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ON KATĚTOV SPACES

BY

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ABSTRACT. Recent work by Krystock, Porter, and Vermeer has emphasized the importance of the concepts of Katětov spaces and H-sets in the theory of H-closed spaces. These properties are closely related to being the θ-closure of some set and being the adherence of an open filter. This relationship is developed by establishing, among other facts, that an H-closed space in which every closed set is the θ-closure of some set is compact and the θ-closure of a subset of an H-closed space is Katětov and characterizing the open filter adherences of a space as precisely those sets which are the image of a closed set of the absolute of the space. Also, examples are given of a countable, scattered space which is not Katětov and an H-closed space with an H-closed subspace which is not the θ-closure of any subset of the given space.

1. **Introduction and Preliminaries.** Katětov spaces and H-sets have been studied in detail recently by Porter and Vermeer [6]. These properties bear an intricate relationship with the properties of being H-closed and being the θ-closure of some set; thus, a deeper analysis is needed. In this paper several results and examples revealing this interrelationship and connection with the open filter adherence property (developed in [5]) are provided. Among other facts, the following results are provided: (1.8) There exists a θ-closed subset of an H-closed space which is not H-closed. (2.2) If X is H-closed and $A \subseteq X$, then $c\ell_\theta A$ is Katětov. (2.5) An H-closed space in which every closed set is the θ-closure of some set is compact. (2.7) There exists an H-closed subspace of an H-closed space which is not the θ-closure of any subset (in the given space). (3.3) There exists a countable, scattered space which is not an H-set in any space.

In this paper all spaces are assumed to be Hausdorff.

Let X and Y be spaces. A function $f : X \longrightarrow Y$ is **θ-continuous** if for each $x \in X$ and open neighborhood U of $f(x)$ in Y, there is an open neighborhood V of x in X such that $f[c\ell V] \subseteq c\ell U$. The function f is **perfect** if f is closed and every fiber is compact and is **irreducible** if for each $A \subseteq X$, A is closed and $f[A] = Y$ imply $A = X$. For $A \subseteq X$, denote $\{y \in Y : f^{\leftarrow}(y) \subseteq A\}$ by $f^{\#}[A]$. Note that for $A \subseteq X$, $f^{\#}[A] = Y \setminus f[X \setminus A]$ (f does not need to be onto). In particular, if f is a closed function and A is open in X, then $f^{\#}[A]$ is open in Y.

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Let X be a space. For $A \subseteq X$, the **θ-closure** of A, denoted by $c\ell_\theta A$, is defined as $\{x \in X : (c\ell U) \cap A \neq \emptyset$ for each open neighborhood U of $x\}$. A subset A of X is **θ-closed** if $A = c\ell_\theta A$. Note that the θ-closure of a subset in a nonregular space may not be θ-closed and that if X is extremely disconnected and $\mathcal{F}$ is an open filter on X, then $\mathrm{ad}_X\, \mathcal{F}$ is θ-closed. The next result is well-known and easy to verify.

PROPOSITION 1.1. *Let X and Y be spaces and $f : X \longrightarrow Y$ be a θ-continuous function. If $A \subseteq X$, then $f|_A : A \longrightarrow Y$ is θ-continuous and $f[c\ell_\theta A] \subseteq c\ell_\theta f[A]$.*

A space X is **H-closed** if X is a closed subset in every space containing X as a subspace. H-closed spaces are characterized (among Hausdorff spaces) by the property that every open cover has a finite subfamily whose union is dense. A related concept is that of an H-set – a subset A of a space X is an **H-set** if for every cover c of A by sets open in X there is a finite subset $\mathcal{F} \subseteq c$ such that $A \subseteq c\ell_X(\bigcup \mathcal{F})$. An H-set with the subspace topology may not be an H-closed subspace; however, an H-closed subspace is an H-set. If A is an H-set in X and $A \subseteq U \subseteq X$ where U is open (resp. X is a subspace of Y), then A is an H-set in U (resp. in Y). If X is an H-closed subspace and $A \subseteq X$, Velicko [8] has shown that $c\ell_\theta A$ is an H-set. The next results are well-known.

PROPOSITION 1.2. [7] *Let X and Y be spaces and $f : X \longrightarrow Y$ a θ-continuous function.*
(a) *If X is H-closed, then so is $f[X]$.*
(b) *If $A \subseteq X$ is an H-set, then so is $f[A]$.*

PROPOSITION 1.3. [5] *If $\mathcal{F}$ is an open filter on an H-closed space X, then* $\mathrm{ad}_X\, \mathcal{F}$ *is an H-set*

A subset A of a space X is **regular open** (resp. **regular closed**) if $A = \mathrm{int}(c\ell A)$ (resp. $A = c\ell(\mathrm{int} A)$). The set of all regular open sets of X form a base for a topology on the underlying set of X, and $X(s)$ denotes this new space. A space is **semiregular** if $X = X(s)$. It is straightforward to verify that $X(s)$ is a semiregular space.

PROPOSITION 1.4. [7] *Let X be an H-closed space and $A \subseteq X$.*
(a) *If A is regular closed, then A is an H-closed space.*
(b) *If A is an H-set in X, then $A = \bigcap\{Y : A \subseteq Y \subseteq X : Y$ is regular closed in $X\}$.*

For a space X, recall that the **Iliadis absolute** is an extremely disconnected, Tychonoff space EX and a perfect, θ-continuous, irreducible surjection $k : EX \longrightarrow X$. The Iliadis absolute is unique in this sense: if Y is an extremely disconnected, Tychonoff space and $f : Y \longrightarrow X$ is a perfect θ-continuous, irreducible surjection, then there is a homeomorphism $h : EX \longrightarrow Y$ such that $f \circ h = k$. The absolute EX can be considered as the set $\{\mathcal{U} : \mathcal{U}$ is an open ultrafilter on X and $\mathrm{ad}_X\, \mathcal{U} \neq \emptyset\}$ where the topology on EX is generated by the $\{OU : U$ is open in $X\}$ which is closed under finite unions and intersections, and $OU = \{\mathcal{U} : U \in \mathcal{U}\}$. For an open set U of X, $\mathrm{int}(c\ell U) = k^\#[OU] \subseteq k[OU] = c\ell U$ [7; Theorem 6.8(f)].

PROPOSITION 1.5. *Let X be a space and $A \subseteq X$.*

(a) [7] *X is H-closed* iff *EX is compact.*
(b) [1] *A is θ-closed in X iff $k^{\leftarrow}[A]$ is closed in EX.*

A space is **minimal Hausdorff** if it contains no strictly Hausdorff topology. Katětov [4] has characterized minimal Hausdorff spaces as those spaces which are H-closed and semiregular and if X is an H-closed space, then $X(s)$ is minimal Hausdorff. A space is **Katětov** if it has a coarser minimal H-closed topology or equivalently, a coarser H-closed topology. A number of facts about Katětov spaces have been established by Porter and Vermeer in [6, 9]. The next result gives two characterizations of Katětov spaces.

PROPOSITION 1.6. [2, 6, 9] *For a space X, the following are equivalent:*
(a) *X is Katětov,*
(b) *X is the remainder of an H-closed extension of a discrete space, and*
(c) *X is the perfect image of a compact space.*

A nice corollary to this result is the following:

COROLLARY 1.7.
(a) *A space is θ-closed in some H-closed space iff it is Katětov.*
(b) *A Katětov space is an H-set in some space.*

PROOF. The proof of one direction of (a) is immediate from 1.6(b). Conversely, suppose A is a θ-closed subspace of an H-closed space X. By 1.5, $k^{\leftarrow}[A]$ is a compact subspace of EX. Since $k|k^{\leftarrow}[A] : k^{\leftarrow}[A] \longrightarrow A$ is a perfect surjection (not necessarily θ-continuous), it follows from 1.6 that A is Katětov. The proof of (b) follows from these two statements: the remainder of an H-closed extension of a discrete space is θ-closed, and a θ-closed subspace of an H-closed space is an H-set (cf. the paragraph preceding 1.2). □

The space $\mathbb{Q}$ of rationals with the usual topology is an example of a space which is not Katětov [3]. Since $\mathbb{Q}$ is θ-closed in $\mathbb{Q}$, it follows immediately that "H-closed" cannot be removed from 1.7(a). On the other hand, a space A is an H-set in some space iff A is an H-set in an H-closed space as each space is contained in an H-closed space (see [4]) and by the comments in paragraph preceding 1.2. Vermeer [9; 5.3] has asked if the converse of 1.7(b) is true. The first space to try for a counterexample to 1.7(b) is $\mathbb{Q}$; however, Vermeer [9, 5.4] has shown that $\mathbb{Q}$ is not an H-set in any space.

The class of Katětov spaces is broad as each complete metric space is Katětov [4, 4.4]. In particular, the discrete space $\mathbb{N}$ is Katětov. But $\mathbb{N}$ is neither the θ-closure of another set nor an H-set in $\beta\mathbb{N}$; of course, by 1.6, $\mathbb{N}$ is an H-set in some H-closed space.

EXAMPLE 1.8. Consider the space $X = \mathbb{R} \cup \{p, q\}$ where $\mathbb{R}$ is the space of real numbers with the usual topology and p and q are distinct elements not in $\mathbb{R}$; $U \subseteq X$ is defined to be open if $U \cap \mathbb{R}$ is open in $\mathbb{R}$ and $p \in U$ (resp. $q \in U$) implies that $\bigcup\{(2n, 2n+1) \cup (-2n-1, -2n) : n \geqq m\} \subseteq U$ (resp. $\bigcup\{(2n-1, 2n) \cup (-2n, -2n+1) :$

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$n \geqq m\} \subseteq U$) for some $m \in \mathbb{N}$. The space X is H-closed and the discrete subspace $\mathbb{N} \cup \{p\}$ is Katětov (and an H-set) but is not the θ-closure of any set; also, $\mathbb{N} \cup \{p, q\}$ is a θ-closed subset of X which is not H-closed.

2. **θ-closure of sets.** In this section, we extend 1.7(a) by showing that if A is a subspace of an H-closed space X, then $c\ell_\theta A$ is a Katětov subspace. Also, a variation of a result first proven by Katětov, is established by showing that an H-closed space in which every closed set is the θ-closure of another set is compact. In the preceding paragraph, an example of a non-H-closed, θ-closed subspace of an H-closed space is given; in this section, an H-closed subspace of a space which is not the θ-closure of any subset is given. First we prove some preliminary results.

LEMMA 2.1. *Let X and Y be spaces, $B \subseteq Y$, and $f : X \longrightarrow Y$ a perfect, θ-continuous function such that $c\ell_\theta B \subseteq f[X]$. Then $f[c\ell_\theta f^{\leftarrow}[B]] = c\ell_\theta B$.*

PROOF. By 1.1, $f[c\ell_\theta f^{\leftarrow}[B]] \subseteq c\ell_\theta B$. Let $p \in Y \backslash f[c\ell_\theta f^{\leftarrow}[B]]$. If $p \in Y \backslash f[X]$, then $p \notin c\ell_\theta B$. So, suppose $p \in f[x]$. Now, $f^{\leftarrow}(p) \cap c\ell_\theta f^{\leftarrow}[B] = \emptyset$. For each $q \in f^{\leftarrow}(p)$, there is an open set U_q such that $q \in U_q$ and $c\ell U_q \cap f^{\leftarrow}[B] = \emptyset$. By compactness of $f^{\leftarrow}(p)$, $f^{\leftarrow}(p) \subseteq \bigcup\{U_q : q \in F\}$ for some finite set $F \subseteq f^{\leftarrow}(p)$. Let $W = \bigcup\{U_q : q \in F\}$. Then $f[c\ell W] \supseteq c\ell f[W] \supseteq c\ell f^{\#}[W]$ where $f^{\#}[W] = Y \backslash f[X \backslash W]$ is open and $p \in f^{\#}[W]$. As $c\ell W \cap f^{\leftarrow}[B] = \emptyset$, it follows that $c\ell f^{\#}[W] \cap B = \emptyset$. So, $p \notin c\ell_\theta B$. This shows that $c\ell_\theta B \subseteq f[c\ell_\theta f^{\leftarrow}[B]]$. □

COMMENT. For a space X, consider the absolute EX and the perfect, θ-continuous surjection $k : EX \longrightarrow X$. For $B \subseteq X$, by 2.1, $k[c\ell_\theta k^{\leftarrow}[B]] = c\ell_\theta B$. Since EX is Tychonoff, $c\ell_\theta k^{\leftarrow}[B] = c\ell k^{\leftarrow}[B]$. So, $k[c\ell k^{\leftarrow}[B]] = c\ell_\theta B$.

PROPOSITION 2.2. *If X is an H-closed space and $A \subseteq X$, then $c\ell_\theta A$ is Katětov.*

PROOF. For $C = c\ell_\theta k^{\leftarrow}[A], k|C : C \longrightarrow c\ell_\theta A$ is a perfect surjection. By 1.5(a) and 1.6, it follows that $c\ell_\theta A$ is Katětov. □

In 1947, Katětov [4] proved that an H-closed space in which every closed set is H-closed is compact. Since an H-closed subspace is Katětov and an H-set, it is natural to inquire whether the result by Katětov can be improved by changing the hypothesis to "every closed set is an H-set" or "every closed set is Katětov." In [10], Viglino gave an example of a non-compact H-closed space in which every closed set is an H-set. To show the second possibility is false, consider this example of a non-compact, H-closed space in which every closed set is Katětov. Let J denote the unit interval $[0, 1]$ with the usual topology enlarged by making $\{1/n : n \in \mathbb{N}\}$ a closed set. Clearly, J is a noncompact, H-closed space. Using that a complete metric space is Katětov [6, 9.4], it follows that every closed set of J is Katětov.

By 2.2 and a result of Velicko [8], if X is an H-closed space and $A \subseteq X$, then $c\ell_\theta A$ is Katětov and an H-set. We now show that an H-closed space with the hypothesis "every closed set is the θ-closure of some set" is compact. First two lemmas are needed.

LEMMA 2.3. *Let X and Y be spaces, $f : X \longrightarrow Y$ a perfect, θ-continuous function, A a closed subset of X, and $f[A] \supseteq c\ell_\theta B$ for some $B \subseteq Y$. Then $f[c\ell_\theta(f^{\leftarrow}[B] \cap A)] = c\ell_\theta f[f^{\leftarrow}[B] \cap A] = c\ell_\theta B$.*

PROOF. The proof follows from 1.1 and 2.1. □

LEMMA 2.4. *Let X be an H-closed space and c a chain of nonempty sets such that for each $C \in c$, $C = c\ell_\theta B$ for some $B \subseteq X$. Then $\cap c$ is a nonempty H-set which is Katětov.*

PROOF. We can assume $c = \{C_\alpha : \alpha < \gamma\}$ is indexed over an ordinal γ with $C_\alpha \supseteq C_\beta$ whenever $\alpha \leqq \beta < \gamma$. By 1.5(a), EX is compact. Let $D_0 = c\ell k^{\leftarrow}[B']$ where $C_0 = c\ell_\theta B'$. By induction, suppose for $\beta < \gamma$, there is a chain $\{D_\alpha : \alpha < \beta\}$ of closed sets such that $k[D_\alpha] = C_\alpha$ for $\alpha < \beta$. Let $D = \cap\{D_\alpha : \alpha < \beta\}$. For $p \in C_\beta$ and $\alpha < \beta, p \in C_\alpha = k[D_\alpha]$; hence, $\{k^{\leftarrow}(p) \cap D_\alpha : \alpha < \beta\}$ is a chain of nonempty compact sets. It follows that $k^{\leftarrow}(p) \cap D \neq \emptyset$ and $k[D] \supseteq C_\beta (= c\ell_\theta B$ for some $B \subseteq X)$. For $D_\beta = c\ell k^{\leftarrow}[B] \cap D, k[D_\beta] = C_\beta$ by 2.3. By induction there is a chain $\{D_\alpha : \alpha < \gamma\}$ of closed subsets of EX such that $k[D_\alpha] = C_\alpha$ for $\alpha < \gamma$. Now $A = \cap\{D_\alpha : \alpha < \gamma\}$ is a nonempty compact set and, by repeating the above argument, $k[A] = \cap\{C_\alpha : \alpha < \gamma\}$. By 1.2 and 1.6, $\cap\{C_\alpha : \alpha < \gamma\}$ is Katětov and an H-set. □

THEOREM 2.5. *An H-closed space in which every closed set is the θ-closure of some set is compact.*

PROOF. This follows from 2.4 and Alexander's subbase theorem for compactness.□

In the first section an example of a θ-closed subset of an H-closed space is given which is not H-closed. An H-closed space is both Katětov and an H-set in every H-closed space containing it as a subspace. In view of 2.2, it is natural to inquire if an H-closed subspace of an H-closed space is the θ-closure of some set. The next example gives a negative answer to this question. First some preliminary results and definitions are needed.

For a space X, κX is used to denote the set $X \cup \{\mathcal{U} : \mathcal{U}$ is a free open ultrafilter on $X\}$ with the topology defined by $U \subseteq \kappa X$ is open if $U \cap X$ is open in X and $\mathcal{U} \in U \backslash X$ implies $U \cap X \in \mathcal{U}$. Katětov [4] showed that κX is an H-closed extension of X. Note that $\kappa X \backslash X$ is a closed discrete subspace of κX. Additional information about κX can be found in [7, 4.8 and 7.2].

Let X be a space and $\mathcal{S}(X) = X \times \omega$. The topology on $\mathcal{S}(X)$ is defined by $U \subseteq \mathcal{S}(X)$ is open iff $(p, 0) \in U$ implies there is some open set W in X and $n(p) \in \mathbb{N}$ such that $p \in W$ and $W \times (\{0\} \cup [n(p), \infty)) \subseteq U$. Note (see [7, 2G(3)]) that $\mathcal{S}(X)$ is a semiregular Hausdorff space and $\{(p, n)\}$ is open for each $p \in X$ and $n \geqq 1$.

LEMMA 2.6. *Let X be a space, $A \subseteq X$ and $\mathcal{F}$ a filter base of closed sets on X such that $\cap\mathcal{F} = \emptyset$ and $F \subseteq A$ for each $F \in \mathcal{F}$. Then there is a point $p \in \kappa(\mathcal{S}(X)) \backslash \mathcal{S}(X)$ such that $p \in c\ell_\theta(A \times \{0\})$ (θ-closure in $\kappa(\mathcal{S}(X))$).*

PROOF. Let $Y = \mathcal{S}(X)$. For each $F \in \mathcal{F}$, let $U_F = F \times \mathbb{N}$. Then U_F is open in Y and $c\ell_Y U_F = F \times \omega$. $\mathcal{T} = \{U_F \backslash T : F \in \mathcal{F}$ and $T \cap (\{q\} \times \mathbb{N})$ is finite for each $q \in F\}$. For $U_F \backslash T \in \mathcal{T}$, it follows that $(c\ell_Y(U_F \backslash T)) \cap (X \times \{0\}) = F$. In particular, $\mathrm{ad}_Y \mathcal{T} = \emptyset$. So, $\mathcal{T}$ is contained in some free open ultrafilter p on Y; now $p \in \kappa Y \backslash Y$. Let $W \in p, F \in \mathcal{F}$, and $T = W \cap (F \times \mathbb{N})$. If $W \cap (\{q\} \times \mathbb{N})$ is finite for each $q \in F$, then $W \cap (U_F \backslash T) = \emptyset$; this is impossible as $\mathcal{T} \subseteq p$. Thus, $W \cap (\{q\} \times \mathbb{N})$ is infinite for some $q \in F$ and $c\ell_Y W \cap (A \times \{0\}) \neq \emptyset$. This shows that $p \in c\ell_\theta(A \times \{0\})$. □

Let Y be the unit interval $[0, 1]$ with the usual topology and $Q = \{x \in Y : x$ rational$\}$. Now Y is an H-closed extension of Q. Define $U \subseteq Y$ to be open iff $U \cap Q$ is open in Q and if $p \in U \backslash Q$, there are an open set V in Y and a closed nowhere dense set N in Q such that $p \in V$ and $(V \cap Q) \backslash N \subseteq U$. Denote Y with this new topology by Z. Note that Z is an extension of X and that $Z \backslash Q$ is a closed discrete subspace of Z. It is now straightforward to show that Z is H-closed.

THEOREM 2.7. *Not every H-closed subspace of an H-closed space X is a θ-closure in X.*

PROOF. Let $X = \kappa(\mathcal{S}(Z))$ where Z is defined in the previous paragraph. Consider the H-closed subspace $Z \times \{0\}$. Assume there is a subset $A \subseteq Z$ such that $c\ell_\theta(A \times \{0\}) = Z \times \{0\}$. If $A \backslash Q$ is infinite, then $\mathcal{F} = \{A \backslash (Q \cup F) : F$ is a finite subset of $A \backslash Q\}$ is a filter base of closed sets on Z such that $\cap \mathcal{F} = \emptyset$. By above fact, $c\ell_\theta(A \times \{0\}) \backslash \mathcal{S}(Z) \neq \emptyset$, a contradiction. So, $A \backslash Q$ is finite. Let $p \in [0, 1] \backslash (A \cup Q))$. Assume there is a sequence $\{q_n\}$ in $A \cap Q$ such that $(q_n) \longrightarrow p$. Let $F_n = \{q_m : m \geqq n\}$. Then F_0 is closed and nowhere dense in Z. Now, $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ is a filter base of closed sets in Z such that $F_n \subseteq A$ for each $n \in \mathbb{N}$ and $\cap \mathcal{F} = \emptyset$. By 2.6, $c\ell_\theta(A \times \{0\}) \backslash \mathcal{S}(Z) \neq \emptyset$, a contradiction. Thus, there is an $\epsilon > 0$ such that $(p, p + 4\epsilon) \cap A = \emptyset$. Then $p + 2\epsilon \in (p + \epsilon, p + 3\epsilon) \subseteq [p + \epsilon, p + 3\epsilon] \subseteq (p, p + 4\epsilon)$ and $(p + 2\epsilon, 0) \in Z \times \{0\}$. □

3. **Compact Preimages.** In the previous section, a subspace of an H-closed space which is the θ-closure of some set is shown to be Katětov. This is accomplished by showing that the θ-closure of a set is the perfect image of a closed subset of the absolute. In this section we characterize those subspaces of a space X which are the image of closed sets of EX. A special case is noted in 1.5(b) – if A is a subset of a space $X, k^{\leftarrow}[A]$ is a closed subset of EX iff A is θ-closed in X. In [9], Vermeer characterizes the H-closed subspaces A of an H-closed space X as those for which there is a compact subspace B of EX such that $k[B] = A$ and $k|B : B \longrightarrow A$ is θ-continuous.

THEOREM 3.1. *Let X be a space and $\emptyset \neq A \subseteq X$. There is a closed $B \subseteq EX$ such that $k[B] = A$ iff $A = \mathrm{ad}\, \mathcal{F}$ for some open filter $\mathcal{F}$ on X.*

PROOF. Suppose there is a closed set B of EX such that $k[B] = A$. For each $p \in X\backslash A, k^{\leftarrow}(p)$ is compact and $k^{\leftarrow}(p) \cap B = \emptyset$. There is an open set U_p in X such that $k^{\leftarrow}(p) \subseteq OU_p \subseteq EX\backslash B (OU_p$ is defined in the paragraph preceding 1.5). Denote $k^{\#}[OU_p]$ by W_p (recall that $W_p = \text{int}(c\ell U_p)$); it is easy to verify that $W_p \cap A = \emptyset$ and $p \in W_p$. Let $\mathcal{F}$ be the filter generated by $\{X\backslash c\ell(\bigcup\{W_p : p \in F\}) : F$ is a finite subset of $X\backslash A\}$. To show $\mathcal{F}$ is a filter and $A = \text{ad}\,\mathcal{F}$, it suffices to establish that for each finite set $F \subseteq X\backslash A, A \cap (\text{int}(c\ell(\cap\{W_p : p \in F\})) = \emptyset$. Now, $B \cap (\bigcup\{OU_p : p \in F\}) = \emptyset$. But $\bigcup\{OU_p : p \in F\} = O(\bigcup\{U_p : p \in F\})$ (see 6.8(f) in [7]). Since $k^{\#}[O(\bigcup\{U_p : p \in F\})] = \text{int}(c\ell(\bigcup\{U_p : p \in F\})]$, it follows that $A \cap \text{int}(c\ell(\bigcup\{U_p : p \in F\})] = \emptyset$. Conversely, suppose $A = \text{ad}\,\mathcal{F}$ for some open filter $\mathcal{F}$ on X. Now $B = \cap\{OU : U \in \mathcal{F}\}$ is a closed set of EX and $k[B] \subseteq \cap\{k[OU] : U \in \mathcal{F}\} = \bigcap\{c\ell U : U \in \mathcal{F}\} = A$. Suppose $q \notin k[B]$. Then $k^{\leftarrow}(q) \cap \cap\{OU : U \in \mathcal{F}\} = \emptyset$. As $k^{\leftarrow}(q)$ is compact and $\mathcal{F}$ is a filter, there is some $V \in \mathcal{F}$ such that $k^{\leftarrow}(q) \cap OV = \emptyset$. So, $q \notin k[OV] = c\ell V$. So $q \notin A$ as $A \subseteq c\ell V$. This shows that $A \subseteq k[B]$. □

If $\mathcal{F}$ is an open filter on a space X, then ad $\mathcal{F}$ is a closed set in $X(s)$. Consider the H-closed space J described in the paragraph following 2.2 and the closed subset $A = \{1/n : n \in \mathbb{N}\}$. Now, A is not a closed subset of $J(s)$ which is the unit interval with the usual topology. So, A is not the image of any closed subset of the compact space EJ.

Let X be a space and $A \subseteq X$. By 2.1, $c\ell_\theta A$ is the adherence of some open filter. By Vermeer's result [9; 4.3], an H-closed subspace of X is the adherence of some open filter on X. We can now extend 1.3.

COROLLARY 3.2. *Let $\mathcal{F}$ be an open filter on an H-closed space X. Then* ad $\mathcal{F}$ *is an H-set of X and is Katětov.*

PROOF. By 1.3, ad $\mathcal{F}$ is an H-set of X. By 3.1, ad $\mathcal{F} = k[B]$ for some closed set B of EX. By 1.5(a) and 1.6, ad $\mathcal{F}$ is a Katětov space. □

It was our hope that by using 3.2 we could prove that an H-set of a countable H-closed space is the adherence of an open filter base and conclude that H-sets in countable semiregular H-closed spaces are Katětov. This would have provided a partial converse to 1.7(b) and a partial solution to Vermeer's question. However, Krystock [5; 2.9] has given an example of a countable semiregular H-closed space with an H-set S which is not the adherence of an open filter base. As S is discrete, it is Katětov, and the problem of the converse of 1.7(b) for countable H-closed spaces remain open.

A related problem was proposed in [6] where it is shown that a countable, Katětov space is scattered (scattered means every nonempty subspace has an isolated point). The problem is to determine if a countable, scattered space is Katětov. We now present an example of a countable, scattered space X which is not an H-set in any space. Hence, by 1.7(b), X is not Katětov.

EXAMPLE 3.3. Let $X = \mathbb{Q} \times (\{0\} \cup \{1/n : n \in \mathbb{N}\})$. Define $U \subseteq X$ to be open iff for each $(r, 0) \in U$, there are $\epsilon > 0$ and $m \in \mathbb{N}$ such that $(r - \epsilon, r + \epsilon) \times 1/n : n \geqq m\} \subseteq U$. Clearly, X is Hausdorff. Since the points of $\mathbb{Q} \times \{1/n : n \in \mathbb{N}\}$ are isolated and $\mathbb{Q} \times \{0\}$ is a discrete space, it is easy to verify that X is scattered. Let $X = \{x_i : i \in \omega\}$, and suppose X is an H-set of a space Y. Let m be the least element of $\omega \backslash \{0\}$ such that $x_m \in \mathbb{Q} \times \{0\}$. There is an open set U_1 in Y such that $x_m \in U_1$ and $\{x_0, \ldots, x_{m-1}\} \subseteq Y \backslash c\ell_Y U_1$. There are $\epsilon_1 > 0$ and $n_1 \in \mathbb{N}$ such that $B_1 = (x_m - \epsilon_1, x_m + \epsilon_1) \times \{1/n : n \geqq n_1\} \subseteq U_1$. Let k be the least element of $\omega \backslash \{0, \ldots, m\}$ such that $x_k \in B_1 \cap \mathbb{Q} \times \{0\}$. There is an open set W_2 such that $x_k \in W_2$ and $\{x_0, \ldots, x_{k-1}\} \subseteq Y \backslash c\ell_Y W_2$. Let $U_2 = W_2 \cap U_1$. There are $\epsilon_2 > 0$ and $n_2 \in \mathbb{N}$ such that $B_2 = (x_k - \epsilon_2, x_k + \epsilon_2) \times \{1/n : n \geqq n_2\} \subseteq U_2$. Continue by induction to obtain a decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of open sets of Y such that $U_n \cap X \neq \emptyset$ for each $n \in \mathbb{N}$ and $X \cap \bigcap\{c\ell_Y U_n : n \in \mathbb{N}\} = \emptyset$. Thus, X is not an H-set in Y. □

We appreciate the help of Judy Roitman and Fred Galvin in constructing the above example.

COMMENT. Vermeer [9] has shown that an H-closed space in which all H-sets are minimal Hausdorff is compact. An infinite discrete space is a noncompact space in which each H-set is compact. A more interesting example of a noncompact (and non-regular) space in which every H-set is compact is described as follows: Let $X = \{(1/n, 1/m) : n, |m| \in \mathbb{N}\} \cup \{(1/n, 0) : n \in \mathbb{N}\} \cup \{(0,1), (0,-1)\}$. Let $\mathcal{U} \in \beta\mathbb{N} \backslash \mathbb{N}$. A set $U \subseteq X$ is defined to be open iff $U \backslash \{(0,1), 0, -1)\}$ is open in the topology induced by the usual topology of the plane $\mathbb{R}^2$ and if $(0,1) \in U$ (resp. $(0,-1) \in U$), then there is a set $K \in \mathcal{U}$ such that $\{(1/n, 1/m)$ (resp. $(1/n, -1/m)) : n \in K, m \in \mathbb{N}\} \subseteq U$.

NOTE. In response to a comment from Mike Girou, here is an example of a non-regular space in which every closed set is the θ-closure of some set. Let ω_1 and $\omega + 1$ have the usual order topology and $\mathcal{F}$ be the unique free closed ultrafilter on ω_1. Let $X = \omega_1 \times (\omega + 1) \cup \{\infty\}$ and define $U \subseteq X$ to be open if $U \cap \omega_1 \times (\omega + 1)$ is open in $\omega_1 \times (\omega + 1)$ and $\infty \in U$ implies $F \times \omega \subseteq U$ for some $F \in \mathcal{F}$. Using that for each $F \in \mathcal{F}$, there is an open set U in ω_1 such that $U \cap F$ is dense in F and $F \backslash U \in \mathcal{F}$, it is straightforward to show that X has the desired properties.

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