e_1$ may be regarded as a continuous section of $\mathcal{H}$ such that $\|g(x)\|^2 = \|f(x)\|^2 - \|c_1(x)\|^2$. This implies that $\mathcal{H}'_1 = \mathcal{M}_1^\perp$ is indeed a continuous subfield of $\mathcal{H}$. The dimension formula given above is obvious, and it follows that $\{x \mid \dim(\mathcal{H}'_n)_x \ge 1\} = U_{n+1}$. Now we proceed by induction on n, working with $\mathcal{H}'_1$ instead of $\mathcal{H}$.

Let $\{f_m\}$ be a sequence of continuous sections of $\mathcal{H}$ such that $\mathcal{H}_x = \overline{\operatorname{span}(f_m(x))}$ for each x. Let $\{g_n\}$ be a sequence that includes each f_m infinitely many times. We choose the e_n's so that for each n and x, the projection of $g_n(x)$ on $(\mathcal{H}'_n)_x$ has norm at most $1/n$. If this is so, then $\mathcal{H} \cong \bigoplus \mathcal{M}_n$, and since $\mathcal{M}_n \cong \mathcal{L}_n$, the result follows.

Assume that e_k has already been constructed for $k < n$. Let h be the $\mathcal{H}'_{n-1}$ component of g_n. Apply Lemma A.3 to $\mathcal{H}'_{n-1}|_{U_n}$ and $h|_{U_n}$ with $\epsilon = 1/n$. (Recall that $U_n = \{x \mid (H'_{n-1})_x \neq 0\}$.) Thus we obtain a non-vanishing section l on U_n, such that $\|h(x) - l(x)\| \leq 1/n$ for every x. If $e_n(x) = l(x)/\|l(x)\|$, then e_n satisfies all our requirements. ■

Corollary A.5 *If X is a separable metric space such that $\dim(X) \leq 1$ and $\mathcal{H}$ is a continuous field of Hilbert spaces over X such that $\dim(H_x) \geq n$ for every x in X, then $\mathcal{H}$ has a trivial subfield of rank n. Equivalently, if p is a strongly continuous projection-valued function on X such that $\operatorname{rank}(p(x)) \geq n$ for every x in X, then there is a norm continuous projection-valued function q such that $q \leq p$ and $\operatorname{rank}(q(x)) = n$ for every $x \in X$.*

Proof This follows from the above proof. The required subfield is $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$. ■

If $\mathcal{H}$ is a trivial field over an interval $[a, b]$, for any unitary u on H_a we always have a continuous path of unitaries connecting u and 1 over $[a, b]$. But the triviality condition of a given continuous field can be loosened as follows. Here again we do not claim any originality.

Lemma A.6 *Let $\mathcal{H}$ be a separable continuous field of Hilbert spaces over an interval $[a, b]$ and u a unitary operator on H_a. Then there is a unitary endomorphism v of $\mathcal{H}$ such that $v(a) = u$ and $v(b) = 1$.*

Proof There is a self-adjoint h in $B(H_a)$ such that $e^{ih} = u$. Let $\mathcal{H}'$ be a trivial infinite rank continuous field on $[a, b]$ and $\mathcal{H}'' = \mathcal{H} \oplus \mathcal{H}'$, so that $\mathcal{H}''$ is also trivial. Then there is a self-adjoint endomorphism k of $\mathcal{H}''$ such that $k(a) = h \oplus 0$. Let p be the obvious projection from $\mathcal{H}''$ to $\mathcal{H}$, and let $\widetilde{h} = pkp$, regarded as an endomorphism of $\mathcal{H}$. Then let $v = e^{i\phi\widetilde{h}}$, where ϕ is a continuous scalar function such that $\phi(a) = 1$ and $\phi(b) = 0$. ■

The following fact, which is needed for examples in Section 4, may be known to experts, but we could not locate any reference, so we give our proof.

Proposition A.7 *Let ϕ and ψ be lower semi-continuous functions on X, taking values in $\{0, 1, \ldots, n, \ldots, \infty\}$. Suppose that $\phi(x) + \psi(x)$ is infinite everywhere. Then there exists a strongly continuous projection-valued function p on X such that for any x in X $\operatorname{rank}(p(x)) = \phi(x)$ and $\operatorname{rank}((1 - p)(x)) = \psi(x)$.*

Proof For $n = 1, \ldots,$ let $U_n = \{x \mid \phi(x) \geq n\}$ and $V_n = \{x \mid \psi(x) \geq n\}$. Then the U_n's and V_n's are open sets. Let $\mathcal{L}_n$ be as in the proof of Proposition A.4, and $\mathcal{L} = \oplus_1^\infty \mathcal{L}_n$. Similarly define $\mathcal{M}_n$ and $\mathcal{M}$ using the V_n's. Then $\mathcal{L} \oplus \mathcal{M}$ has infinite rank everywhere and hence is isomorphic to H^∞. Thus $\mathcal{L}$ may be regarded as a complemented subfield of H^∞ and is therefore given by a projection-valued function with the required properties. ■

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Department of Mathematics, Purdue University, West Lafayette, USA 47907
e-mail: lgb@math.purdue.edu

Department of Mathematics, University of Ulsan, Ulsan, Korea 680-749
e-mail: hadamard@ulsan.ac.kr