Can. J. Math., Vol. XL, No. 5, 1988, pp. 1144-1155

SOME REMARKS ON IA AUTOMORPHISMS OF FREE GROUPS

J. McCOOL

1. Introduction. Let A_n be the automorphism group of the free group F_n of rank n, and let K_n be the normal subgroup of A_n consisting of those elements which induce the identity automorphism in the commutator quotient group F_n/F_n'. The group K_n has been called the group of IA automorphisms of F_n (see e.g. [**1**]). It was shown by Magnus [**7**] using earlier work of Nielsen [**11**] that K_n is finitely generated, with generating set the automorphisms

$$x_{ij}:x_i \rightarrow x_j x_i \bar{x}_j \qquad (i \neq j)$$
$$x_k \rightarrow x_k \qquad (k \neq i),$$

and

$$x_{ijk}:x_i \rightarrow x_i x_j x_k \bar{x}_j \bar{x}_k \qquad (i \neq j < k \neq i)$$
$$x_m \rightarrow x_m \qquad (m \neq i),$$

where $x_1, x_2, \dots, x_n$ is a chosen basis of F_n.

A presentation of the subgroup C_n of K_n generated by the x_{ij} was found in [**10**]; the case $n = 3$ is given already in [**4**] and [**5**]. In [**4**] Chein also found a (rather awkward) presentation for $K_3(1)$, where $K_n(1)$ denotes the intersection in A_n of K_n with the subgroup $S(x_2, \dots, x_n)$ consisting of those automorphisms which fix each of $x_2, \dots, x_n$. In particular, Chein showed that $K_3(1)$ is generated by the set $\{x_{12}, x_{13}, x_{123}\}$. The first result we wish to report in the present paper is a description of $K_n(1)$ for all $n \geqq 3$, namely

THEOREM 1. *Let Y, Z be free groups of rank $n - 1$, with bases $y_2, \dots, y_n$ and $z_2, \dots, z_n$ respectively, and let θ be the homomorphism of the direct product $Y \times Z$ onto the free abelian group with basis $a_2, \dots, a_n$ given by $\theta(y_i) = a_i$ and $\theta z_i = \bar{a}_i$ $(2 \leqq i \leqq n)$. Then*

(a) *$K_n(1)$ is isomorphic to the kernel of θ,*
and

(b) *$K_n(1)$ is finitely generated (by the set of all x_{1j} and x_{1jk}), but is not finitely presentable.*

Received September 30, 1987.

Here in fact y_i represents the automorphism which send x_1 to x_1x_i and fixes all other x_t, while z_j maps x_1 to $\bar{x}_jx_1$ and fixes the other x_t, so that

$$\bar{x}_{1i} = y_iz_i \quad \text{and} \quad x_{1jk} = \bar{y}_k\bar{y}_jy_ky_j$$

(our convention being that automorphisms of F_n are applied on the right).

The theorem gives a reasonable description of the structure of $K_n(1)$, namely that $K_n(1)$ is the semidirect product of the commutator subgroup Y' of the free group Y, by the free group on $x_{12}, \dots, x_{1n}$, where each x_{1j} acts on Y' just as the corresponding y_j. In the case $n = 3$, the group has a simple presentation:

COROLLARY 1. *The group* $K_3(1)$ *has presentation*

$$\langle a_n, b_m \ (n, m \in \mathbf{Z});\ a_nb_m = b_{m-1}a_{n+1} \ (n, m \in Z) \rangle.$$

Let us write $S(x_2^0, \dots, x_n^0)$ for the elements of A_n which fix each of the conjugacy classes x_i^0 $(2 \leqq i \leqq n)$, and $K_n^0(1)$ for the intersection of K_n and $S(x_2^0, \dots, x_n^0)$. Then, denoting by I_n the group of inner automorphisms, we have

THEOREM 2.

(a) $K_n^0(1)$ *is generated by the set of all* x_{ij} *and* x_{1jk}.

(b) $K_n^0(1)$ *is not* f.p.

(c) *The quotient* $K_3^0(1)/I_3$ *is the free product of* $K_3(1)$ *and the infinite cycle generated by* x_{31}.

We note that $K_3(1)$ embeds in the quotient $K_3^0(1)/I_3$, since its intersection with I_3 is trivial. Now $K_3^0 = K_3/I_3$ is generated by (the image of) the set $V = x_{12}, x_{23}, x_{31}, x_{123}, x_{213}, x_{312}$, and Theorems 1 and 2 enable us to describe the relations satisfied by any subset of V containing just one of the x_{ijk}. Also, it has been shown by Bachmuth [**1**] that the subgroup T_3 of K_3^0 generated by $x_{123}, x_{213}, x_{312}$ is free of rank three. It could be asked therefore if we have obtained enough relations to present K_3^0 on the generating set V. We shall show later that this is not the case, and then make use of our result to disprove a conjecture of Chein [**4**]. The conjecture, which is repeated as a question in problem 5 of [**2**], is to the effect that the normal closure N of C_3 in K_3 has trivial intersection with the subgroup T_3. In view of the result of Bachmuth cited above, this is equivalent to the assertion that the quotient group K_3/N is isomorphic to F_3. We show

THEOREM 3. *The group* K_3/N *is a quotient of the group* L *with presentation*

$$L = \langle x, y, z;\ [yx, x^ry^r] = [zy, y^rz^r] = [xz, z^rx^r] = 1 \quad (r \in \mathbf{Z}) \rangle.$$

The group L *is not* f.p.

https://doi.org/10.4153/CJM-1988-047-4 Published online by Cambridge University Press

2. Presentations of $S(x_2, \dots, x_n)$ and $S(x_2^0, \dots, x_n^0)$. We shall need, in order to obtain our main results, presentations of $S(x_2, \dots, x_n)$ and $S(x_2^0, \dots, x_n^0)$. These are given in the following results, whose proofs will be given later:

PROPOSITION A. *$S(x_2, \dots, x_n)$ has presentation with*

$$\textit{generators}: \quad \tau, y_i, z_i \quad (2 \leqq i \leqq n)$$

and

$$\textit{relations}: \tau^2 = 1, \tau y_i \tau = z_i, [y_i, z_j] = 1 \quad (2 \leqq i, j \leqq n).$$

Here τ is the automorphism sending x_1 to $\bar{x}_1$ and fixing the other x_t, while y_i, z_i are as described previously. We note that $S(x_2, \dots, x_n)$ is the semidirect product of $Y \times Z$ by the two-cycle τ.

Next we have

PROPOSITION B. *$S(x_2^0, \dots, x_n^0)$ has presentation T_n with*

$$\textit{generators}: \quad \tau, y_i, z_j, x_{rs} \quad (2 \leqq i, j \leqq n, 1 \leqq r \neq s \leqq n)$$

and

relations:

$$\left.\begin{array}{ll} y_j z_j = x_{1j}, [y_j, z_j] = 1 & \\ {[x_{ij}, x_{kj}] = 1,} & \\ {[y_j, x_{ij}] = 1, [z_j, x_{ij}] = 1} & \end{array}\right\} Q2$$

$$\left.\begin{array}{ll} {[y_i, z_j] = 1} & (i \neq j) \\ {[x_{ij}, x_{rs}] = 1} & (i, j, r, s \text{ distinct}) \\ {[y_j, x_{rs}] = 1, [z_j, x_{rs}] = 1} & (1, j \neq r, s) \end{array}\right\} Q3$$

$$x_{i1} y_s \bar{x}_{i1} = y_s x_{is}, \bar{x}_{i1} z_s x_{i1} = z_s x_{is} \quad (i \neq s) \quad \} Q4$$

$$\tau y_i \tau = z_i, \tau x_{ij} \tau = x_{ij} \quad (j > 1), \tau x_{k1} \tau = \bar{x}_{k1} \quad \} Q6$$

$$\tau^2 = 1 \quad \} Q7$$

$$x_{sj} y_s \bar{x}_{sj} = \bar{y}_j y_s y_j, x_{sj} z_s \bar{x}_{sj} = \bar{z}_j z_s z_j \quad (j \neq 1, s \neq j) \quad \} Q9$$

$$y_s x_{s1} \bar{y}_s = x_{s1} \bar{x}_{1s}, z_s x_{s1} \bar{z}_s = x_{1s} x_{s1} \quad \} Q10.$$

The presentation given has a number of redundancies, which occur naturally in the course of the proof. We note that the presentation exhibits $S(x_2^0, \dots, x_n^0)$ as the semidirect product of the subgroup $S^+(x_2^0, \dots, x_n^0)$ generated by the y_i, z_j and x_{rs}, by the cycle τ, and that a presentation of $S^+(x_2^0, \dots, x_n^0)$ is obtained from the above merely by deleting the generator τ and the relations $Q6$ and $Q7$.

3. Proof of theorem 1. To prove Theorem 1, we note that if

$$g = v(y_2, \dots, y_n)w(z_2, \dots, z_n)$$

is any element of $Y \times Z$, then

$$x_1 g = w(\bar{x}_2, \dots, \bar{x}_n)x_1\tilde{v}(x_2, \dots, x_n),$$

where if $v(y_2, \dots, y_n) = y_{i_1}^{\epsilon_1} \dots y_{i_k}^{\epsilon_k}$ then $\tilde{v}$ is the reverse word $y_{i_k}^{\epsilon_k} \dots y_{i_1}^{\epsilon_1}$. Since $x_1 g\tau$ is $x_1 g$ with x_1 replaced by $\bar{x}_1$, it follows that $S(x_2, \dots, x_n \cap K_n$ consists of those $g = vw$ in $Y \times Z$ as above such that, for each $i(2 \leqq i \leqq n)$, the exponent sum of z_i in w is equal to the exponent sum of y_i in v. This is precisely the kernel of the homomorphism θ described in the theorem, and hence part (a) has been established.

To show that $K_n(1)$ is the subgroup H (say) generated by the $x_{1i} = y_i z_i$ together with the $x_{1jk} = [\bar{y}_k, \bar{y}_j]$, we note that x_{1i} acts on Y just as y_i, so that clearly H contains Y'. Now the subgroup generated by the x_{1i} and Y' contains Z' also. Hence H is a normal subgroup of $Y \times Z$ contained in $K_n(1)$, and with the same quotient group as $K_n(1)$. It follows that $H = K_n(1)$, proving the first statement in part (b) of the theorem. The discussion of this paragraph also substantiates the remark that $K_n(1)$ is the semidirect product of Y' by the x_{1i}.

To prove that $K_n(1)$ is not f.p., we may apply the result of Bieri (see e.g. [**3**], p. 118) that if N is a f.p. normal subgroup of a finitely generated group G of cohomological dimension two, then either N is free or N is of finite index in G. Since $K_n(1)$ is clearly not free, and not of finite index in $Y \times Z$, it is not f.p.

To prove Corollary 1, we exploit the fact that when $n = 3$ the homomorphism θ splits, with e.g. the subgroup generated by y_2 and z_3 being a splitting subgroup. Thus we have the standard presentation

$$\langle y_2, y_3, z_2, z_3; [y_i, z_j] = 1 \quad (2 \leqq i, j \leqq 3) \rangle$$

of $Y \times Z$. We now add the generators a_0, b_0, where $a_0 = y_2 z_2$ and $b_0 = y_3 z_3$, and delete the generators y_3, z_2 to obtain the presentation

$$\langle y_2, z_3, a_0, b_0; [y_2, z_3] = [y_2, a_0] = [z_3, b_0] = [a_0\bar{y}_2, b_0\bar{z}_3] = 1 \rangle.$$

Now if we define $a_n = z_3^{-n} a_0 z_3^n$ and $b_n = y_2^{-n} b_0 y_2^n$ then the relation

$$[a_0\bar{y}_2, b_0\bar{z}_3] = 1$$

can be rewritten as $a_0 b_1 = b_0 a_1$, and conjugation of this by $y_2^n z_3^m$ yields $a_m b_{n+1} = b_n a_{m+1}$. Thus $Y \times Z$ has presentation

generators: $y_2, z_3, a_n, b_n \qquad (n \in Z)$

relations:
$$[y_2, z_3] = [y_2, a_n] = [z_3, b_n] = 1$$
$$\bar{z}_3 a_n z_3 = a_{n+1}, \ \bar{y}_2 b_n y_2 = b_{n+1}$$
$$a_n b_m = b_{m-1} a_{n+1} \qquad (n, m \in Z).$$

https://doi.org/10.4153/CJM-1988-047-4 Published online by Cambridge University Press

This exhibits $Y \times Z$ as the semidirect product of the group H with generators a_n, b_n and defining relations $a_n b_m = b_{m-1} a_{n+1}$ $(n, m \in Z)$, by the free abelian group on y_2, z_3. Since a_0, $b_0 \in K_3(1)$, it is clear that $H = K_3(1)$. This proves the corollary.

We note that in H we have $a_n = b_0^{-n} a_0 b_1^n$ and $b_n = a_0^{-n} b_0 a_1^n$. It is not difficult to show that H can be presented on a_0 and the b_m by

$$\langle a_0, b_m \ (m \in z);\ a_0 b_1^n b_m b_1^{-(n+1)} \bar{a}_0 = b_0^n b_{m-1} b_0^{-(n+1)} \quad (n, m \in Z) \rangle,$$

and from this a presentation on the generators a_0, b_0, b_1 can be obtained. The fact that the above presentation is an HNN-extension of the free group on the b_n can be used to give an easy direct proof of the fact that $K_3(1)$ is not f.p.

4. Proof of theorem 2. It is clear that $K_n^0(1)$ is contained in the subgroup $S^+(x_2^0, \dots, x_n^0)$ of $S(x_2^0, \dots, x_n^0)$, and that the x_{ij} and x_{1jk} are in $K_n^0(1)$. It now follows that $K_n^0(1)$ contains the subgroup L of $S^+(x_2^0, \dots, x_n^0)$ generated by the x_{rs}, Y' and (therefore) Z'. We show that L is a normal subgroup of $S^+(x_2^0, \dots, x_n^0)$. Since $S^+(x_2^0, \dots, x_n^0)$ is generated by the x_{rs} and y_j, it is enough to show that L is closed under conjugation by the $y_j^{\pm 1}$. Now the following relations are obtained easily from the indicated relations of Proposition B:

$$y_j x_{1s} \bar{y}_j = [y_j, y_s] x_{1s} \qquad \text{(from } Q2)$$

$$y_j x_{rs} \bar{y}_j = x_{rs} \quad \text{if } 1, j \neq r, s \qquad (Q3)$$

$$y_j x_{r1} \bar{y}_j = \bar{x}_{rj} x_{r1} \quad \text{if } j \neq r \qquad (Q4)$$

$$y_j x_{js} \bar{y}_j = [y_j, \bar{y}_s] x_{js} \quad \text{if } s \neq 1, j \neq s \qquad (Q9)$$

$$y_j x_{j1} \bar{y}_j = x_{j1} \bar{x}_{1j} \qquad (Q10),$$

and the desired result follows. Thus L is a normal subgroup, and the corresponding quotient group is obviously free abelian of rank $n - 1$. Since this is also the quotient of $S^+(x_2^0, \dots, x_n^0)$ by $K_n^0(1)$, it follows that $L = K_n^0(1)$, and this proves part (a) of Theorem 2.

To prove that $K_n^0(1)$ is not f.p., we note that the natural homomorphism from F_n to F_{n-1} with kernel the normal closure of x_n induces a homomorphism Ψ_n from $K_n^0(1)$ to $K_{n-1}^0(1)$, and that each x_{jn} and x_{nj} is in ker Ψ_n, as is each x_{1nj} and x_{1jn} $(1 \leqq j \leqq n - 1)$. Now the remaining x_{rs} and x_{1rs} generate $K_{n-1}^0(1)$, so that clearly ker Ψ_n is the normal closure in K_n^0 of the finite set of x_{jn}, x_{nj}, x_{1nj} and x_{1jn} $(1 \leqq j \leqq n - 1)$. Hence it will follow that $K_n^0(1)$ is not f.p. provided this is true when $n = 3$. Thus part (b) of the theorem will follow once we have established part (c).

https://doi.org/10.4153/CJM-1988-047-4 Published online by Cambridge University Press

We now take the presentation of $S^+(x_2^0, x_3^0)$ obtained from the presentation of Proposition B (with $n = 3$) by deleting the generator τ and the relations $Q6$ and $Q7$. To this presentation we add the relations

$$x_{21}x_{31} = x_{12}x_{32} = x_{13}x_{23} = 1$$

in order to factor out the group I_3 of inner automorphisms. If we then eliminate x_{21}, x_{32} and x_{23} using the above relations, we obtain the following presentation of $S^+(x_2^0, x_3^0)/I_3$:

generators: $y_2, y_3, z_2, z_3, x_{12}, x_{13}, x_{31}$

and

$$\text{relations:} \quad [y_i, z_j] = 1, \ y_iz_j = x_{1j} \qquad (2 \leqq i, j \leqq 3)$$
$$x_{31}y_2\bar{x}_{31} = y_2\bar{x}_{12}, \ \bar{x}_{31}z_2x_{31} = z_2\bar{x}_{12}$$
$$\bar{x}_{31}y_3x_{31} = y_3\bar{x}_{13}, \ x_{31}z_3\bar{x}_{31} = z_3\bar{x}_{13}.$$

Here the first line of relations comes from the first lines of $Q2$ and $Q3$; the remaining lines of $Q2$ and $Q3$ are superfluous. The second line above arises from $Q4$ with $i = 3$ and $s = 2$, while the third line arises from $Q4$ with $i = 2$, $s = 3$, and x_{21} replaced by $\bar{x}_{31}$. This yields all $Q4$ relations. The relations from $Q9$ and $Q10$ are easily seen to be superfluous.

We note that the presentation obtained exhibits $S^+(x_2^0, x_3^0)/I_3$ as an HNN-extension with base $Y \times Z$ and stable letter x_{31}, where $x_{31}y_2\bar{x}_{31} = \bar{z}_2$ and $x_{31}z_3\bar{x}_{31} = \bar{y}_3$; i.e., the 'associated subgroups' are the (free abelian) groups $\langle y_2, z_3\rangle$ and $\langle \bar{z}_2, \bar{y}_3\rangle$. In terms of the presentation of $Y \times Z$ on the generating set y_2, z_3, a_n, b_n ($n \in \mathbf{Z}$) which we obtained in section 3, we can describe $S^+(x_2^0, x_3^0)/I_3$ as having the following presentation:

generators: $y_2, z_3, a_n, b_n, x_{31} \qquad (n \in Z)$

and

$$\text{relations:} \quad [a_n, y_2] = [b_n, z_3] = [y_2, z_3] = 1$$
$$a_nb_m = b_{m-1}a_{n+1}$$
$$\bar{z}_3a_nz_3 = a_{n+1}, \quad \bar{y}_2b_ny_2 = b_{n+1}$$
$$\bar{z}_3x_{31}z_3 = \bar{b}_0x_{31}, \quad \bar{y}_2x_{31}y_2 = \bar{a}_0x_{31} \quad (n, m \in Z).$$

This exhibits $S^+(x_2^0, x_3^0)/I_3$ as the semidirect product of the free product $\langle x_{31}\rangle * K_3(1)$ by the free abelian group on y_2, z_3. Thus $K_3^0(1)/I_3$ is the free product of $K_3(1)$ and the cycle generated by x_{31}, as claimed.

5. Proof of theorem 3. Let us write y_{ij} for the element of A_n which maps x_i to x_ix_j and fixes the other x_t's, and z_{ij} for the element sending x_i to $\bar{x}_jx_i$

and fixing the remaining x_t's (so that our previous y_i, z_i are now denoted by y_{1i}, z_{1i} respectively). We have

$$a_n = z_{13}^{-n} x_{12} z_{13}^n, \quad b_n = y_{12}^{-n} x_{13} y_{12}^n.$$

We now define elements c_n, d_n of $K_3^0(2)$, and elements e_n, f_n of $K_3^0(3)$ by

$$c_n = z_{21}^{-n} x_{23} z_{21}^n, \quad d_n = y_{23}^{-n} x_{21} y_{23}^n$$

and

$$e_n = z_{23}^{-n} x_{31} z_{32}^n, \quad f_n = y_{31}^{-n} x_{32} y_{31}^n.$$

Now $b_0c_0 = d_0e_0 = f_0a_0 = 1$ in K_3/I_3, and it follows from Theorem 1 that K_3/I_3 is a quotient of the group $\hat{K}_3$ with presentation

generators: $a_n, b_n, c_n, d_n, e_n, f_n$

and

relations: $b_0c_0 = d_0e_0 = f_0a_0 = 1$

$$a_nb_m = b_{m-1}a_{n+1},\ c_nd_m = d_{m-1}c_{n+1},$$
$$e_nf_m = f_{m-1}e_{n+1} \qquad (n, m \in Z).$$

We shall show that K_3/I_3 is a proper quotient of $\hat{K}_3$. For this purpose, we use the following table

	$\mathbf{a_n}$	$\mathbf{b_n}$	$\mathbf{c_0}$	$\mathbf{d_0}$	$\mathbf{d_1}$
y_{12}	a_n	b_{n-1}	$\bar{b}_{-1}$	$d_0\bar{a}_0$	$\bar{b}_0 d_1 \bar{a}_0 \bar{b}_{-1}$
ρ	$\bar{a}_n$	b_{-n}	c_0	d_0	$\bar{c}_0 d_{-1} c_0$
θ	$\bar{a}_n$	b_{1-n}	$\bar{b}_1$	$d_0 a_0$	$d_{-1} c_0 a_0 \bar{b}_1$

The entries of the table are elements of A_3/I_3, where ρ is the element taking x_2 to $\bar{x}_2$ and fixing x_1 and x_3, and $\theta = \rho y_{12}$. The entries are obtained by conjugation of the top elements by the elements at the left; thus, e.g. $y_{12}d_1\bar{y}_{12} = \bar{b}_0 d_1 \bar{a}_0 \bar{b}_{-1}$, $\rho b_n \bar{\rho} = b_{-n}$, etc. We now use the table to compute $\theta c_{-1}\bar{\theta}$ and $\theta d_n \bar{\theta}$. We have

$$\theta c_{-1}\bar{\theta} = \theta d_0 c_0 \bar{d}_1 \bar{\theta} = d_0 a_0 \bar{b}_1 b_1 \bar{a}_0 \bar{c}_0 \bar{d}_{-1}$$
$$= d_0 \bar{c}_0 \bar{d}_{-1} = \bar{c}_{-1}$$

and

$$\theta d_n \bar{\theta} = \theta(\bar{c}_{-1}^n d_0 c_0^n)\bar{\theta} = c_{-1}^n d_0 a_0 \bar{b}_1^n$$
$$= c_{-1}^n d_0 b_0^{-n} a_{-n} = c_{-1}^n d_0 c_0^n a_{-n}$$
$$= c_{-1}^{2n} d_n a_{-n}.$$

Now in K_3/I_3 we have the relation

$$\bar{c}_{-1}\bar{d}_r d_0 c_{-1} = \bar{d}_{r+1} d_1 \quad (r \in Z).$$

Conjugating by θ yields

$$(5.1) \quad c_{-1}\bar{a}_{-r}\bar{d}_r\bar{c}_{-1}^{2r} d_0 a_0 \bar{c}_{-1} = \bar{a}_{-r-1}\bar{d}_{r+1}\bar{c}_{-1}^{2(r+1)} c_{-1}^2 d_1 a_{-1}.$$

We now note that in the quotient of $K_3(1)$ by the normal closure H of the set $\{a_0, b_0\}$, we have

$$b_r = \bar{a}_0^r b_0 a_1^r = a_1^r, \quad \text{and} \quad a_t = \bar{b}_0^t a_0 b_1^t = b_1^t = a_1^t.$$

Hence $K_3(1)/H$ is infinite cyclic, and generated by a_1. It is now clear that adding the relations $a_0 = c_0 = e_0 = 1$ to the group $\hat{K}_3$ yields the free group on a_1, c_1, e_1. However, adding the same relations to K_3/I_3 yields, in view of relation (5.1) above, the relation

$$\bar{c}_1 a_1^r c_1^{-r} c_1^{2r} c_1 = a_1^{r+1} c_1^{-(r+1)} c_1^{2(r+1)} c_1^{-2} c_1 \bar{a}_1,$$

so that

$$a_1^r c_1^r c_1 a_1 = c_1 a_1 a_1^r c_1^r,$$

i.e.,

$$[c_1 a_1, a_1^r c_1^r] = 1 \quad (r \in Z).$$

This establishes the first part of Theorem 3, since by symmetry we will have the relations

$$[e_1 c_1, c_1^r e_1^r] = [a_1 e_1, e_1^r a_1^r] = 1$$

in the group K_3/N.

To show that the group L of Theorem 3 is not f.p., we consider the quotient group L_1 obtained by adding the relation $z = 1$ to L. We have

$$L_1 = \langle x, y; [yx, x^r y^r] = 1 \quad (r \in Z) \rangle.$$

If we put $w = yx$ and replace y by $w\bar{x}$ we obtain

$$L_1 = \langle x, w; [w, x^r (w\bar{x})^r] = 1 \quad (r \leqq Z) \rangle.$$

Thus $[w, xw\bar{x}] = 1$ in L_1, and then using

$$[w, x^{s+1}(w\bar{x})^{s+1}] = [w, x^{s+1} w\bar{x}^{s+1} x^s w \bar{x}^s \ldots xw\bar{x}]$$

if $s \geq 1$, it follows that $[w, x^s w\bar{x}^s] = 1$ for all $s \in Z$. It is then clear that

$$L_1 = \langle x, w; [w, x^s w \bar{x}^s] = 1 \quad (s \in Z) \rangle.$$

Thus L_1 is the restricted wreath product of the infinite cycle on w by the infinite cycle on x. This group is easily seen to be non f.p.

6. Proof of the propositions. In this section we shall assume familiarity with the notation and results of [**9**] (see also [**6**]). In [**10**] we used (the improved version of) Theorem 1 of [**9**] to obtain a presentation of $C_n = S(x_1^0, \dots, x_n^0)$. It is not difficult to extend the analysis of [**10**] to obtain a result for the subgroup $S(x_{r+1}^0, \dots, x_n^0)$ of A_n consisting of those elements of A_n which fix the conjugacy classes $x_{r+1}^0, \dots, x_n^0$ (where r is an integer with $0 \leqq r \leqq n$). We state this as

PROPOSITION C. *$S(x_{r+1}^0, \dots, x_n^0)$ has presentation with*
generators: the union of
(a) *the set Ω_r, and*
(b) *those type 2 whitehead automorphisms $(A; a)$ of A_n such that for each i with $r + 1 \leqq i \leqq n$ we have $x_i \in A - a$ if and only if $\bar{x}_i \in A - a$.*
And
relations: All relations of type $R1$-$R10$ in [**8**] *which involve only the above generators, together with the multiplication table for the group Ω_r.*

Proof. Let M_2 be the complex described in Section 4 of [**9**] for the tuple $U = x_{r+1}^0, \dots, x_n^0$. Then it is easy to see (as in [**10**]) that each type 2 edge of M_2 is in fact a loop. This observation enables us to construct M_2 as follows:

Let M_1 be the one-point (labelled) complex corresponding to the presentation in the statement of the proposition, and let $P(n)$ be the one-point complex corresponding to the multiplication table of Ω_n. Now take $P_r(n)$ to be the covering complex for $P(n)$ corresponding to the subgroup Ω_r of Ω_n. At each point p of $P_r(n)$ there is a (unique) copy of $P(r)$. Note that M_1 also contains a (unique) copy of $P(r)$. To each point p of $P_r(n)$ we attach a copy of M_1, identifying the copy of $P(r)$ in M_1 with the copy of $P(r)$ at p. The resulting complex, M_2' say, is a subcomplex of M_2 which contains the 1-skeleton of M_2. Now M_2 is a labelled complex; if we take this same labelling on M_2', then we obtain M_2 from M_2' merely by adding 2-cells corresponding to all loops of M_2' with boundary label the $R6$ relator $T^{-1}(A; a)T(AT; aT)^{-1}$ of [**8**] which are not in the attached copies of M_1 (noting that the excluded loops already correspond to 2-cells).

From the above construction it is easy to see that $\pi_1(M_2, U) = \pi_1(M_1)$, as required.

We now specialize the above result to the case $r = 1$, in order to prove Proposition B. We note firstly that from part (a) of the generating set we obtain only the generator τ of T_n. The generators y_i, z_j, x_{ts} of T_n are included in those supplied by part (b) of the generating set, and if $(A; a)$ is a generator coming from (b) then $(A; a)^{\pm 1}$ will either be a product (without repetition) of elements of the set $x_{21}, \dots, x_{n1}$ or a similar product of elements of the set y_j, z_j, $x_{2j}, \dots, x_{nj}$, for some $j \geqq 2$; moreover, the fact that this is so will be conveyed by the relations $R1$ and $R2$. Thus

$S(x_2^0, \ldots, x_n^0)$ is generated by τ and the y_i, z_j, x_{ts}. We shall present the group on this set, and in determining the defining relations required we must therefore suitably modify those provided by Proposition C. We now examine the list $R1$-$R10$ of [**8**] to do this:

From $R2$: we obtain the relations $Q2$. We note that the additional relations from $R1$ and $R2$ merely enable us to eliminate the 'superfluous' $(A; a)$ generators.

From $R3$: we obtain the relations $Q3$.

From $R4$: The general $R4$ relation may be written

$$(B - b + \bar{b}; \bar{b})(A; a) = (A + B - b; a)(B - b + \bar{b}; \bar{b})$$

where $A \cap B = \emptyset, \bar{b} \in A, \bar{a} \notin B$. In our case we must have $b = x_1^{\pm 1}$, since otherwise the condition of (b) of Proposition C is not satisfied. There is no real loss of generality in taking $A = x_1, x_2, x_3, \ldots, x_k, \bar{x}_3, \ldots, \bar{x}_k$, $a = x_2$, and $B = \bar{x}_1, x_{k+1}, \ldots, x_s, \bar{x}_{k+1}, \ldots, \bar{x}_s$, $b = \bar{x}_1$ for some s, k with $k < s$. Then the $R4$ relation can be written as

$$\left(\prod_{i=k+1}^{s} x_{i1}\right)\left(\prod_{j=3}^{k} x_{j2}\right)y_2 = y_2\left(\prod_{i=k+1}^{s} x_{i2}\right)\left(\prod_{j=3}^{k} x_{j2}\right)\left(\prod_{i=k+1}^{s} x_{i1}\right),$$

and we have to show that this holds in T_n. We can delete the term $\Pi_{j=3}^{k} x_{j2}$ from both sides, since the relations of T_n imply that this term commutes with the others. Now repeated use of the relation

$$x_{i1}y_2 = y_2 x_{i2} x_{i1}$$

of T_n gives

$$\left(\prod_{i=k+1}^{s} x_{i1}\right)y_2 = y_2 \prod_{i=k+1}^{s} (x_{i2}x_{i1})$$

$$= y_1\left(\prod_{i=k+1}^{s} x_{i2}\right)\left(\prod_{i=k+1}^{s} x_{i1}\right)$$

as required (where the last equality is obtained using the relation $[x_{j2}, x_{i1}] = 1$ if $j \neq i$).

From $R5$: no relations arise (since otherwise some $\left(\begin{smallmatrix} a & b \\ \bar{b} & a \end{smallmatrix}\right)$ would belong to $S(x_2^0, \ldots, x_n^0)$).

From $R6$: We obtain $Q6$.

From $R7$: We obtain $Q7$.

From $R8$: We obtain only consequences of $Q2$.

From $R9$: The general $R9$ relation is

$$(A; a)j(b)(A; a)^{-1} = j(b)$$

where $j(b)$ is conjugation by the letter b, and $b, \bar{b} \in A'$. If $(A; a)^{\pm 1}$ is a product of the x_{ij}, then the deduction on page 1528 of [**10**] shows that the required relation holds in T_n. Otherwise we may suppose, with no essential loss of generality, that $(A; a) = y_j$ and that we have to show

$$y_j\left(\prod_{\substack{i=1\\ i\neq k}}^{n} x_{ik}\right)\bar{y}_j = \prod_{\substack{i=1\\ i\neq k}}^{n} x_{ik}$$

in T_n, where $k \neq 1, j$. Using the relations $Q2$ and $Q3$, this reduces to showing that

$$y_j x_{1k} x_{jk} \bar{y}_j = x_{1k} x_{jk}.$$

Now $\bar{x}_{1k} y_j x_{1k} = \bar{y}_k y_j y_k$ in T_n, so we need

$$\bar{y}_k y_j y_k = x_{jk} y_j \bar{x}_{jk},$$

and this is in $Q9$.

From $R10$: The conditions $b \neq a$, $b \in A$ and $\bar{b} \in A'$ ensure that $b = x_1^{\pm 1}$. Now the general $R10$ relation may be written

$$(A; a) j(b) (A; a)^{-1} = j(b) j(\bar{a}),$$

(if the $(A'; \bar{a})$ term is rewritten as $j(\bar{a})(A; a)$). There is no real loss of generality in taking $b = x_1$ and $a = x_2$. We then have

$$(A; a) = y_2 \prod_{s \in S} x_{s2}$$

for some subset S of $1, 3, \ldots, n$, and we have to show, in T_n, that

$$y_2\left(\prod_{s\in S} x_{s2}\right) \prod_{r=2}^{n} x_{r1} \left(\prod_{s\in S} x_{s2}\right)^{-1} y_2^{-1} = \prod_{r=2}^{n} x_{r1} \prod_{\substack{t=1\\ t\neq 2}}^{n} \bar{x}_{t2}.$$

Now using $R9$ the terms $(\prod_{s\in S} x_{s2})^{\pm 1}$ on the left-hand side of this may be deleted. We then have

$$y_2\left(\prod_{r=2}^{n} x_{r1}\right)\bar{y}_2 = x_{21}\, \bar{x}_{12} \prod_{r=3}^{n} \bar{x}_{r2} x_{r1}$$

in T_n (using $Q10$ and $Q4$). Now by repeated use of the relation $[x_{12}x_{r2}, x_{r1}] = 1$ and of $[x_{ij}, x_{rs}] = 1$ if i, j, r, s, are distinct, we can write the right-hand side of the last relation in the desired form. This concludes the proof of Proposition B.

Finally we consider the proof of Proposition A. The results of [**9**] show easily that $S(x_2, \ldots, x_n)$ is generated by τ and the y_i, z_j. The methods of [**9**] can also be used to present the group, but in fact consideration of the observations in the first paragraph of Section 3 is enough to provide an easy verification of the proposition.

References

1. S. Bachmuth, *Automorphisms of free metabelian groups*, Trans. Amer. Math. Soc. *118* (1965), 93-104.
2. ——— *Automorphisms of solvable groups, Part 1*, Proceedings of Groups – St. Andrews (1985), 1-13 (London Mathematical Society, Lecture Notes Series #121, Cambridge Univ. Press 1986).
3. R. Bieri, *Homological dimension of discrete groups*, Queen Mary College Mathematics Notes (1976).
4. O. Chein, *Subgroups of IA automorphisms of a free group*, Acta Mathematica *123* (1969), 1-12.
5. B. Levinger, *A generalisation of the braid group*, Ph.D. thesis, New York University (1968).
6. R. Lyndon and P. E. Schupp, *Combinatorial group theory* (Springer, 1977).
7. W. Magnus, *Über n-dimensionale gittertransformationen*, Acta Math. *64* (1934), 353-367.
8. J. McCool, *A presentation for the automorphism group of a free group of finite rank*, J. London Math. Soc. (2) *8* (1974), 259-266.
9. ——— *Some finitely presented subgroups of the automorphism group of a free group*, J. of Alg. *35* (1975), 205-213.
10. ——— *On basis-conjugating automorphisms of free groups*, Can. J. Math. *38* (1986), 1525-1529.
11. J. Nielsen, *Die gruppe der dreidimensionalen Gittertransformationen, Kgl. Danske Videnskabernes Selskeb.*, Math. Fys. Meddelelser v. *12* (1924), 1-29.

University of Toronto,
Toronto, Ontario

https://doi.org/10.4153/CJM-1988-047-4 Published online by Cambridge University Press