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GENERALISED SPECTRAL ANALYSIS OF FRACTIONAL RANDOM FIELDS

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Abstract

This paper considers a large class of non-stationary random fields which have fractal characteristics and may exhibit long-range dependence. Its motivation comes from a Lipschitz-Holder-type condition in the spectral domain.

The paper develops a spectral theory for the random fields, including a spectral decomposition, a covariance representation and a fractal index. From the covariance representation, the covariance function and spectral density of these fields are defined. These concepts are useful in multiscaling analysis of random fields with long-range dependence.

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1. Introduction

A random field $\{X(t),\ t \in \mathbb{R}^n\}$ is said to exhibit long-range dependence (LRD) if its 'spectral density' has the form

$$f(\omega) = f_*(\omega)|\omega|^{-2\beta}, \quad \beta > 0,\ \omega \in \mathbb{R}^n, \tag{1.1}$$

where $f_*(\omega)$ is slowly varying as $\omega \to 0$. The spectral density has an integrable pole at the origin when $\beta < 1/2$ with the characteristic effect that the covariance function of $X(t)$ decays to zero at a very slow rate. The phenomenon of LRD has now been observed in a large number of different areas including hydrology, geophysics, agriculture, meteorology, economics and telecommunications. A good perspective on the occurrence of LRD is given in Hampel (1987). Modelling LRD in a space-time context is attempted in Haslett and Raftery (1989). A recent survey on statistical methods for data with LRD is presented in Beran (1992). Central limit theorems for

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LRD processes and fields are different from the classical theorems, so that standard results of statistical inference do not hold. Recent works on statistical inference of random fields with LRD include Heyde and Gay (1993), Anh and Lunney (1995).

With $f_*(\omega)$ equal to a constant in (1.1), the random field is identified as $1/f$ noise (or flicker noise) in signal and image processing, where f stands for frequency. $1/f$ noise is known to display fractal characteristics (partially meaning that its Hausdorff dimension is strictly greater that the topological dimension), and has been used to model natural phenomena (such as terrain, clouds, water) and texture data (see Mandelbrot and van Ness (1968), Wornell (1993) and Anh et al. (1994), for example).

For $\beta > 1/2$ the 'spectral density' is not integrable, suggesting that $1/f$ noise can have infinite variance; hence it may not be second-order stationary and, as a result, may not have a spectral density in the usual sense. This is the difficulty noted in Mandelbrot and van Ness (1968) where an approach was attempted to define a 'spectral density' of fractional Brownian motion (fBm). A more precise concept of spectral density of fBm based on a time-frequency analysis was given in Flandrin (1989). Solo (1992) presents another approach to define a spectral density for the larger class of intrinsic random functions of order zero. It should be noted that a similar approach was given in Anh and Lunney (1991) for asymptotically stationary random fields (see their Theorem 1).

In this paper, we study the spectral representation and spectral density of a large class of non-stationary random fields using the tools of generalised harmonic analysis as developed in Beurling (1964), Henniger (1970), Anh and Lunney (1992). These random fields may exhibit LRD and have fractal characteristics, hence can be used to model fractal phenomena. Furthermore, a fractal index can be defined and used in the role of fractal dimension of these random fields. In order to motivate the consideration of these random fields, let us briefly recall some concepts relating to fractals.

A random field $\{X(t),\ t \in \mathfrak{R}^n\}$ is said to satisfy a uniform Lipschitz-Hölder condition of order α in a domain D of $\mathfrak{R}^n$ if there exists a constant A such that, for every $t, t+h \in D$ and $\|h\|$ small enough,

$$|X(t+h) - X(t)| \le A\|h\|^{\alpha} \qquad \text{a.s.} \tag{1.2}$$

A class of random fields which satisfy the uniform Lipschitz-Hölder condition is that of index-β Gaussian fields which are characterised by the existence of a number β_0 such that

$$\begin{aligned}\beta_0 &= \sup\left\{\beta :\ \sigma(t) = o(\|t\|^{\beta}),\ \|t\| \downarrow 0\right\} \\ &= \inf\left\{\beta :\ \|t\|^{\beta} = o(\sigma(t)),\ \|t\| \downarrow 0\right\}\end{aligned}$$

where the random fields are assumed to have stationary increments and $\sigma^2(t)$ is the incremental variance $\sigma^2(t) = E|X(t) - X(0)|^2$.

A specific example of an index-β field is the isotropic fractional Brownian field (fBf), which is characterised by

$$EX(s)X(t) = \frac{1}{2}c\left[\|s\|^{2\beta} + \|t\|^{2\beta} - \|s-t\|^{2\beta}\right]. \tag{1.3}$$

fBf is known to be fractal and exhibit LRD. As shown in Adler (1981), p. 204, the Hausdorff dimension of the graph of an index-β field is

$$\dim(\mathrm{graph}(X)) = \min\{n/\beta,\ n+1-\beta\}. \tag{1.4}$$

To our knowledge, apart from fBf, a spectral analysis has not been attempted for random fields which satisfy the Lipschitz-Holder condition (1.2) or a subclass such as that of index-β fields.

In this paper, we shall give a spectral theory for a large class of random fields which satisfy a kind of Lipschitz-Holder condition in the spectral domain. Let $B_r(x)$ denote the ball of radius r with centre at x. A positive σ-finite Borel measure μ on $\mathfrak{R}^n$ is said to be *locally uniformly α-dimensional*, $0 \le \alpha \le n$, if

$$\mu(B_r(x)) \le Cr^{\alpha}, \qquad 0 < r < 1, \quad \forall x \in \mathfrak{R}^n. \tag{1.5}$$

Strichartz (1990) showed that if μ is such a measure, then there exists $C_1 > 0$ such that

(1.6)
$$\limsup_{T\to\infty} \frac{1}{T^{n-\alpha}} \int_{B_T(0)} \left|(X\,d\mu)\check{}(\xi)\right|^2 d\xi \le C_1 \int |X|^2\,d\mu, \qquad \forall X \in L^2(d\mu).$$

For a positive σ-finite Borel measure μ on $\mathfrak{R}^n$ let

$$V_\alpha(\epsilon;\mu) = \frac{1}{\epsilon^{n+\alpha}} \int_{\mathfrak{R}^n} |\mu(B_\epsilon(x))|^2\,dx$$

and define

$$\alpha_0 = \inf\left\{\alpha;\ \limsup_{\epsilon\to 0} V_\alpha(\epsilon;\mu) > 0\right\}.$$

Lau and Wang (1993) showed that α_0 plays the role of Hausdorff dimension of the measure μ.

Motivated by the result (1.6), we shall consider random fields which are characterised by the condition

(1.7)
$$\sup_{T_j>0,\ j=0,\dots,n} \frac{1}{1+\prod_{j=1}^n 2T_j^{1-\alpha_j}} \int_{-T_n}^{T_n} \cdots \int_{-T_1}^{T_1} |X(t_1,\cdots,t_n)|^2 dt_1,\dots,dt_n < \infty,$$
$$0 \le \alpha_j < 1,\ j = 1,\dots,n.$$

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Here, the scaling exponents α_j are allowed to be different for each j. This is useful in practice, where fractal characteristics of the random field may be distinct in different directions. As will be seen later, these random fields may also display LRD. The class defined by (1.7) generalises the class $\mathscr{B}$ of non-stationary random fields studied in Anh and Lunney (1992), which corresponds to $\alpha = 0$. It is noted that, by definition, the class $\mathscr{B}$ contains the class of asymptotically stationary random fields, which in turn contains the well-known class of harmonizable random fields. Fractal measures of the form (1.5) and (1.6) were investigated extensively in Lau (1992) and Lau and Wang (1993). In this paper, we develop a spectral theory for random fields defined by condition (1.7). In particular, a useful characterisation of these random fields is given in Section 2. Their spectral representation and fractal index are developed in Section 3. Section 4 gives a definition of the covariance function and spectral density of these fields. These concepts are useful in multifractal analysis of random fields with LRD. This application together with empirical results to back up the theory is reported in Anh and Lunney (1995).

2. Fractional random fields

Let $T = (T_1, \dots, T_n)$, $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$ with $T_j > 0$, $\epsilon_j > 0$, $0 \le \alpha < 1$, $j = 1, \dots, n$. Put

$$C(T) = \left\{(t_1, \dots, t_n) : |t_j| \le T_j,\ j = 1, \dots, n\right\},$$
$$V^{1-\alpha}(T) = \textstyle\prod_{j=1}^n (2T_j^{1-\alpha_j}), \qquad V(T) = \prod_{j=1}^n (2T_j),$$
$$C(\epsilon, \alpha) = \left\{(x_1, \dots, x_n) : |x_j| \le \epsilon_j^{1+\alpha_j},\ j = 1, \dots, n\right\},$$
$$V^{1+\alpha}(\epsilon) = \textstyle\prod_{j=1}^n (2\epsilon_j^{1+\alpha_j}).$$

We define the class of random fields

$$\mathscr{B}_\alpha = \left\{X(t) \in L^2_{loc}(\mathfrak{R}^n);\ \sup_{T>0} \frac{1}{1 + V^{1-\alpha}(T)} \int_{C(T)} |X(t)|^2\, dt < \infty\right\}.$$

Since $T_j^{1-\alpha_j} \le T_j$ for each j, we have

$$\frac{1}{1 + V(T)} \int_{C(T)} |X(t)|^2\, dt \le \frac{1}{1 + V^{1-\alpha}(T)} \int_{C(T)} |X(t)|^2\, dt.$$

Hence,

$$\mathscr{B}_\alpha \subset \mathscr{B}_o. \tag{2.1}$$

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The norm of the class $\mathscr{B}_\alpha$ is defined as

$$\|X\|_{\mathscr{B}_\alpha} = \sup_{T>0}\left(\frac{1}{1+V^{1-\alpha}(T)}\int_{C(T)}|X(t)|^2\,dt\right)^{1/2}. \tag{2.2}$$

The equivalent of $\mathscr{B}_\alpha$ for $n = 1$ is the class of stochastic processes

$$\mathscr{B}_\alpha^1 = \left\{X(t) \in L^2_{loc}(\mathfrak{R}) : \sup_{T>0}\frac{1}{1+2T^{1-\alpha}}\int_{-T}^{T}|X(t)|^2\,dt < \infty\right\}$$

where $0 \le \alpha < 1$. A characterisation condition for this class is given by the following

THEOREM 1. *Let $X(t) \in L^2_{loc}(\mathfrak{R})$. Then $X(t) \in \mathscr{B}_\alpha^1$ if and only if*

$$S(X) = \sup_{0<\mu<1/2}\frac{1}{\mu^{1+\alpha}}\int_{\mathfrak{R}}|X(t)|^2\left(\frac{\sin\mu^{1+\alpha}t}{t}\right)^2 dt < \infty. \tag{2.3}$$

Furthermore, there exist constants k_1, k_2 independent of X such that

$$k_1\|X\|_{\mathscr{B}_\alpha^1} \le S^{1/2}(X) \le k_2\|X\|_{\mathscr{B}_\alpha^1}. \tag{2.4}$$

PROOF. We follow the method of Henniger (1970). We first note that, by integration by parts,

(2.5)

$$\begin{aligned}
&\frac{1}{\mu^{1+\alpha}}\int_{-\infty}^{\infty}|X(t)|^2\left(\frac{\sin\mu^{1+\alpha}t}{t}\right)^2 dt \\
&= \frac{1}{\mu^{1+\alpha}}\left[\int_{-1}^{1}|X(t)|^2\left(\frac{\sin\mu^{1+\alpha}t}{t}\right)^2 dt\right. \\
&\qquad \left.+\left(\int_{-\infty}^{-1}+\int_{1}^{\infty}\right)\left(\frac{1}{x^2}\frac{d}{dx}\int_0^x|X(t)|^2(\sin\mu^{1+\alpha}t)^2\,dt\,dx\right)\right] \\
&= \frac{1}{\mu^{1+\alpha}}\left[\int_{-1}^{1}|X(t)|^2\left(\frac{\sin\mu^{1+\alpha}t}{t}\right)^2 dt + 2\int_1^\infty\frac{1}{x^3}\int_{-x}^{x}|X(t)|^2(\sin\mu^{1+\alpha}t)^2\,dt\,dx\right. \\
&\qquad \left.+\lim_{x\to\infty}\frac{1}{x^2}\int_{-x}^{x}|X(t)|^2(\sin\mu^{1+\alpha}t)^2\,dt - \int_{-1}^{1}|X(t)|^2(\sin\mu^{1+\alpha}t)^2\,dt\right].
\end{aligned}$$

Put $\|X\|^2_{\mathscr{B}_\alpha^1} = \sup_{T>0}(1+2T^{1-\alpha})^{-1}\int_{-T}^{T}|X(t)|^2\,dt = M$. Then, as

$$\frac{\sin\mu^{1+\alpha}t}{\mu^{1+\alpha}t} = \frac{1}{2\mu^{1+\alpha}}\int_{-\mu^{1+\alpha}}^{\mu^{1+\alpha}} e^{ixt}\,dx,$$

we get $(\sin \mu^{1+\alpha} t)^2 \leq (\mu^{1+\alpha} t)^2$. Hence for $0 \leq \mu \leq 1/2$,

$$\frac{1}{\mu^{1+\alpha}} \int_{-1}^{1} |X(t)|^2 \left(\frac{\sin \mu^{1+\alpha} t}{t} \right)^2 dt \leq \left(\frac{1}{2} \right)^{1+\alpha} 3M. \tag{2.6}$$

Now

$$0 \leq \lim_{x \to \infty} \frac{1}{x^2} \int_{-x}^{x} |X(t)|^2 (\sin \mu^{1+\alpha} t)^2 \, dt \leq \lim_{x \to 0} \frac{1 + 2x^{1-\alpha}}{x^2} M = 0. \tag{2.7}$$

For the last term in (2.5), we have

$$\begin{aligned} &\frac{1}{\mu^{1+\alpha}} \int_{1}^{\infty} \frac{1}{x^3} \int_{-x}^{x} |X(t)|^2 (\sin \mu^{1+\alpha} t)^2 \, dt dx \\ &\quad = \frac{1}{\mu^{1+\alpha}} \left(\int_{1}^{1/\mu^{1+\alpha}} + \int_{1/\mu^{1+\alpha}}^{\infty} \right) \frac{1}{x^3} \int_{-x}^{x} |X(t)|^2 (\sin \mu^{1+\alpha} t)^2 \, dt dx. \end{aligned}$$

For the range $1 \leq x \leq 1/\mu^{1+\alpha}$, use of the inequality $(\sin \mu^{1+\alpha} t)^2 \leq (\mu^{1+\alpha} t)^2$ for $-x \leq t \leq x$ implies

$$\begin{aligned} &\frac{1}{\mu^{1+\alpha}} \int_{1}^{1/\mu^{1+\alpha}} \frac{1}{x^3} \int_{-x}^{x} |X(t)|^2 (\sin \mu^{1+\alpha} t)^2 \, dt dx \\ &\quad \leq \mu^{1+\alpha} \int_{1}^{1/\mu^{1+\alpha}} \frac{x^2(1 + 2x^{1-\alpha})}{x^3} \frac{1}{1 + 2x^{1-\alpha}} \int_{-x}^{x} |X(t)|^2 \, dt dx \\ &\quad \leq \mu^{1+\alpha} M \int_{1}^{1/\mu^{1+\alpha}} \left(\frac{1}{x} + 2 \right) dx \\ &\quad = M(-\mu^{1+\alpha} \log \mu^{1+\alpha} + 2 - 2\mu^{1+\alpha}) \\ &\quad \leq M(-\beta \log \beta + 2) \end{aligned} \tag{2.8}$$

where $\beta = \mu^{1+\alpha}, 0 < \beta < (1/2)^{1+\alpha}$. The function $-\beta \log \beta$ is maximized at $\beta = 1/e$ and $0 < 1/e < (1/2)^{1+\alpha}$. Thus (2.8) is bounded above by $(2 + 1/e)M$. For the range $x \geq 1/\mu^{1+\alpha}$ we simply use $|\sin \mu^{1+\alpha} t| \leq 1$. Then

$$\begin{aligned} &\frac{1}{\mu^{1+\alpha}} \int_{1/\mu^{1+\alpha}}^{\infty} \frac{1}{x^3} \int_{-x}^{x} |X(t)|^2 (\sin \mu^{1+\alpha} t)^2 \, dt dx \\ &\quad \leq \frac{1}{\mu^{1+\alpha}} \int_{1/\mu^{1+\alpha}}^{\infty} \frac{1 + 2x^{1-\alpha}}{x^3} \frac{1}{1 + 2x^{1-\alpha}} \int_{-x}^{x} |X(t)|^2 \, dt dx \\ &\quad \leq \frac{M}{\mu^{1+\alpha}} \int_{1/\mu^{1+\alpha}}^{\infty} \left(\frac{1}{x^3} + \frac{2}{x^2} \right) dx \\ &\quad \leq \left(\frac{1}{2} \left(\frac{1}{2} \right)^{1+\alpha} + 2 \right) M. \end{aligned} \tag{2.9}$$

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Summarising, we get from (2.5) – (2.9) that $S^{1/2}(X) \leq k_2\|X\|_{B_\alpha}$, where, since $4(1/2)^{1+\alpha} + 8 + 2/e \leq 11$ we may take $k_2 = \sqrt{11}$.

Conversely, assume that (2.3) holds. We want to show that $X(t) \in \mathscr{B}^1_\alpha$. Suppose that for some T_0

$$\frac{1}{1+2T_0^{1-\alpha}} \int_{-T_0}^{T_0} |X(t)|^2\, dt > B, \quad B > 0. \tag{2.10}$$

We can assume that $T_0 > 2$. Take $\mu = 1/T_0$. Then $0 < \mu < 1/2$ and the last term of (2.5) is greater than

$$2T_0^{1+\alpha} \int_{T_0}^{2T_0} \frac{1}{x^3} \int_{-x}^{x} |X(t)|^2 \sin^2\left(\frac{t}{T_0^{1+\alpha}}\right) dt dx. \tag{2.11}$$

Let $c = \sup\left\{x : (1+2T_0^{1-\alpha})^{-1} \int_{-x}^{x} |X(t)|^2\, dt \leq b < \infty\right\}$.

Then $c < T_0$, $c/T_0^{1+\alpha} < 1/T_0^\alpha < 1/2^\alpha$ and the expression (2.11) is greater than

$$\begin{aligned} &2T_0^{1+\alpha} \int_{T_0}^{2T_0} \frac{1}{x^3} \int_{-T_0}^{T_0} |X(t)|^2 \sin^2\left(\frac{t}{T_0^{1+\alpha}}\right) dt dx \\ &> 2T_0^{1+\alpha} \sin^2 \frac{c}{T_0^{1+\alpha}} \left[\left(\int_{-T_0}^{-c} + \int_{c}^{T_0}\right) |X(t)|^2\, dt\right] \int_{T_0}^{2T_0} \frac{dx}{x^3} \\ &= \frac{3}{4}\frac{1}{T_0^{1-\alpha}} \sin^2 \frac{c}{T_0^{1+\alpha}} \left(\int_{-T_0}^{-c} + \int_{c}^{T_0}\right) |X(t)|^2\, dt. \end{aligned} \tag{2.12}$$

But

$$\left(\int_{-T_0}^{-c} + \int_{c}^{T_0}\right) |X(t)|^2\, dt = \left(\int_{-T_0}^{T_0} - \int_{-c}^{c}\right) |X(t)|^2\, dt > (B-b)(1+2T_0^{1-\alpha}).$$

Thus, the last expression in (2.12) is greater than $(3/2)(B-b)\sin^2(c/T_0^{1+\alpha})$.

Consequently, if $(1+2T^{1-\alpha})^{-1} \int_{-T}^{T} |X(t)|^2\, dt$ is unbounded, then $S(X) = \infty$, contradicting (2.3). Hence $X(t) \in \mathscr{B}^1_\alpha$.

Now, take $\mu^{1+\alpha} = 1/2^{1+\alpha}T^{1-\alpha}$. Then, for a fixed α, we have in view of (2.3),

$$S(X) \geq \lim_{\mu\to 0} \mu^{1+\alpha} \int_{\mathbb{R}} |X(t)|^2 \left(\frac{\sin \mu^{1+\alpha} t}{\mu^{1+\alpha} t}\right)^2 dt = \lim_{T\to\infty} \frac{1}{2^\alpha} \frac{1}{1+2T^{1-\alpha}} \int_{-T}^{T} |X(t)|^2\, dt$$

Hence we may take $k_1 = 1/\sqrt{2}$ in (2.4).

An extension of Theorem 1 to random fields is

THEOREM 2. *$X(t)$ belongs to $\mathscr{B}_\alpha$ if and only if*

(2.13)

$$S(X) = \sup_{0<\mu_j<1/2,\ j=1,\dots,n} \int_{\mathbb{R}^n} |X(t_1,\dots,t_n)|^2 \prod_{j=1}^n \left(\frac{\sin^2 \mu_j^{1+\alpha_j} t_j}{\mu_j^{1+\alpha_j} t_j^2} \right) dt_1 \cdots dt_n < \infty.$$

PROOF. Suppose that $X(t) \in \mathscr{B}_\alpha$. Then, as $\prod_{j=1}^n (1 + 2T_j^{1-\alpha_j}) \geq 1 + V^{1-\alpha}(T)$, we get

$$\begin{aligned} &\frac{1}{1+2T_n^{1-\alpha_n}} \int_{-T_n}^{T_n} \cdots \frac{1}{1+2T_1^{1-\alpha_1}} \int_{-T_1}^{T_1} |X(t_1,\dots,t_n)|^2 \, dt_1 \cdots dt_n \\ &\qquad \leq \frac{1}{1+V^{1-\alpha}(T)} \int_{C(T)} |X(t)|^2 \, dt. \end{aligned} \tag{2.14}$$

This implies that, for $j = 1, \dots, n$,

$$\sup_{T_j>0} \frac{1}{1+2T_j^{1-\alpha_j}} \int_{-T_j}^{T_j} |X(t_1,\dots,t_n)|^2 \, dt_j < \infty. \tag{2.15}$$

Therefore, in view of Theorem 1 above and for $j = 1, \dots, n$,

$$\begin{aligned} &\sup_{0<\mu_j<1/2} \int_{-\infty}^{\infty} |X(t_1,\dots,t_n)|^2 \frac{\sin^2 \mu_j^{1+\alpha_j} t_j}{\mu_j^{1+\alpha_j} t_j^2} \, dt_j \\ &\qquad \leq k_j \sup_{T_j>0} \frac{1}{1+2T_j^{1-\alpha_j}} \int_{-T_j}^{T_j} |X(t_1,\dots,t_n)|^2 \, dt_j, \end{aligned} \tag{2.16}$$

for some constant k_j independent of X. A recursive argument for j from 1 to n then yields

$$\begin{aligned} &\sup_{0<\mu_j<1/2, j=1,\dots,n} \int_{\mathbb{R}^n} |X(t_1,\dots,t_n)|^2 \prod_{j=1}^n \frac{\sin^2 \mu_j^{1+\alpha_j} t_j}{\mu_j^{1+\alpha_j} t_j^2} \, dt_1 \cdots dt_n \\ &\leq k_1 \dots k_n \sup_{T_n>0} \frac{1}{1+2T_n^{1-\alpha_n}} \int_{-T_n}^{T_n} \cdots \sup_{T_1>0} \frac{1}{1+2T_1^{1-\alpha_1}} \int_{-T_1}^{T_1} |X(t_1,\dots,t_n)|^2 dt_1 \cdots dt_n \\ &< \infty \end{aligned}$$

in view of (2.14) and the assumption $X(t) \in \mathscr{B}_\alpha$.

Conversely, suppose that for some T_{j_0} and all $-T_i \leq t_i \leq T_i$, $i \neq j$,

$$\frac{1}{1+2T_{j_0}^{1-\alpha_j}} \int_{-T_{j_0}}^{T_{j_0}} |X(t_1,\dots,t_n)|^2 \, dt_j > D, \quad D > 0. \tag{2.17}$$

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In view of the inequality $(1+2T_{j_0}^{1-\alpha_j})/(1+2T_j^{1-\alpha_j}) \geq 1/2^{1-\alpha_j}$ for $T_j \leq 2T_{j_0}$, we get from (2.17) that

$$\frac{1}{1+2T_j^{1-\alpha_j}} \int_{-T_j}^{T_j} |X(t_1, \ldots, t_n)|^2 \, dt_j > \frac{D}{2^{1-\alpha_j}} \tag{2.18}$$

for $T_{j_0} \leq T_j \leq 2T_{j_0}$. Note also that $1 + 2T_1^{1-\alpha_1} \cdots 2T_n^{1-\alpha_1} < (1+2T_1^{1-\alpha_1}) \cdots (1 + 2T_n^{1-\alpha_n})$. Thus (2.18) implies

$$\begin{aligned} &\frac{1}{1+V^{1-\alpha}(T)} \int_{C(T)} |X(t)|^2 \, dt \\ &\quad > \frac{1}{1+2T_n^{1-\alpha_n}} \int_{-T_n}^{T_n} \cdots \frac{1}{1+2T_1^{1-\alpha_1}} \int_{-T_1}^{T_1} |X(t_1, \ldots, t_n)|^2 \, dt_1 \ldots dt_n \\ &\quad > \frac{D}{2^{1-\alpha_j}} \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{2T_i}{1+2T_i^{1-\alpha_i}} > \frac{D}{2} \end{aligned} \tag{2.19}$$

for large T_i, $i = 1, \ldots, n$, $i \neq j$ and $T_{j_0} \leq T_j \leq 2T_{j_0}$. As in the proof of Theorem 1, the inequality (2.17) yields that

$$\sup_{0<\mu_j \leq \frac{1}{2}} \int_{-\infty}^{\infty} |X(t_1, \ldots, t_j)|^2 \frac{\sin^2 \mu_j^{1+\alpha_j} t}{\mu_j^{1+\alpha_j} t^2} \, dt_j > \frac{3}{2}(D-d) \sin^2 \frac{C}{T_{j_0}^{1+\alpha_j}},$$

where

$$C = \sup \left\{ x : \frac{1}{1+2T_{j_0}^{1-\alpha_j}} \int_{-x}^{x} |X(t_1, \ldots, t_n)|^2 \, dt_j \leq d < \infty \right\}.$$

Consequently,

$$\begin{aligned} &\sup_{0<\mu_j<1/2,\ j=1,\ldots,n} \int_{\mathfrak{R}^n} |X(t_1, \ldots, t_n)|^2 \prod_{j=1}^{n} \frac{\sin^2 \mu_j^{1+\alpha_j} t}{\mu_j^{1+\alpha_j} t^2} \, dt_1 \cdots dt_n \\ &\quad > \frac{3}{2} \pi^{n-1} (D-d) \sin^2 \frac{C}{T_{j_0}^{1+\alpha_j}}, \end{aligned} \tag{2.20}$$

where we have used the result that $\int_0^\infty \sin^2 x / x^2 \, dx = \pi/2$. In view of (2.17), (2.19) and (2.20), it is seen that if (2.17) holds for all $D > 0$, then $S(X) = \infty$, contradicting (2.13). Hence $X(t)$ must belong to $\mathscr{B}_\alpha$ if (2.13) is assumed.

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3. Spectral representation and fractal index

In this section, we obtain a spectral representation and a characterisation of the spectral measure of fractional random fields in the class $\mathscr{B}_\alpha$.

We first consider

$$W = \{w : \mathfrak{R}^n \to \mathfrak{R};\ w(t) > 0 \text{ non-increasing in } |t|,$$
$$\int_{\mathfrak{R}^n} w(t)\,dt < \infty \text{ and } \omega(0) = \lim_{t\to 0} w(t) < \infty\}.$$

Define the norm in W as

$$N(w) = w(0) + \int_{\mathfrak{R}^n} w(t)\,dt, \tag{3.1}$$

and consider $W_0 = \{w \in W : N(w) = 1\}$. We next define

$$L^2(w(t)\,dt) = \left\{X : \mathfrak{R}^n \to C : \int_{\mathfrak{R}^n} |X(t)^2| w(t)\,dt < \infty\right\},$$
$$B = \textstyle\bigcap_{w\in W_0} L^2(w(t)\,dt),$$
$$\|X\|_B = \sup_{w\in W_0} \left(\int_{\mathfrak{R}^n} |X(t)|^2 w(t)\,dt\right)^{1/2}. \tag{3.2}$$

Then B is a Banach space in the norm (3.2) (see Beurling (1964), Theorem 1). We further define

$$A = \textstyle\bigcup_{w\in W_0} L^2(dt/w(t)),$$
$$\|X\|_A = \inf_{w\in W_0} \left(\int_{\mathfrak{R}^n} |X(t)|^2\,dt/w(t)\right)^{1/2}. \tag{3.3}$$

Then A is a Banach algebra under addition and convolution with the norm (3.3) and $\|X_1 * X_2\|_A \le \|X_1\|_A \|X_2\|_A$ (see Beurling (1964), Theorem 1).

An important result of Beurling (1964) that we require is the following.

THEOREM (Beurling). (i) *$\mathscr{B}_0 = B = A^*$, where A^* is the dual of A in the Banach space sense;*

(ii) *each linear functional X on A has the form $X(\varphi) = \int_{\mathfrak{R}^n} \varphi(t)\overline{Y(t)}\,dt$ for a unique Y in B and*

$$\|X\|_{A^*} = \sup_{\|\varphi\|_A = 1} \left|\int_{\mathfrak{R}^n} \varphi(t)\overline{Y(t)}\,dt\right| = \|Y\|_B$$

(see Beurling (1964), Theorem 2).

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From Beurling's Theorem, it is seen that, for $X \in \mathscr{B}_\alpha$, $X \in L^2(dt/(1+|t|^{n+1}))$ since $1/(1+|t|^{n+1}) \in W$. Also, $\int_{C(\epsilon,\alpha)} e^{i(t,x)}\,dx = O(|t|^{-(n+1)/2})$.

Thus, for large $|t|$,

$$\int_{\mathfrak{R}^n} |X(t)|^2 \left| \int_{C(\epsilon,\alpha)} e^{i(t,x)}\,dx \right|^2 dt \le c \int_{\mathfrak{R}^n} |X(t)|^2 \frac{dt}{1+|t|^{n+1}} < \infty$$

for some constant c; that is,

$$X(t) \int_{C(\epsilon,\alpha)} e^{i(t,x)}\,dx \in L^2(dt).$$

Its Fourier transform is then

$$Z_{\epsilon,\alpha}(\lambda) = \frac{1}{(2\pi)^n} \int_{\mathfrak{R}^n} X(t) \left(\int_{C(\epsilon,\alpha)} e^{i(t,x)}\,dx \right) e^{i(t,\lambda)}\,dt. \tag{3.4}$$

The quantity $Z_{\alpha,\epsilon}(\lambda)$ is called the *generalised Fourier transform* of $X(t)$. We now put

$$X_\epsilon(t) = X(t) \frac{1}{V^{1+\alpha_{(\epsilon)}}} \int_{C(\epsilon,\alpha)} e^{i(t,x)}\,dx. \tag{3.5}$$

Then, for a fixed ϵ,

$$\frac{Z_{\epsilon,\alpha}(\lambda)}{V^{1+\alpha}(\epsilon)} = \frac{1}{(2\pi)^n} \int_{\mathfrak{R}^n} X_\epsilon(t) e^{i(t,\lambda)}\,dt. \tag{3.6}$$

Let $\varphi \in A$. Then Parseval's identity implies that, for $\epsilon \neq 0$,

$$\int_{\mathfrak{R}^n} \varphi(t) X_\epsilon(t)\,dt = \int_{\mathfrak{R}^n} \hat{\varphi}(\lambda) \frac{Z_{\epsilon,\alpha}(\lambda)}{V^{1+\alpha}(\epsilon)}\,d\lambda,$$

where $\hat{\varphi}$ is the Fourier transform of φ. It is clear that $\|X_\epsilon\|_{\mathscr{B}_0} \le \|X\|_{\mathscr{B}_0}$ so that $\{X_\epsilon\}$ is a bounded set in $\mathscr{B}_0$. Also $X_\epsilon \to X$ in $L^2(C(T))$ for each $T > 0$ as $\epsilon \to 0$ since

$$\lim_{\epsilon \to 0} \frac{1}{V^{1+\alpha}(\epsilon)} \int_{C(\epsilon,\alpha)} e^{i(t,x)}\,dx = 1.$$

Thus, an extension of Theorem 2.1 of Henniger (1970) to $\mathfrak{R}^n$ yields $X_\epsilon \to X$ in the weak-star topology of $\mathscr{B}_0$. Consequently, Beurling's theorem implies that

$$\int_{\mathfrak{R}^n} \varphi(t) X_\epsilon(t)\,dt \to X(\varphi)$$

and

$$\hat{X}(\hat{\varphi}) = \lim_{\epsilon \to 0} = \int_{\mathfrak{R}^n} \hat{\varphi}(\lambda) \frac{Z_{\epsilon,\alpha}(\lambda)}{V^{1+\alpha}(\epsilon)}\,d\lambda = \int_{\mathfrak{R}^n} \hat{\varphi}(\lambda) Z(d\lambda). \tag{3.7}$$

Equation (3.7) defines the Fourier transform $\hat{X}$ of $X \in \mathscr{B}_\alpha$. It is represented by the measure $Z(d\lambda)$ generated by the generalised Fourier transform $Z_{\epsilon,\alpha}$ of X_ϵ.

For the representation of $X \in \mathscr{B}_\alpha$, we note that by Plancherel's theorem,

$$X_\epsilon(t) = \int_{\mathbb{R}^n} \frac{Z_{\epsilon,\alpha}(\lambda)}{V^{1+\alpha}(\epsilon)} e^{-i(t,\lambda)}\, d\lambda.$$

As seen above, $X_\epsilon \to X$ in $\mathscr{B}_0$. It is not true in general that this limit is in $\mathscr{B}_\alpha$. We now want to find the condition under which $X \in \mathscr{B}_\alpha$. Then

$$X(t) = \lim_{\epsilon\to 0} \int_{\mathbb{R}^n} \frac{Z_{\epsilon,\alpha}(\lambda)}{V^{1+\alpha}(\epsilon)} e^{-i(t,\lambda)}\, d\lambda, \tag{3.8}$$

that is,

$$X(t) = \int_{\mathbb{R}^n} e^{-i(t,\lambda)} Z(d\lambda), \tag{3.9}$$

which gives the spectral representation of $X(t) \in \mathscr{B}_\alpha$. The condition for X to belong to $\mathscr{B}_\alpha$ is given by the following

THEOREM 3. *The measure $Z(d\lambda)$ of* (3.7) *and* (3.8) *is the spectral measure of $X(t) \in \mathscr{B}_\alpha$ if and only if $Z_{\epsilon,\alpha}(\lambda)$ is locally in $L^2(d\lambda)$ and satisfies*

$$\sup_{0<\epsilon_j<1/2,\ j=1,\dots,n} \frac{1}{V^{1+\alpha}(\epsilon)} \int_{\mathbb{R}^n} |Z_{\epsilon,\alpha}(\lambda)|^2\, d\lambda < \infty. \tag{3.10}$$

PROOF. Since $Z_{\epsilon,\alpha}(\lambda)/V^{1+\alpha}(\epsilon)$ is the Fourier transform of $X_\epsilon(t)$, we get

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |X_\epsilon(t)|^2\, dt = \int_{\mathbb{R}^n} \left| \frac{Z_{\epsilon,\alpha}(\lambda)}{V^{1+\alpha}(\epsilon)} \right|^2 d\lambda. \tag{3.11}$$

Noting that $\int_{-\epsilon^{1+\alpha}}^{\epsilon^{1+\alpha}} e^{itx}\, dx = 2 \sin \epsilon^{1+\alpha} t / t$, Equation (3.11) can be rewritten as

$$\frac{1}{\pi^n} \int_{\mathbb{R}^n} |X_\epsilon(t)|^2 \prod_{j=1}^n \frac{\sin^2 \epsilon_j^{1+\alpha_j} t_j}{\epsilon_j^{1+\alpha_j} t_j^2}\, dt = \frac{1}{V^{1+\alpha}(\epsilon)} \int_{\mathbb{R}^n} |Z_{\epsilon,\alpha}(\lambda)|^2\, d\lambda. \tag{3.12}$$

Taking $\epsilon_j = \mu_j$, $j = 1, \dots, n$, the condition (3.10) now follows directly from Theorem 2.

Conversely, suppose that $Z_{\epsilon,\alpha}$ is a field locally in $L^2(d\lambda)$ and satisfies (3.10). Then, using Plancherel's theorem, we can define a function $X(t, \epsilon)$ such that $X(t, \epsilon)$ is locally in $L^2(dt)$ and $X(t, \epsilon) \int_{C(\epsilon,\alpha)} e^{i(t,x)}\, dx$ is the inverse Fourier transform of $Z_{\epsilon,\alpha}(\lambda)$, that is,

$$Z_{\epsilon,\alpha}(\lambda) = \frac{1}{(2\pi)^n} \lim_{T\to\infty} \int_{C(T)} \prod_{j=1}^n \left(\frac{e^{i\epsilon_j^{1+\alpha_j} t_j} - e^{-i\epsilon_j^{1+\alpha_j} t_j}}{it_j} \right) X(t, \epsilon) e^{i(t,\lambda)}\, dt. \tag{3.13}$$

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We now want to show that $X(t,\epsilon) = \int_{\mathbb{R}^n} e^{-i(t,u)} Z(du)$ as $\epsilon \to 0$. In fact, put

$$(3.14)\quad g(T) = \int_{C(T)} \prod_{j=1}^{n} \left(\frac{e^{i(\lambda_j + \epsilon_j^{1+\alpha_j})t_j} - e^{i(\lambda_j - \epsilon_j^{1+\alpha_j})t_j}}{it_j} \right) \left(\int_{\mathbb{R}^n} e^{-i(t,u)} Z(du) \right) dt.$$

Then

$$\begin{aligned} g(T) &= \int_{\mathbb{R}^n} \int_{C(T)} \prod_{j=1}^{n} \left(\frac{e^{i(\lambda_j + \epsilon_j^{1+\alpha_j} - u_j)t_j} - e^{i(\lambda_j - \epsilon_j^{1+\alpha_j} + u_j)t_j}}{it_j} \right) dt\, Z(du) \quad \text{(Fubini)} \\ &= 2^n \int_{\mathbb{R}^n} h(u,T) Z(du), \quad \text{where} \\ h(u,T) &= \prod_{j=1}^{n} \int_0^{T_j} \left(\frac{\sin(\lambda_j + \epsilon_j^{1+\alpha_j} - u_j)t_j - \sin(\lambda_j - \epsilon_j^{1+\alpha_j} - u_j)t_j}{t_j} \right) dt_j. \end{aligned}$$

If we denote by C the cube

$$\left\{ u = (u_1, \ldots, u_n) : -\epsilon_j^{1+\alpha_j} + \lambda_j < u_j < \epsilon_j^{1+\alpha_j} + \lambda_j,\ j = 1, \ldots, n \right\},$$

then $g(T) = 2^n \left(\int_{\bar{C}^c} + \int_{\partial C} + \int_C \right) h(u,T) Z(du) = 2^n (I_1 + I_2 + I_3)$, where $\bar{C}^c$ is the complement of the closure of C and ∂C is the boundary of C. Since C is a continuity set of $Z(u)$, $I_2 = 0$. Also, for $u \in C$, we have for each j $\lambda_j + \epsilon_j^{1+\alpha_j} - u_j > 0$ and $\lambda_j - \epsilon_j^{1+\alpha_j} - u_j < 0$. Thus

$$\lim_{T_j \to 0} \int_0^{T_j} \left(\frac{\sin(\lambda_j + \epsilon_j^{1+\alpha_j} - u_j)t_j - \sin(\lambda_j - \epsilon_j^{1+\alpha_j} - u_j)t_j}{t_j} \right) dt_j = \frac{\pi}{2} - \left(\frac{-\pi}{2} \right) = \pi$$

For $u \ni \bar{C}$, there exists some j such that $u_j > \lambda_j + \epsilon_j^{1+\alpha_j}$ or some i such that $u_i < \lambda_i - \epsilon_i^{1+\alpha_i}$. The former inequality is equivalent to $\lambda_j + \epsilon_j^{1+\alpha_j} - u_j < 0$, which implies $\lambda_j - \epsilon_j^{1+\alpha_j} - u_j < 0$, and therefore,

$$\lim_{t_j \to 0} \int_0^{T_j} \left(\frac{\sin(\lambda_j + \epsilon_j^{1+\alpha_j} - u_j)t_j - \sin(\lambda_j - \epsilon_j^{1+\alpha_j} - u_j)t_j}{t_j} \right) dt_j = -\frac{\pi}{2} - \left(\frac{-\pi}{2} \right) = 0.$$

The latter inequality is equivalent to $\lambda_i - \epsilon_i^{1+\alpha_i} - u_i > 0$, which implies $\lambda_i + \epsilon_i^{1+\alpha_i} - u_i > 0$, and therefore

$$\lim_{T_i \to 0} \int_0^{T_i} \left(\frac{\sin(\lambda_i + \epsilon_i^{1+\alpha_i} - u_i)t_i - \sin(\lambda_i - \epsilon_i^{1+\alpha_i} - u_i)t_i}{t_i} \right) dt_i = \frac{\pi}{2} - \frac{\pi}{2} = 0.$$

Summarising, we have

$$\lim_{T\to\infty} h(u,T) = \begin{cases} \pi^n, & u \in C, \\ 0, & u \ni \bar{C}. \end{cases}$$

Consequently, in view of the dominated convergence theorem, we get

$$\lim_{T\to\infty} I_1 = \int_{\bar{C}^c} \left(\lim_{T\to\infty} h(u,T)\right) Z(du) = 0,$$

$$\lim_{T\to\infty} I_3 = \int_{C} \left(\lim_{T\to\infty} h(u,T)\right) Z(du)$$
$$= \pi^n \left(Z(\lambda_1+\epsilon_1^{1+\alpha_1},\dots,\lambda_n+\epsilon_n^{1+\alpha_n}) - Z(\lambda_1-\epsilon_1^{1+\alpha_1},\dots,\lambda_n-\epsilon_n^{1+\alpha_n})\right)$$

so that

(3.15)
$$\lim_{T\to\infty} \frac{1}{(2\pi)^n} g(T) = Z(\lambda_1+\epsilon_1^{1+\alpha_1},\dots,\lambda_n+\epsilon_n^{1+\alpha_n}) - Z(\lambda_1-\epsilon_1^{1+\alpha_1},\dots,\lambda_n-\epsilon_n^{1+\alpha_n}).$$

By the definition of Stieltjes integrals, we can write

(3.16)
$$\lim_{\epsilon\to 0} \int_{\mathfrak{R}^n} e^{-i(t,\lambda)} \frac{Z(\lambda_1+\epsilon_1^{1+\alpha_1},\dots,\lambda_n+\epsilon_n^{1+\alpha_n}) - Z(\lambda_1-\epsilon_1^{1+\alpha_1},\dots,\lambda_n-\epsilon_n^{1+\alpha_n})}{2\epsilon_1^{1+\alpha_1}\cdots 2\epsilon_n^{1+\alpha_n}}\, d\lambda$$
$$= \int_{\mathfrak{R}^n} e^{-i(t,\lambda)} Z(d\lambda).$$

But the right-hand side of (3.16) is equal to $\lim_{\epsilon\to 0} \int_{\mathfrak{R}^n} e^{-i(t,\lambda)} Z_{\epsilon,\alpha}(\lambda)/V^{1+\alpha}(\epsilon)\, d\lambda$ by (3.8) and (3.9). Thus, as $\epsilon \to 0$,

$$Z(\lambda_1+\epsilon_1^{1+\alpha_1},\dots,\lambda_n+\epsilon_n^{1+\alpha_n}) - Z(\lambda_1-\epsilon_1^{1+\alpha_1},\dots,\lambda_n-\epsilon_n^{1+\alpha_n}) = Z_{\epsilon,\alpha}(\lambda).$$

This result and (3.15) yields

$$\lim_{T\to\infty} \frac{1}{(2\pi)^n} g(T) = Z_{\epsilon,\alpha}(\lambda). \tag{3.17}$$

In view of (3.13), (3.14) and (3.17), we get, as $\epsilon \to 0$,

$$X(t,\epsilon) = \int_{\mathfrak{R}^n} e^{-i(t,u)} Z(du). \tag{3.18}$$

Denoting the right-hand side of (3.18) by $X(t)$, Plancherel's theorem then implies

$$\int_{\mathfrak{R}^n} |Z_{\epsilon,\alpha}(\lambda)|^2\, d\lambda = \frac{1}{(2\pi)^n} \int_{\mathfrak{R}^n} |X(t)|^2 \prod_{j=1}^n \frac{4\sin^2 \epsilon_j^{1+\alpha_j} t_j}{t_j^2}\, dt,$$

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that is,

$$\frac{1}{V^{1+\alpha}(\epsilon)}\int_{\mathfrak{R}^n}|Z_{\epsilon,\alpha}(\lambda)|^2\,d\lambda=\frac{1}{\pi^n}\int_{\mathfrak{R}^n}|X(t)|^2\prod_{j=1}^{n}\frac{\sin^2\epsilon_j^{1+\alpha_j}t_j}{\epsilon_j^{1+\alpha_j}t_j^2}.$$

Condition (3.10) then yields that $X(t)\in\mathscr{B}_\alpha$ in view of Theorem 2.

Throughout this paper, inequality on $\mathfrak{R}^n$ means componentwise inequality; that is, for $x=(x_1,\dots,x_n),\ y=(y_1,\dots,y_n)\in\mathfrak{R}^n$,

$$x<y \text{ (respectively } x>y) \quad \text{if and only if} \quad x_i<y_i \text{ (respectively } x_i>y_i),\quad i=1,\dots,n.$$

In view of Theorem 3, let

$$\begin{aligned}\alpha^0=(\alpha_1^0,\dots,\alpha_n^0)&=\sup\left\{(\alpha_1,\dots,\alpha_n):\limsup_{\epsilon\to0}\frac{1}{V^{1+\alpha}(\epsilon)}\int_{\mathfrak{R}^n}|Z_{\epsilon,\alpha}(\lambda)|^2\,d\lambda<\infty\right\}\\&=\inf\left\{(\alpha_1,\dots,\alpha_n):\limsup_{\epsilon\to0}\frac{1}{V^{1+\alpha}(\epsilon)}\int_{\mathfrak{R}^n}|Z_{\epsilon,\alpha}(\lambda)|^2\,d\lambda>0\right\}.\end{aligned}$$

It follows that, if α^0 exists, then it can be defined as the fractal index of $X(t)$. In the one-dimensional case, this number corresponds to the Hausdorff dimension of $X(t)$ and it has been found useful in studying the multiscaling behaviour of LRD time series in Anh and Lunney (1994).

4. Spectral density of fractional random fields

Section 3 defines the spectral measure $Z(d\lambda)$ of a random field $X(t)$ in $\mathscr{B}_\alpha$ and establishes its spectral decomposition. This section will give a definition of the covariance function and spectral density of $X(t)\in\mathscr{B}_\alpha$. For this purpose, we assume that

(4.1) $$\lim_{T\to\infty}\frac{1}{V^{1-\alpha}(T)}\int_{C(T)}X(t+k)\overline{X(t)}\,dt \quad \text{exists for } k\in\mathfrak{R}^n.$$

Denote this limit by R(k). Then it will be shown below that

(4.2) $$R(k)=\lim_{\epsilon\to0}\frac{1}{V^{1+\alpha}(\epsilon)}\int_{\mathfrak{R}^n}e^{-i(k,\lambda)}|Z_{\epsilon,\alpha}(\lambda)|^2\,d\lambda.$$

Consequently, $R(k)$ can be defined as the covariance function and $\lim_{\epsilon\to0}|Z_{\epsilon,\alpha}(\lambda)|^2/V^{1+\alpha}(\epsilon)$ the spectral density of $X(t)\in\mathscr{B}_\alpha$. We shall require

WIENER'S TAUBERIAN THEOREM *Suppose that* $\psi \in L^{\infty}(\mathfrak{R}^n)$, $f \in L^1(\mathfrak{R}^n)$, $\hat{f}(\lambda) \neq 0$ *for every* $\lambda \in \mathfrak{R}^n$ *and* $\lim_{|t|\to\infty}(f * \psi)(t) = a\hat{f}(0)$. *Then* $\lim_{|t|\to\infty}(g * \psi)(t) = a\hat{g}(0)$ *for every* $g \in L^1(\mathfrak{R}^n)$.

PROOF. See Rudin (1973), pp. 211–212.

THEOREM 4. *For* $X(t) \in \mathscr{B}_\alpha$ *under condition* (4.1),

$$\lim_{T\to\infty} \frac{1}{V^{1-\alpha}(T)} \int_{C(t)} |X(t)|^2\, dt = \lim_{\epsilon\to 0} \frac{1}{V^{1+\alpha}(\epsilon)} \int_{\mathfrak{R}^n} |Z_{\epsilon,\alpha}(\lambda)|^2\, d\lambda. \tag{4.3}$$

PROOF. Let $\psi \in L^{\infty}(\mathfrak{R}^n)$. Put $t_j = e^{\xi_j}$, $T_j = e^{\eta_j}$, $\epsilon_j = e^{-\eta_j}$, $j = 1, \dots, n$, and define $\varphi(t) = \psi(\xi)$, $t, \xi \in \mathfrak{R}^n$. Then

$$\begin{aligned} &\frac{1}{T_1^{1-\alpha_1} \dots T_n^{1-\alpha_n}} \int_0^{T_n} \dots \int_0^{T_1} \varphi(t)\, dt_1 \dots dt_n \\ &\qquad = \int_{-\infty}^{\eta_n} \dots \int_{-\infty}^{\eta_1} e^{\xi_1+\dots+\xi_n-(\eta_1(1-\alpha_1)+\dots+\eta_n(1-\alpha_n))} \psi(\xi)\, d\xi_1 \dots d\xi_n, \\ &\left(\frac{2}{\pi}\right)^n \int_0^{\infty} \dots \int_0^{\infty} \varphi(t) \left(\prod_{j=1}^n \frac{\sin^2 \epsilon_j^{1+\alpha_j} t_j}{\epsilon_j^{1+\alpha_j} t_j^2} \right) dt_1 \dots dt_n \\ &\qquad = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\prod_{j=1}^n \frac{2}{\pi} \frac{\sin^2(e^{\xi_j - \eta_j(1+\alpha_j)})}{e^{\xi_j-\eta_j(1+\alpha_j)}} \right) \psi(\epsilon)\, d\xi_1 \dots d\xi_n. \end{aligned}$$

We next define

$$f(\xi) = \prod_{j=1}^n \frac{2}{\pi} e^{\xi_j} \sin^2(e^{-\xi_j}), \quad \xi = (\xi_1, \dots, \xi_n) \in \mathfrak{R}^n,$$

$$g(\xi) = \begin{cases} e^{-(\xi_1+\dots+\xi_n)}, & 0 < \xi_j < \infty,\ j = 1, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} &\int_{\mathfrak{R}^n} f(\xi)\, d\xi = \prod_{j=1}^n \int_{-\infty}^{\infty} \frac{2}{\pi} e^{\xi_j} \sin^2(e^{-\xi_j})\, d\xi_j = \prod_{j=1}^n \frac{2}{\pi} \int_0^{\infty} \frac{\sin^2 x_j}{x_j^2}\, dx_j = 1, \\ &\int_{\mathfrak{R}^n} g(\xi)\, d\xi = \prod_{j=1}^n \int_0^{\infty} e^{-\epsilon_j}\, d\xi_j = 1. \end{aligned}$$

Also,

(4.4)

$$\lim_{\epsilon_j\to 0,\ j=1,\dots,n}\left(\frac{2}{\pi}\right)^n\int_0^\infty\cdots\int_0^\infty\varphi(t)\left(\prod_{j=1}^n\frac{\sin^2\epsilon_j^{1+\alpha_j}t_j}{\epsilon_j^{1+\alpha_j}t_j^2}\right)dt$$

$$=\lim_{\eta_j\to\infty,\ j=1,\dots,n}\int_{\mathfrak{R}^n}f(\eta_1(1+\alpha_1)-\xi_1,\dots,\eta_n(1+\alpha_n)-\xi_n)\psi(\xi_1,\dots,\xi_n)\,d\xi_1\dots d\xi_n$$

(4.5)

$$\lim_{T_j\to\infty,\ j=1,\dots,n}\frac{1}{T_1^{1-\alpha_1}\dots T_n^{1-\alpha_n}}\int_0^{T_n}\cdots\int_0^{T_1}\varphi(t_1,\dots,t_n)\,dt_1\dots dt_n$$

$$=\lim_{\eta\to\infty}\int_{\mathfrak{R}^n}g(\eta_1(1-\alpha_1)-\xi_1,\dots,\eta_n(1-\alpha_n)-\xi_n)\psi(\xi_1,\dots,\xi_n)\,d\xi_1\dots d\xi_n.$$

As shown by Wiener (1958), pp. 142-143, for each $j = 1,\dots,n$, $(2/\pi)\exp(\xi_j)\sin^2\exp(-\xi_j)\in L^1(\mathfrak{R})$ and its Fourier transform does not vanish on $\mathfrak{R}$. Thus, $f\in L^1(\mathfrak{R}^n)$ and $\hat{f}(\lambda)\neq 0$, $\lambda\in\mathfrak{R}^n$. Consequently, the two limits in (4.4) and (4.5) assume the same value, if they exist, by Wiener's Tauberian theorem. By choosing $\varphi(t)=|X(t)|^2$, we then get

$$\lim_{T\to\infty}\frac{1}{V^{1-\alpha}(T)}\int_{C(T)}|X(t)|^2\,dt=\lim_{\epsilon\to 0}\frac{1}{\pi^n}\int_{\mathfrak{R}^n}|X(t)|^2\prod_{j=1}^n\frac{\sin^2\epsilon_j^{1+\alpha_j}t_j}{\epsilon_j^{1+\alpha_j}t_j^2}\,dt. \tag{4.6}$$

Equality (4.3) now follows from (3.12) and (4.6).

REMARK 1. The spherical form of (4.3) was obtained in Lau (1992), Theorem 3.1, where the integral with respect to t is taken over the ball of radius T and centre 0. The result is therefore more suitable to isotropic random fields with uniform fractal characteristics in all directions. The method of proof is inevitably different from the method of this paper.

THEOREM 5. *Let* $X(t)\in\mathscr{B}_\alpha$ *and* $R(k)$ *be defined by condition* (4.1). *Then* $R(k)$ *has the representation* (4.2).

PROOF. The proof follows the same lines of that of Theorem 2 of Anh and Lunney (1992). We give an outline here. $R(k)$ can be rewritten as

$$R(k)=\frac{1}{4}\lim_{T\to\infty}\frac{1}{V^{1-\alpha}(T)}\int_{C(T)}|X(t+k)+X(t)|^2-|X(t+k)-X(t)|^2 + i|X(t+k)+iX(t)|^2-i|X(t+k)-iX(t)|^2\,dt. \tag{4.7}$$

Denote the generalised Fourier transform of $X(t+k)$ by $Z_{\epsilon,\alpha}(\lambda; k)$. Then

$$\begin{aligned}Z_{\epsilon,\alpha}(\lambda; k) &= \frac{1}{(2\pi)^n}\int_{\mathfrak{R}^n} X(t+k)\prod_{j=1}^{n}\left(\frac{2\sin\epsilon_j^{1+\alpha_j}t_j}{t_j}e^{it_j\lambda_j}\right)dt\\ &= \frac{1}{(2\pi)^n}\int_{\mathfrak{R}^n} X(t)\prod_{j=1}^{n}\left(\frac{2\sin\epsilon_j^{1+\alpha_j}(t_j-k_j)}{t_j-k_j}e^{i(t_j-k_j)\lambda_j}\right)dt.\end{aligned}$$

Plancherel's theorem now yields

$$\begin{aligned}&\int_{\mathfrak{R}^n}\left|Z_{\epsilon,\alpha}(\lambda; k) - e^{-i(k,\lambda)}Z_{\epsilon,\alpha}(\lambda; 0)\right|^2\\ &\quad= \frac{1}{\pi^n}\int_{\mathfrak{R}^n}|X(t)|^2\prod_{j=1}^{n}\left(\frac{\sin\epsilon_j^{1+\alpha_j}(t_j-k_j)}{t_j-k_j} - \frac{\sin\epsilon_j^{1+\alpha_j}t_j}{t_j}\right)^2 dt\\ &\quad= O\left(\epsilon_1^{2(1+\alpha_1)}\cdots\epsilon_n^{2(1+\alpha_n)}\right).\end{aligned}\tag{4.8}$$

Using the equality (4.3) and Minkowski's inequality, we get

$$\begin{aligned}&\left(\lim_{T\to\infty}\frac{1}{V^{1-\alpha}(T)}\int_{C(T)}|X(t+k)+zX(t)|^2\,dt\right)^{1/2}\\ &\quad= \left(\lim_{\epsilon\to 0}\frac{1}{V^{1+\alpha}(\epsilon)}\int_{\mathfrak{R}^n}|Z_{\epsilon,\alpha}(\lambda; k)+zZ_{\epsilon,\alpha}(\lambda; 0)|^2\,d\lambda\right)^{1/2}, \qquad |z|=1\\ &\quad\le \left(\lim_{\epsilon\to 0}\frac{1}{V^{1+\alpha}(\epsilon)}\int_{\mathfrak{R}^n}|Z_{\epsilon,\alpha}(\lambda; k)-e^{-i(k,\lambda)}Z_{\epsilon,\alpha}(\lambda; 0)|^2\,d\lambda\right)^{1/2}\\ &\qquad+ \left(\lim_{\epsilon\to 0}\frac{1}{V^{1+\alpha}(\epsilon)}\int_{\mathfrak{R}^n}|z+e^{-i(k,\lambda)}|^2|Z_{\epsilon,\alpha}(\lambda; 0)|^2\,d\lambda\right)^{1/2}\\ &\quad= \lim_{\epsilon\to 0}\left(\frac{1}{V^{1+\alpha}(\epsilon)}\int_{\mathfrak{R}^n}\left(2+ze^{i(k,\lambda)}+\bar{z}e^{-i(k,\lambda)}\right)|Z_{\epsilon,\alpha}(\lambda; 0)|^2\,d\lambda\right)^{1/2}\end{aligned}\tag{4.9}$$

in view of (4.8). Taking $z = 1, -1, i, -i$ successively in (4.9) and substituting the four values in (4.7) yields (4.2) as required.

REMARK 2. From the representation (4.2), the spectral density of the fields $X(t) \in \mathscr{B}_\alpha$ satisfying condition (4.1) has the form

$$\begin{aligned}f(\lambda) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon_1^{\alpha_1}\cdots\epsilon_n^{\alpha_n}}\frac{|Z_{\epsilon,\alpha}(\lambda)|^2}{2\epsilon_1\cdots 2\epsilon_n}\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon_1^{\alpha_1}\cdots\epsilon_n^{\alpha_n}}f_*(\lambda), \qquad \epsilon_j > 0,\ 0\le\alpha_j<1,\ j=1,\dots,n.\end{aligned}$$

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We may consider $\epsilon_j \sim \lambda_j$, $j = 1, \dots, n$. Then, for $f_*(\lambda)$ slowly varying as $\lambda \downarrow 0$, $f(\lambda)$ may be considered as the spectral density of a LRD random field as defined in (1.1).

As specified in the proof of Theorem 4, we may take $\epsilon_j = 1/T_j$, $j = 1, \dots, n$. Then the periodogram of the random field may be taken as

$$\frac{1}{(2\pi)^n}\frac{1}{T_1^{1+\alpha_1}\cdots T_n^{1+\alpha_n}}\frac{1}{\pi^n}\left|\int_{\mathfrak{R}^n} X(t)\prod_{j=1}^{n}\left(\frac{\sin T_j^{(1+\alpha_j)}t_j}{T_j^{-(1+\alpha_j)}t_j}\right)e^{i(t,\lambda)}\,dt\right|.$$

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