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On the Multiplicities of Characters in Table Algebras

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Abstract. In this paper we show that every module of a table algebra can be considered as a faithful module of some quotient table algebra. Also we prove that every faithful module of a table algebra determines a closed subset that is a cyclic group. As a main result we give some information about multiplicities of characters in table algebras.

1 Introduction

In [6], Hanaki proved that every character of an association scheme can be considered as a faithful character of some quotient scheme. Also he showed that a faithful character of an association scheme determines a thin closed subset that is cyclic as a finite group. In this paper we first generalize the above facts for table algebras. Then as an application of them, we give some information about multiplicities of characters for a table algebra. More precisely, we first show that for every irreducible character χ of a table algebra (A, B) we have

$$\zeta_\chi \leq \frac{|B^+|}{\chi(1)|Z(\chi)^+|}, \tag{1.1}$$

where ζ_χ is the multiplicity of χ. Then we give a condition for which the equality occurs in (1.1). More precisely, we show that if (A, B) is a table algebra and $\chi \in \mathrm{Irr}(A)$ such that $B/\!\!/Z(\chi)$ is an abelian group, then

$$\zeta_\chi = \frac{|B^+|}{\chi(1)|Z(\chi)^+|}.$$

In particular, if (A, B) is commutative, then

$$\zeta_\chi = \frac{|B^+|}{|Z(\chi)^+|}.$$

This is a generalization of [5, Theorem 2.31] in the character theory of finite groups which states that if G is a finite group and $\chi \in \mathrm{Irr}(G)$ such that $G/Z(\chi)$ is abelian, then $\chi(1)^2 = |G : Z(\chi)|$.

Throughout this paper we follow [1] for the definition of *table algebras* and related notions. Hence we deal with non-commutative table algebras defined as follows.

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A table algebra (A, B) is a finite dimensional algebra A over the complex field $\mathbb{C}$ and a distinguished basis $B = \{b_1 = 1, \cdots, b_d\}$ for A, where 1 is a unit, such that the following properties hold:

(a) The structure constants of B are nonnegative real numbers, *i.e.*, for $a, b \in B$,

$$ab = \sum_{c \in B} \lambda_{abc} c, \qquad \lambda_{abc} \in \mathbb{R}^+ \cup \{0\}.$$

(b) There is a semilinear involutory anti-automorphism of A (denoted by *) such that $B^* = B$.

(c) For all $a, b \in B$, $\lambda_{aa^*1} > 0$ and $\lambda_{ab1} = 0$ if $b \neq a^*$.

Remark 1.1 (i) Let (A, B) be a table algebra. Then [1, Theorem 3.11] implies that A is semisimple.

(ii) For any table algebra (A, B), there is a unique algebra homomorphism $|\cdot| \colon A \to \mathbb{C}$, called the *degree map*, such that $|b| = |b^*| > 0$ for all $b \in B$ (see [1, Theorem 3.14]).

(iii) If $|b| = \lambda_{bb^*1}$ for all $b \in B$, then (A, B) is called the *standard table algebra*.

Without loss of generality, in this paper we will assume that (A, B) is a standard table algebra.

The value $|b|$ is called the *degree* of the basis element b. For an arbitrary element $\sum_{b \in B} x_b b \in A$, we have $|\sum_{b \in B} x_b b| = \sum_{b \in B} x_b |b|$.

For any $x = \sum_{b \in B} x_b b \in A$ we denote by $\operatorname{Supp}(x)$ the set of all basis elements $b \in B$ such that $x_b \neq 0$. If $E, D \subseteq B$, then we set $ED = \bigcup_{e \in E, d \in D} \operatorname{Supp}(ed)$.

A nonempty subset $C \subseteq B$ is called *closed* if $C^* C \subseteq C$, where $C^* = \{c^* \mid c \in C\}$. We denote by $\mathcal{C}(B)$ the set of all closed subsets of B. In addition, a closed subset C of B is called *strongly normal* if for every $b \in B$, $b^* C b \subseteq C$. An element $b \in B$ is called *linear* if $bb^* = |b| 1$. From [1, Proposition 4.6], the set of all linear elements of B is a closed subset of B that forms a finite group.

Let (A, B) be a table algebra with basis B and let $C \in \mathcal{C}(B)$. From [1, Proposition 4.7], it follows that $\{CbC \mid b \in B\}$ is a partition of B. The subset CbC is called a *C-double coset* or *double coset* with respect to the closed subset C. Let

$$b /\!\!/ C := |C^+|^{-1} (CbC)^+ = |C^+|^{-1} \sum_{x \in CbC} x,$$

where $C^+ = \sum_{c \in C} c$ and $|C^+| = \sum_{c \in C} |c|$. Then the following theorem is an immediate consequence of [1, Theorem 4.9].

Theorem 1.2 *Let (A, B) be a table algebra and let $C \in \mathcal{C}(B)$. Suppose that $\{b_1 = 1, \ldots, b_k\}$ is a complete set of representatives of C-double cosets. Then the vector space spanned by the elements $b_i /\!\!/ C$, $1 \leq i \leq k$, is a table algebra (which is denoted by $A /\!\!/ C$) with a distinguished basis $B /\!\!/ C = \{b_i /\!\!/ C \mid 1 \leq i \leq k\}$. The structure constants are given by*

$$\gamma_{ijk} = |C^+|^{-1} \sum_{\substack{r \in C b_i C, \\ s \in C b_j C}} \lambda_{rst},$$

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where $t \in Cb_kC$ is an arbitrary element.

The table algebra $(A/\!\!/C, B/\!\!/C)$ is called the *quotient table algebra* of (A, B) modulo C. One can see that a closed subset C of table algebra (A, B) is strongly normal if and only if $(A/\!\!/C, B/\!\!/C)$ is a group algebra; see [2, Corollary 2.9].

Let (A, B) be a table algebra and $C \in \mathcal{C}(B)$. Set $e = |C^+|^{-1}C^+$. Then e is an idempotent of A, and the subalgebra eAe is equal to the quotient table algebra $(A/\!\!/C, B/\!\!/C)$ modulo C; see [1].

2 Faithful Modules

Let (A, B) be a table algebra. The *kernel* of an A-module V in B is defined by

$$\ker_B V = \left\{ b \in B \;\middle|\; bx = |b|x, \forall x \in V \right\}.$$

From [8, Proposition 4.5], it follows that $\ker_B V$ is a closed subset in B and if χ is the character of A afforded by the A-module V, then $\ker_B V = \ker_B \chi$, where $\ker_B \chi = \{b \in B \mid \chi(b) = |b|\chi(1)\}$. Furthermore, the A-module V or character χ is called *faithful* if $\ker_B V = \{1\}$.

In the theorem below we show that every A-module V can be considered as a faithful $A/\!\!/K$-module, where $K = \ker_B V$. It might be mentioned that this is an analog of the association schemes that was done by Hanaki in [6].

Theorem 2.1 *Let (A, B) be a table algebra and V be an A-module. Suppose that L is a closed subset of B contained in $K = \ker_B V$. Then V can be considered as an $A/\!\!/L$-module by the multiplication*

$$(b/\!\!/L)v = \frac{|b/\!\!/L|}{|b|}bv, \quad v \in V.$$

Moreover, if $L = K$, then V is faithful as an $A/\!\!/K$-module.

Proof Put $e = |L^+|^{-1}L^+$. Then e is an idempotent of A and for every $b \in B$ we have

$$b/\!\!/L = \frac{|b/\!\!/L|}{|b|}ebe.$$

Moreover, by assumption of the theorem one can see that for $v \in V$,

$$ev = (|L^+|^{-1}L^+)v = |L^+|^{-1}(L^+v) = |L^+|^{-1}|L^+|v = v.$$

This implies that for every $b \in B$, $ebe(v) = eb(ev) = (eb)v = e(bv) = bv$. Now for every $v \in V$, we consider the multiplication

$$(b/\!\!/L)v = \frac{|b/\!\!/L|}{|b|}ebe(v) = \frac{|b/\!\!/L|}{|b|}bv.$$

We will show that the above multiplication is well defined. Suppose $b/\!\!/L = c/\!\!/L$. Then we have

$$\frac{|b/\!\!/L|}{|b|}ebe = \frac{|c/\!\!/L|}{|c|}ece.$$

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So for every $v \in V$,

$$\frac{|b/\!\!/L|}{|b|}bv = \frac{|b/\!\!/L|}{|b|}ebe(v) = \frac{|c/\!\!/L|}{|c|}ece(v) = \frac{|c/\!\!/L|}{|c|}cv.$$

Now since for every $b, c \in B$ and $v \in V$,

$$(b/\!\!/L\, c/\!\!/L)v = \frac{|b/\!\!/L|}{|b|}\frac{|c/\!\!/L|}{|c|}(ebece)v = \frac{|b/\!\!/L|}{|b|}\frac{|c/\!\!/L|}{|c|}ebe\big(ece(v)\big) = b/\!\!/L\big(c/\!\!/L(v)\big),$$

it follows that V is an $A/\!\!/L$-module. Moreover, suppose that $L = K$ and $b/\!\!/K \in \ker_B V$ as an $A/\!\!/K$-module. Then for every $v \in V$, $(b/\!\!/K)v = |b/\!\!/K|v$. So

$$\frac{|b/\!\!/K|}{|b|}bv = |b/\!\!/K|v,$$

and hence $bv = |b|v$. This implies that $b \in \ker_B V = K$ and then $b/\!\!/K = 1/\!\!/K$. Therefore, V is faithful as an $A/\!\!/K$-module, as desired. ■

The following corollary is a generalization of [6, Theorem 2.1] for table algebras.

Corollary 2.2 *Let (A, B) be a table algebra and χ be a character of A afforded by an A-module V. Suppose that T is a closed subset of B contained in $\ker_B V$. Then we can define a character χ' of $A/\!\!/L$ such that*

$$\chi'(b/\!\!/L) = \frac{|b/\!\!/L|}{|b|}\chi(b).$$

Moreover, χ' is faithful if $T = \ker_B V$.

Let (A, B) be a table algebra and V be an A-module. We define

$$Z(V) = \left\{ b \in B \;\middle|\; \forall v \in V,\ bv = \lambda_b v,\ \text{where } \lambda_b \in \mathbb{C} \text{ and } |\lambda_b| = |b| \right\}.$$

Clearly $\ker_B V \subseteq Z(V)$. In the following lemma we show that $Z(V)$ is a closed subset of B.

Lemma 2.3 *For every A-module V, $Z(V)$ is a closed subset of B.*

Proof Let $b, c \in Z(V)$. Then for every $v \in V$

$$b(cv) = b(\lambda_c v) = \lambda_b \lambda_c v, \tag{2.1}$$

where $\lambda_b, \lambda_c \in \mathbb{C}$ such that $|\lambda_b| = |b|$ and $|\lambda_c| = |c|$. On the other hand, suppose that

$$bc = \sum_{d \in B} \lambda_{bcd} d.$$

Then for every $v \in V$

$$(bc)v = \sum_{d \in B} \lambda_{bcd} dv = \sum_{d \in B} \lambda_{bcd} \sum_{w \in T} \mu_{dw} w, \tag{2.2}$$

where T is a $\mathbb{C}$-basis of V and for every $w \in T$, $\mu_{dw} \in \mathbb{C}$. But since $b(cv) = (bc)v$, from (2.1) and (2.2) we get

$$\lambda_b \lambda_c v = \sum_{d \in B} \lambda_{bcd} \mu_{dv} v,$$

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and so

$$\lambda_b\lambda_c = \sum_{d\in B} \lambda_{bcd}\mu_{dv}.$$

Since from [8, Proposition 4.1] it follows that $|\mu_{dv}| \le |d|$, by the latter equality we get

$$|b||c| = |\lambda_b||\lambda_c| = |\lambda_b\lambda_c| = \Big|\sum_{d\in B} \lambda_{bcd}\mu_{dv}\Big| \le \sum_{d\in B} \lambda_{bcd}|\mu_{dv}| \le \sum_{d\in B} \lambda_{bcd}|d| = |b||c|.$$

Hence we conclude that for every $d \in B$, where $\lambda_{bcd} \neq 0$, $|\mu_{dv}| = |d|$. This implies that for every $d \in B$, where $\lambda_{bcd} \neq 0$, $dv = \mu_{dv}v$ and $|\mu_{dv}| = |d|$. So $d \in Z(V)$. Therefore, $Z(V)$ is a closed subset of B, as desired. ■

Let (A, B) be a table algebra and V be an A-module. Let D be the representation of A corresponding to V. Then one can see that

$$Z(V) = \left\{ b \in B \;\middle|\; D(b) = \lambda_b I, \text{ where } \lambda_b \in \mathbb{C} \text{ and } |\lambda_b| = |b| \right\}.$$

In the lemma below we show that if V is a faithful A-module, then $Z(V)$ is cyclic as a finite group.

Lemma 2.4 *Let V be a faithful A-module of a table algebra (A, B). Then every element of $Z(V)$ is linear. In particular, $Z(V)$ is cyclic as a finite group.*

Proof Let D be the representation of A corresponding to V and let $b \in Z(V)$. Then we have

$$D(bb^*) = D(b)D(b^*) = \lambda_b\overline{\lambda_b}I = |\lambda_b|^2 I = |b|^2 I, \tag{2.3}$$

where I is the identity matrix. On the other hand, suppose that

$$bb^* = \sum_{d\in B} \lambda_{bb^*d}\, d.$$

Then we have

$$D(bb^*) = \sum_{d\in B} \lambda_{bb^*d} D(d). \tag{2.4}$$

From (2.3) and (2.4) we conclude that for every $d \in \mathrm{Supp}(bb^*)$, $D(d) = |d|I$. So for every $d \in \mathrm{Supp}(bb^*)$, $d \in \ker_B(V) = \{1\}$. Thus $bb^* = \{1\}$ and b is a linear element of B. Moreover, since for every $b, c \in B$, $D(bcb^*c^*) = D(b)D(c)D(b^*)D(c^*) = I$, we have $bcb^*c^* \in \ker_B(V) = \{1\}$ and hence $bc = cb$. So we conclude that $Z(V)$ is an abelian group.

To prove the second statement, since for every irreducible constituent W of V we have $Z(V) \subseteq Z(W)$, we can assume that V is irreducible. Now we define $\lambda\colon Z(V) \mapsto \mathbb{C} - \{0\}$ by $\lambda(b) = \lambda_b$. Suppose that $b, c \in B$ such that $\lambda_b = \lambda_c$. Then $D(bc^*) = D(b)D(c^*) = \lambda_b\overline{\lambda_c} = |b||c|I$, where I is the identity matrix. It follows that $D(d) = |d|I$, for every $d \in \mathrm{Supp}(bc^*)$. This implies that $d \in \ker_B(V) = \{1\}$, for every $d \in \mathrm{Supp}(bc^*)$ and hence $b = c$. Thus λ is a faithful irreducible representation of abelian group $Z(V)$. Now from [5, Theorem 2.32(a)], $Z(V)$ is cyclic as a finite group. ■

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Definition 2.5 Let (A,B) be a table algebra and χ be a character of A. We define

$$Z(\chi)=\left\{ b\in B \;\middle|\; |\chi(b)|=|b|\chi(1)\right\}.$$

One can see that if χ is afforded by an A-module V, then $Z(\chi)=Z(V)$, and so $Z(\chi)$ is a closed subset of B.

In the corollary below we give a generalization of [6, Theorem 3.1] for table algebras.

Corollary 2.6 *Let χ be a character of table algebra (A,B). Then every element of $Z(\chi)$ is linear. In particular, $Z(\chi)$ is cyclic as a finite group.*

3 Multiplicities of Characters

Let (A,B) be a table algebra. Define a linear function ζ on A by $\zeta(b)=\delta_{b,1}|B^+|$ for $b\in B$, where $|B^+|=\sum_{b\in B}|b|$. Then ζ is a non-degenerate feasible trace on A, and from [7] it follows that

$$\zeta=\sum_{\chi\in\mathrm{Irr}(A)}\zeta_\chi\chi,$$

where $\zeta_\chi\in\mathbb{C}$ and all ζ_χ are nonzero. The feasible trace ζ is called the *standard feasible trace*, and ζ_χ is called the *standard feasible multiplicity* or briefly the *multiplicity* of character χ.

For every $\chi,\varphi\in\mathrm{Hom}_{\mathbb{C}}(A,\mathbb{C})$, we define the inner product of χ and φ as follows:

$$[\chi,\varphi]=\frac{1}{|B^+|}\sum_{b\in B}\frac{1}{|b|}\chi(b)\varphi(b^*).$$

From [7, Lemma 3.1(ii)], one can see that

$$[\chi,\psi]=\delta_{\chi,\psi}\frac{\chi(1)}{\zeta_\chi},$$

for any $\chi,\psi\in\mathrm{Irr}(A)$.

Now let H be a closed subset of B. Then for every character χ of A we have

$$|H^+|[\chi_H,\chi_H]=\sum_{b\in H}\frac{\chi(b)\chi(b^*)}{|b|}\le\sum_{b\in B}\frac{\chi(b)\chi(b^*)}{|b|}=|B^+|[\chi,\chi].$$

Then

$$[\chi_H,\chi_H]\le\frac{|B^+|}{|H^+|}[\chi,\chi] \tag{3.1}$$

with equality if and only if $\chi(b)=0$, for every $b\in B-H$. Now let $H=Z(\chi)$ for some irreducible character of A. Then since

$$[\chi_{Z(\chi)},\chi_{Z(\chi)}]=\frac{1}{|Z(\chi)^+|}\sum_{b\in Z(\chi)}\frac{\chi(b)\chi(b^*)}{|b|}=\frac{1}{|Z(\chi)^+|}\sum_{b\in Z(\chi)}\frac{|\chi(b)|^2}{|b|}=\chi(1)^2,$$

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from inequality (3.1) we get

$$\chi(1)^2 = [\chi_{Z(\chi)}, \chi_{Z(\chi)}] \leq \frac{|B^+|}{|Z(\chi)^+|}[\chi, \chi] = \frac{|B^+|}{|Z(\chi)^+|}\frac{\chi(1)}{\zeta_\chi},$$

and thus

$$\chi(1)\zeta_\chi \leq \frac{|B^+|}{|Z(\chi)^+|}. \tag{3.2}$$

Equality occurs if and only if for every $b \in B - Z(\chi)$, $\chi(b) = 0$.

The following theorem gives a condition under which the equality occurs in (3.2).

Theorem 3.1 *Let (A, B) be a table algebra and $\chi \in \mathrm{Irr}(A)$. Suppose that $B/\!\!/Z(\chi)$ is an abelian group. Then $\chi(1)\zeta_\chi = |B^+|/|Z(\chi)^+|$.*

Proof From the above remark it is enough to show that for every $b \in B - Z(\chi)$, $\chi(b) = 0$. To do so, let $b \in B - Z(\chi)$. First we assume that χ is faithful. Then from Lemma 2.4, $Z = Z(\chi)$ is cyclic as a finite group. Since

$$1 = |b/\!\!/Z| = |Z^+|^{-1}|(ZbZ)^+| = |Z^+|^{-1}|(bZ)^+| = |Z^+|^{-1}\frac{|b||Z^+|}{|\,\mathrm{St}_Z(b)|} = \frac{|b|}{|\,\mathrm{St}_Z(b)|},$$

where $\mathrm{St}_Z(b) = \{t \in Z \mid bt = b\}$, we conclude that $|\,\mathrm{St}_Z(b)| > 1$. Thus there is an element $t \in Z$ such that $bt = b$. Then $\chi(b) = \chi(bt) = \mathrm{tr}\, D(bt) = \mathrm{tr}(D(b)D(t)) = \lambda_b\chi(b)$. Since $\lambda_b \neq 1$, from the latter equality we have $\chi(b) = 0$, as desired.

Now we suppose that $K = \ker(\chi) \neq \{1\}$. Then from Theorem 2.1, we can consider faithful irreducible character χ' of quotient table algebra $(A/\!\!/K, B/\!\!/K)$ such that

$$\chi'(b/\!\!/K) = \frac{|b/\!\!/K|}{|b|}\chi(b).$$

Furthermore,

$$\begin{aligned} Z(\chi') &= \left\{ b/\!\!/K \in B/\!\!/K \;\middle|\; |\chi'(b/\!\!/K)| = |b/\!\!/K|\chi'(1) \right\} \\ &= \left\{ b/\!\!/K \in B/\!\!/K \;\middle|\; \frac{|b/\!\!/K|}{|b|}|\chi(b)| = |b/\!\!/K|\chi'(1/\!\!/K) \right\} \\ &= \left\{ b/\!\!/K \in B/\!\!/K \;\middle|\; |\chi(b)| = |b|\chi(1) \right\} \\ &= \left\{ b/\!\!/K \in B/\!\!/K \;\middle|\; b \in Z \right\} = Z/\!\!/K. \end{aligned}$$

Since from [4, Proposition 2.13] we have $(B/\!\!/K)/\!\!/(Z/\!\!/K) = B/\!\!/Z$, we conclude that $(B/\!\!/K)/\!\!/Z(\chi')$ is an abelian group. Then from the first part of proof we conclude that for every $b/\!\!/K \in B/\!\!/K - Z(\chi')$, $\chi'(b/\!\!/K) = 0$. This implies for every $b \in B - Z$, $\chi(b) = 0$. ■

The following corollary is a generalization of [5, Theorem 2.31].

Corollary 3.2 *Let (A, B) be a commutative table algebra. Suppose that $\chi \in \mathrm{Irr}(A)$ such that $Z(\chi)$ is a strongly normal closed subset of B. Then*

$$\zeta_\chi = \frac{|B^+|}{|Z(\chi)^+|}.$$

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Corollary 3.3 ([5, Theorem 2.31]) *Let G be a finite group. Suppose that $\chi \in \mathrm{Irr}(G)$ such that $G/Z(\chi)$ is abelian. Then $\chi(1)^2 = |G : Z(\chi)|$.*

Proof Let $C_1, \dots, C_h$ be the conjugacy classes of G. Put $\mathrm{Cla}(G) = \{K_1, \dots, K_h\}$, where $K_i = \sum_{g \in C_i} g$. Then it is known that $(Z(\mathbb{C}G), \mathrm{Cla}(G))$ is a commutative table algebra, where $Z(\mathbb{C}G)$ is the center of group algebra $\mathbb{C}G$. One can see that the degree map of $(Z(\mathbb{C}G), \mathrm{Cla}(G))$ is defined by $K_i \to |C_i|$, for every $1 \le i \le h$, and $\{\omega_\chi \mid \chi \in \mathrm{Irr}(A)\}$ is the set of irreducible characters of $Z(\mathbb{C}(G))$, where

$$\omega_\chi(K_i) = \frac{\chi(g)|C_i|}{\chi(1)}, \tag{3.3}$$

for some $g \in C_i$. Moreover, for every $\chi \in \mathrm{Irr}(A)$, we have $\zeta_{\omega_\chi} = \chi(1)^2$. From (3.3) it follows that the closed subset $Z(\omega_\chi)$ corresponds to $Z(\chi) \trianglelefteq G$. Then one can see that

$$\mathrm{Cla}\big(G/Z(\chi)\big) \simeq \mathrm{Cla}(G)/\!\!/Z(\omega_\chi)$$

(see [3]). So our assumption implies that $\mathrm{Cla}(G)/\!\!/Z(\omega_\chi)$ is a finite group. Hence by Corollary 3.2 it follows that $\chi(1)^2 = \zeta_{\omega_\chi} = |G : Z(\omega_\chi)^+| = |G : Z(\chi)|$, as desired. ∎

Example 3.4 Let A be a $\mathbb{C}$-linear space with the basis $B = \{b_0 = 1, b_1, b_2, b_3\}$ such that

$$\begin{aligned} b_1^2 &= b_0, & b_1 b_2 &= b_2,\\ b_2^2 &= 2b_3, & b_1 b_3 &= b_3,\\ b_3^2 &= 2b_2, & b_2 b_3 &= 2b_0 + 2b_1. \end{aligned}$$

Then one can see that the pair (A, B) is a commutative table algebra, and an easy computation shows that the character table of (A, B) is

	b_0	b_1	b_2	b_3	ζ_{χ_i}
χ_1	1	1	2	2	1
χ_2	1	1	2ω	$2\omega^2$	1
χ_3	1	1	$2\omega^2$	2ω	1
χ_4	1	-1	0	0	3

,

where ω is a primitive third root of unity. One can see that $Z(\chi_4) = \{b_0, b_1\}$ is a strongly normal closed subset of B and $B/\!\!/Z(\chi_4)$ is an abelian group. Then the assertion of Corollary 3.2 holds.

Remark 3.5 The strongly normal condition in Corollary 3.2 is a necessary condition. In the example below we give a commutative table algebra for which the assertion of Corollary 3.2 does not hold.

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Example 3.6 Let A be a $\mathbb{C}$-linear space with the basis $B = \{b_0 = 1, b_1, b_2, b_3\}$ such that

$$b_1^2 = b_0, \qquad b_1 b_2 = b_3,$$
$$b_2^2 = 2b_0 + b_2, \qquad b_1 b_3 = b_2,$$
$$b_3^2 = 2b_0 + b_2, \qquad b_2 b_3 = 2b_1 + b_3.$$

Then one can see that the pair (A, B) is a commutative table algebra and an easy computation shows that the character table of (A, B) is

	b_0	b_1	b_2	b_3	ζ_{χ_i}
χ_1	1	1	2	2	1
χ_2	1	-1	2	-2	1
χ_3	1	-1	-1	1	2
χ_4	1	1	-1	-1	2

.

One can see that $Z(\chi_3) = \{b_0, b_1\}$ is not a strongly normal closed subset of B, and thus the assertion of Corollary 3.2 does not hold.

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