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ON A THEOREM OF RAV CONCERNING EGYPTIAN FRACTIONS

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Problems involving Egyptian fractions (rationals whose numerator is 1 and whose denominator is a positive integer) have been extensively studied. (See [1] for a more complete bibliography). Some of the most interesting questions, many still unsolved, concern the solvability of

$$\frac{m}{n}=\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}$$

where k is fixed.

In [2] Rav proved necessary and sufficient conditions for the solvabilty of the above equation, as a consequence of some other theorems which are rather complicated in their proofs. In this note we give a short, elementary proof of this theorem, and at the same time generalize it slightly.

THEOREM 1.

$$\frac{m}{n}=\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k} \tag{1}$$

if and only if there exist positive integers M and N and divisors $D_1,\ldots,D_k$ of N such that $M/N=m/n$ and $D_1+D_2+\cdots+D_k\equiv 0 \pmod{M}$. Also, the last condition can be replaced by $D_1+D_2+\cdots+D_k=M$; and the condition $(D_1, D_2,\ldots, D_k)=1$ may be added without affecting the validity of the theorem.

Proof. Suppose there exist M and N and divisors $D_1,\ldots,D_k$ of N such that $M/N=m/n$ and $D_1+D_2+\cdots+D_k=rM$. Then

$$\sum_{i=1}^{k}\frac{1}{rN/D_i}=\frac{rM}{rN}=\frac{m}{n} \tag{2}$$

and so (1) is solvable. This holds regardless of whether $r=1$ or whether $(D_1, D_2,\ldots, D_k)=1$.

Now suppose that (1) is solvable, then

$$\frac{m}{n}=\sum_{i=1}^{k}\frac{1}{x_i}=\frac{\sum_{i=1}^{k}x_1\cdots x_{i-1}x_{i+1}\cdots x_k}{x_1x_2\cdots x_k}. \tag{3}$$

Letting M and N be the numerator and denominator of the right hand fraction in (3) and letting $D_i=x_1\cdots x_{i-1}x_{i+1}\cdots x_k$ our theorem is satisfied. Furthermore

$D_1+D_2+\cdots D_k=M$. Also, letting $d=(D_1, D_2, \ldots, D_k)$, $M'=M/d$, $N'=N/d$, $D_i'=D_i/d$ then M', N' and the D_i' also satisfy the theorem, and $(D_1', D_2', \ldots, D_k')=1$.

This theorem would give an effective means of deciding the solvability of (1) if we could obtain an upper bound for the least M and N satisfying the theorem. Letting $M=Cm$ and $N=Cn$ we want a bound B such that if (1) is solvable, then our theorem is satisfied for some $C\leq B$. This is possible to do inductively, although the bounds obtained are rather cumbersome. We illustrate with the cases $k=2$ and 3—these cases are the most important and have been most extensively studied in other contexts.

THEOREM 2. *Letting $M=Cm$ and $N=Cn$, theorem* 1 *is satisfied with*

(i) $C \leq (n+1)/m$ if $k = 2$

(ii) $C \leq \max_{x\in(n/m,3n/m]} \dfrac{nx^2+x}{mx-n}$ if $k = 3$.

Proof. Suppose without loss of generality that $(m, n)=1$ and $m/n=(1/x_1)+(1/x_2)$. In this case we know that there exist $d_1, d_2 \mid n$ such that $d_1+d_2=tm$. Then theorem 1 is satisfied with $C=t$ and $t\leq(n+1)/m$ since $(d_1, d_2)=1$ which implies $d_1+d_2\leq n+1$. Examples such as 3/11 show that this is the best possible bound for C in the case $k=2$.

Now suppose $m/n=1/x_1+1/x_2+1/x_3$, where $x_1\leq x_2\leq x_3$. By applying part (i) to $m/n-1/x_1$ we get the bound given in (ii) since obviously $x_1 \in (n/m, 3n/m]$. The maximum occurs at either $x=[n/m]+1$ or $[3n/m]$ depending on the values of m and n in the specific case considered. A simpler bound for C such as $(n+m)(n^2+nm+m)/m^2$ could be used in (ii) although some precision would be lost.

The case where the x_i are allowed to be negative is also of considerable interest. A result completely analogous to Theorem 1 can be proved in this case with only the most minor changes. This extends Lemmas 2 and 3 of [3].

REFERENCES

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