On Absolute Summability for any Positive Order

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§ 1. Absolute summability according to Cesàro's method has been defined by Fekete[1] for positive integral orders, as follows:—

Denoting the rth partial sum of a series Σu_n by S_n^r and its rth mean, namely[2] S_n^r/A_n^r, by s_n^r, we can regard s_n^r as the sum of the series

$$\sum_{\nu=0}^{n} u_\nu^r \equiv \sum_{\nu=0}^{n} (s_\nu^r - s_{\nu-1}^r) \qquad (s_{-1}^r = 0).$$

Thus the convergence of Σu_n^r is equivalent to summability (C, r). If now this series is *absolutely convergent*, we say that Σu_n is *absolutely summable* (C, r), *or summable* $|C, r|$.

The above definition can be adapted without change for non-integral orders; it is the object of the present paper to extend to all positive orders the consistency and product results given by Fekete. As a preliminary investigation bearing on the question appears an analogue of Toeplitz's theorem for sequences of bounded variation, the sufficiency only, and not the necessity, being taken into consideration.

§ 2. *If s_n is a sequence of bounded variation, i.e. for which*

$$\Sigma u_n = \Sigma (s_n - s_{n-1})$$

is absolutely convergent, then

$$t_n = \sum_{\nu=0}^{n} a_{n,\nu} s_\nu \qquad (a_{n,\nu} = 0, \text{ if } \nu > n)$$

is also of bounded variation, provided that the double sum

$$(1) \qquad \sum_{0 \leqq \nu \leqq n \leqq N} |a_{n,\nu} - a_{n-1,\nu}| < K$$

for all values of N.

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[1] *Math. ès Term. Ért.* (1911), pp. 719-726.

[2] A_n^r stands for $\binom{n+r}{n}$.

For we have

$$v_n = t_n - t_{n-1} = \sum_{\nu=0}^{n} (a_{n,\nu} - a_{n-1,\nu})(u_0 + u_1 + \dots + u_\nu)$$
$$= \sum_{i=0}^{n} u_i \sum_{\nu=i}^{n} (a_{n,\nu} - a_{n-1,\nu});$$

and

$$\sum_{n=0}^{N} |v_n| \leqq \sum_{n=0}^{N} \sum_{i=0}^{n} |u_i| \sum_{\nu=i}^{n} |a_{n,\nu} - a_{n-1,\nu}|$$
$$= \sum_{i=0}^{N} |u_i| \sum_{i \leqq \nu \leqq n \leqq N} |a_{n,\nu} - a_{n-1,\nu}|$$
$$< K \sum_{i=0}^{N} |u_i|.$$

A simple case presents itself, when

$$\sum_{\nu=i}^{n} (a_{n,\nu} - a_{n-1,\nu}) \geqq 0 \qquad (0 \leqq i \leqq n).$$

For then

$$|v_n| \leqq \sum_{i=0}^{n} |u_i| \sum_{\nu=i}^{n} (a_{n,\nu} - a_{n-1,\nu})$$
$$= \sum_{\nu=0}^{n} (a_{n,\nu} - a_{n-1,\nu}) \sigma_\nu,$$

where $\sigma_\nu = \sum_{i=0}^{\nu} |u_i|$. It follows that

$$\sum_{n=0}^{N} |v_n| \leqq \sum_{\nu=0}^{N} a_{N,\nu} \sigma_\nu.$$

Consequently, if the ordinary Toeplitz conditions hold, we can deduce, without assuming (1), that Σv_n converges absolutely.

§ 3. We next prove two lemmas.

LEMMA I.

$$\frac{1}{A_n^{p+q+1}} \sum_{\nu=i}^{n-j} A_\nu^p A_{n-\nu}^q$$

does not increase as n increases $(p > -1,\ q > -1,\ p+q > -1;\ i, j, \geqq 0)$.

We have

$$0 < a_\nu = \frac{A_\nu^p A_{n-\nu}^q}{A_n^{p+q+1}} \underset{>}{\leqq} \frac{A_\nu^p A_{n-\nu-1}^q}{A_{n-1}^{p+q+1}} = \beta_\nu,$$

according as

$$\nu \underset{>}{\leqq} \frac{p+1}{p+q+1} n = \nu_1, \text{ say.}$$

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Hence, for $0 \leqq \nu \leqq \nu_1$,

$$\alpha_0 + \alpha_1 + \ldots . + \alpha_\nu < \beta_0 + \beta_1 + \ldots . + \beta_\nu ;$$

and for $\nu_1 < \nu < n$, noting that

$$\sum_{\nu=0}^{n} \alpha_\nu = \sum_{\nu=0}^{n} \beta_\nu = 1,$$

we get

$$\begin{aligned} \alpha_0 + \alpha_1 + \ldots . + \alpha_\nu &= 1 - (\alpha_{\nu+1} + \alpha_{\nu+2} + \ldots . + \alpha_n) \\ &< 1 - (\beta_{\nu+1} + \beta_{\nu+2} + \ldots . + \beta_{n-1}) \\ &= \beta_0 + \beta_1 + \ldots . + \beta_\nu . \end{aligned}$$

Therefore

$$\frac{1}{A_n^{p+q+1}} \sum_{\nu=0}^{i-1} A_\nu^p A_{n-\nu}^q$$

decreases, as n increases $(i > 0)$. And in the same way

$$\frac{1}{A_n^{p+q+1}} \sum_{\nu=0}^{j-1} A_\nu^q A_{n-\nu}^p$$

decreases, as n increases $(j > 0)$. Our result thus follows, since[1]

$$\frac{1}{A_n^{p+q+1}} \sum_{\nu=i}^{n-j} A_\nu^p A_{n-\nu}^q = \frac{1}{A_n^{p+q+1}} \left\{ \sum_{\nu=0}^{n} - \sum_{\nu=0}^{i-1} - \sum_{\nu=n-j-1}^{n} \right\} A_\nu^p A_{n-\nu}^q .$$

LEMMA II.

$$A_j^l \sum_{\nu=0}^{\infty} \frac{A_\nu^{k-1}}{A_{j+\nu}^{k+l}}$$

is finite for all $j \geqq 0,\ k \geqq 0,\ l > 0.$

When $k = 0$, the sum reduces to unity. Otherwise we have $\sum_{\nu=0}^{j} A_\nu^{k-1} = A_j^k$, and as i becomes great,

$$A_i^r \simeq \frac{i^r}{\Gamma(r+1)} \quad (r > -1),$$

so that

$$\begin{aligned} \sum_{\nu=0}^{\infty} \frac{A_\nu^{k-1}}{A_{j+\nu}^{k+l}} &= \left(\sum_{\nu=0}^{j} + \sum_{\nu=j+1}^{\infty} \right) \frac{A_\nu^{k-1}}{A_{j+\nu}^{k+l}} \\ &> \frac{A_j^k}{A_j^{k+l}} + O \sum_{\nu=j+1}^{\infty} \left(\frac{\nu}{j+\nu} \right)^{k-1} \frac{1}{(j+\nu)^{l+1}} = O\left(\frac{1}{j^l}\right), \end{aligned}$$

[1] If i or j is zero, we omit the corresponding sum.

which yields the required result. We conclude with our two theorems.

§4. THEOREM I. *If Σu_n is summable $|C, k|$, then it is also summable $|C, k+l|$,* $(k \geqq 0, \quad l > 0)$.

We have

$$s_n^{k+l} = \frac{S_n^{k+l}}{A_n^{k+l}} = \sum_{\nu=0}^{n} \frac{A_{n-\nu}^{l-1} \, A_\nu^k}{A_n^{k+l}} \, s_\nu^k$$

$$= \sum_{\nu=0}^{n} a_{n\nu} \, s_\nu^k, \text{ say;}$$

and from Lemma I, with $j = 0$, it appears that[1]

$$\sum_{\nu=i}^{n} (a_{n,\nu} - a_{n-1,\nu}) \geqq 0.$$

Consequently, in view of our earlier remarks, the series Σu_n^{k+l} converges absolutely.

THEOREM II. *If Σu_n is summable $|C, k|$ and Σv_n is summable $|C, l|$, then the product series*

$$\Sigma w_n \equiv \Sigma (u_0 v_n + u_1 v_{n-1} + \ldots + u_n v_0)$$

is summable $|C, k+l|$, $(k \geqq 0, \; l \geqq 0)$.

When k and l are both zero, this reduces to the well-known theorem on the multiplication of two absolutely convergent series. We may suppose then that $l > 0$. Denoting the partial sums of order r for Σu_n and Σv_n by S_n^r and T_n^r respectively, we have, when $|x| < 1$,

$$(1-x)^{-k-l-1} \sum_{n=0}^{\infty} u_n x^n \, . \sum_{n=0}^{\infty} v_n x^n = (1-x)^{-k} \sum_{n=0}^{\infty} u_n x^n \, . \, (1-x)^{-l-1} \sum_{n=0}^{\infty} v_n x^n$$

$$= \sum_{n=0}^{\infty} S_n^{k-1} x^n \, . \sum_{n=0}^{\infty} T_n^l x^n.$$

Hence the partial sum of order $k+l$ for the product series is

$$P_n^{k+l} = \sum_{\nu=0}^{n} (S_\nu^k - S_{\nu-1}^k) \, T_{n-\nu}^l$$

$$= \sum_{\nu=0}^{n} (A_\nu^k \, s_\nu^k - A_{\nu-1}^k \, s_{\nu-1}^k) \, A_{n-\nu}^l \, t_{n-\nu}^l$$

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[1] $a_{n-1,n}$ vanishes, if we take $A_{-1}^r = 0$ $(r = 0, \; -1, \; -2 \ldots)$. This follows, if we suppose $A_{n-1}^r = A_n^r \dfrac{n}{n+r}$ true for $n = 0$.

and

$$w_n^{k+l} = \frac{P_n^{k+l}}{A_n^{k+l}} - \frac{P_{n-1}^{k+l}}{A_{n-1}^{k+l}}$$

$$= \sum_{\nu=0}^{n} (A_\nu^k s_\nu^k - A_{\nu-1}^k s_{\nu-1}^k)\left(\frac{A_{n-\nu}^l t_{n-\nu}^l}{A_n^{k+l}} - \frac{A_{n-\nu-1}^l t_{n-\nu-1}^l}{A_{n-1}^{k+l}}\right)$$

$$= \sum_{\nu=0}^{n} (A_\nu^{k-1} s_\nu^k + A_{\nu-1}^k u_\nu^k)\left\{\left(\frac{A_{n-\nu}^l}{A_n^{k+l}} - \frac{A_{n-\nu-1}^l}{A_{n-1}^{k+l}}\right) t_{n-\nu}^l + \frac{A_{n-\nu-1}^l}{A_{n-1}^{k+l}} v_{n-\nu}^l\right\}$$

$$= \sum_{\nu=0}^{n} A_\nu^{k-1} s_\nu^k \left(\frac{A_{n-\nu}^l}{A_n^{k+l}} - \frac{A_{n-\nu-1}^l}{A_{n-1}^{k+l}}\right) t_{n-\nu}^l + \sum_{\nu=0}^{n-1} A_\nu^{k-1} s_\nu^k \frac{A_{n-\nu-1}^l}{A_{n-1}^{k+l}} v_{n-\nu}^l$$

$$+ \sum_{\nu=1}^{n} A_{\nu-1}^k u_\nu^k \left(\frac{A_{n-\nu}^l}{A_n^{k+l}} - \frac{A_{n-\nu-1}^l}{A_{n-1}^{k+l}}\right) t_{n-\nu}^l + \sum_{\nu=0}^{n-1} A_{\nu-1}^k u_\nu^k \frac{A_{n-\nu-1}^l}{A_{n-1}^{k+l}} v_{n-\nu}^l$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \quad \text{say.}^{1}$$

Now

$$\Sigma_1 = \sum_{\nu=0}^{n} A_\nu^{k-1} \sum_{i=0}^{\nu} u_i^k \sum_{j=0}^{n-\nu} v_j^l \left(\frac{A_{n-\nu}^l}{A_n^{k+l}} - \frac{A_{n-\nu-1}^l}{A_{n-1}^{k+l}}\right)$$

$$= \sum_{0 \leqq i+j \leqq n} u_i^k v_j^l \left\{\sum_{\nu=i}^{n-j} \frac{A_\nu^{k-1} A_{n-\nu}^l}{A_n^{k+l}} - \sum_{\nu=i}^{n-j-1} \frac{A_\nu^{k-1} A_{n-\nu-1}^l}{A_{n-1}^{k+l}}\right\}.$$

But the expression in brackets is $\geqq 0$ by Lemma I. Hence

$$|\Sigma_1| \leqq \sum_{0 \leqq i+j \leqq n} |u_i^k|\,|v_j^l| \left\{\sum_{\nu=i}^{n-j} \frac{A_\nu^{k-1} A_{n-\nu}^l}{A_n^{k+l}} - \sum_{\nu=i}^{n-j-1} \frac{A_\nu^{k-1} A_{n-\nu-1}^l}{A_{n-1}^{k+l}}\right\},$$

$$\sum_{n=1}^{N} |\Sigma_1| \leqq \sum_{0 \leqq i+j \leqq N} |u_i^k|\,|v_j^l| \sum_{\nu=i}^{N-j} \frac{A_\nu^{k-1} A_{N-\nu}^l}{A_N^{k+l}}$$

$$\leqq \sum_{0 \leqq i+j \leqq N} |u_i^k|\,|v_j^l| \leqq \sum_{i=0}^{\infty} |u_i^k| \sum_{j=0}^{\infty} |v_j^l| = O(1).$$

Next, using Lemma II, we get, when $\bar{s}$ is the upper bound of $|s_\nu|$,

$$\sum_{n=1}^{N} |\Sigma_2| \leqq \bar{s} \sum_{n=1}^{N} \frac{1}{A_n^{k+l}} \sum_{\nu=0}^{n-1} A_\nu^{k-1} A_{n-\nu-1}^l |v_{n-\nu}^l|$$

$$= \bar{s} \sum_{j=1}^{N} |v_j^l| O(1) = O(1).$$

In Σ_3 we notice that, for $k > 0$,

$$\left|\frac{A_{n-\nu}^l}{A_n^{k+l}} - \frac{A_{n-\nu-1}^l}{A_{n-1}^{k+l}}\right| = \left|\frac{A_{n-\nu}^{l-1}}{A_n^{k+l}} - \frac{k+l}{n} \frac{A_{n-\nu-1}^l}{A_n^{k+l}}\right| < K \frac{A_{n-\nu}^{l-1}}{A_n^{k+l}}$$

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[1] It should be observed that $\sum_1$ and $\sum_2$ do not occur when $k = 0$.

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and, for $k = 0$, the expression is equal to

$$\frac{\nu A_{n-\nu}^{l-1}}{n A_n^l} < K' \frac{A_{\nu-1}^1 A_{n-\nu}^{l-1}}{A_n^{l+1}}$$

Σ_3 can therefore be treated like Σ_2, so that $\sum_{n=1}^{N} |\Sigma_3| = O(1)$.

Finally

$$|\Sigma_4| \leqq \sum_{\nu=0}^{n-1} |u_\nu^k| \, |v_{n-\nu}^l|,$$

$$\sum_{n=1}^{N} |\Sigma_4| \leqq \sum_{\nu=0}^{\infty} |u_\nu^k| \sum_{\nu=0}^{\infty} |v_\nu^l| = O(1).$$

Combining results we see that $\sum_{n=1}^{N} |w_n^{k+l}|$ is finite for all N, *i.e.* that the product series is summable $|C, k+l|$.

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