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ON UNIQUENESS POLYNOMIALS FOR MEROMORPHIC FUNCTIONS

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Abstract. A polynomial $P(w)$ is called a uniqueness polynomial (or a uiqueness polynomial in a broad sense) if $P(f) = cP(g)$ (or $P(f) = P(g)$) implies $f = g$ for any nonzero constant c and nonconstant meromorphic functions f and g on $\mathbf{C}$. We consider a monic polynomial $P(w)$ without multiple zero whose derivative has mutually distinct k zeros e_j with multiplicities q_j. Under the assumption that $P(e_\ell) \neq P(e_m)$ for all distinct ℓ and m, we prove that $P(w)$ is a uniqueness polynomial in a broad sense if and only if $\sum_{\ell<m} q_\ell q_m > \sum_\ell q_\ell$. We also give some sufficient conditions for uniqueness polynomials.

§1. Introduction

In this paper, a meromorphic function means a meromorphic function on the complex plane $\mathbf{C}$. A discrete subset S of $\mathbf{C}$ is called a uniqueness range set for meromorphic (or entire) functions if there exists no pair of two distinct nonconstant meromorphic (or entire) functions such that they have the same inverse images of S counted with multiplicities. Since F. Gross and C. C. Yang proved that the set $S := \{w\ ;\ w + e^w = 0\}$ is a uniqueness range set for entire functions ([4]), many efforts were made to find uniqueness range sets which are as small as possible ([5], [9], [10]). In relation to this problem, B. Shiffman, C. C. Yang and X. Hua studied polynomials $P(w)$ satisfying the condition that there exists no pair of two distinct nonconstant meromorphic (or entire) functions f and g with $P(f) = P(g)$ in their papers [7] and [8]. For a finite set $S = \{a_1, a_2, \ldots, a_q\}$, it is necessary for S to be a uniqueness range set for meromorphic (or entire) functions that the associated polynomial

$$P_S(w) = (w - a_1)(w - a_2) \cdots (w - a_q)$$

satisfies this condition.

In this paper, we use the following terminology.

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Definition 1.1. Let $P(w)$ be a nonconstant monic polynomial. We call $P(w)$ a uniqueness polynomial if $P(f) = cP(g)$ implies $f = g$ for any nonconstant meromorphic functions f, g and any nonzero constant c. We also call $P(w)$ a uniqueness polynomial in a broad sense if $P(f) = P(g)$ implies $f = g$ for any nonconstant meromorphic functions f, g.

In the previous paper [1], the author gave some sufficient conditions for uniqueness polynomials as well as for uniqueness range sets.

Let $P(w)$ be a monic polynomial without multiple zero whose derivative has mutually distinct k zeros $e_1, e_2, \ldots, e_k$ with multiplicities $q_1, q_2, \ldots, q_k$ respectively. Under the assumption that

$$\text{(H)} \qquad P(e_\ell) \neq P(e_m) \quad \text{for } 1 \leq \ell < m \leq k,$$

he proved the following:

Theorem 1.2. *If $k \geq 4$, $P(w)$ is a uniqueness polynomial in a broad sense.*

He also gave the following theorem for uniqueness polynomials:

Theorem 1.3. *For a polynomial $P(w)$ with $k \geq 4$ satisfying the hypothesis* (H), *if*

$$P(e_1) + P(e_2) + \cdots + P(e_k) \neq 0,$$

then $P(w)$ is a uniqueness polynomial.

Moreover, he obtained some partial results for the case where $k = 3$.

The main purpose of this paper is to give new geometric proofs of the above results in [1], which is due to the ideas used in [7, Section 4], and some improvements in [1] for the case where $k = 2, 3$.

We first investigate uniqueness polynomials in a broad sense. For a given nonconstant polynomial $P(z)$, we consider the algebraic curve C in $P^2(\mathbf{C})$ which is the closur of a plane curve $\{(z, w)\ ;\ (P(z) - P(w))/(z-w) = 0\}$ in $\mathbf{C}^2(\subset P^2(\mathbf{C}))$. We can show that $P(z)$ is a uniqueness polynomial in a broad sense if and only if every irreducible component of C is of genus greater than one. Under the condition (H), we prove that C is irreducible and give a formula for the genus of C. These enable us to obtain the following improvement of the above results:

THEOREM 1.4. *Let $P(w)$ be a polynomial satisfying the above assumption* (H). *Then, $P(w)$ is a uniqueness polynomial in a broad sense if and only if*

$$\sum_{1\le \ell<m\le k} q_\ell q_m > \sum_{\ell=1}^{k} q_\ell. \tag{1.5}$$

We can show that, for the case $k \geq 4$, the condition (1.5) is always satisfied. Moreover, (1.5) holds when $\max(q_1, q_2, q_3) \geq 2$ for the case $k = 3$ and when $\min(q_1, q_2) \geq 2$ and $q_1 + q_2 \geq 5$ for the case $k = 2$.

Next, we try to obtain some improvements of the results in [1] for uniqueness polynomials with $k = 3$. We prove the following:

THEOREM 1.6. *Let $P(w)$ be a monic polynomial with $k = 3$ satisfying the condition* (H). *Assume that* $\max(q_1, q_2, q_3) \geq 2$ *and*

$$\frac{P(e_\ell)}{P(e_m)} \neq \pm 1 \quad \text{for } 1 \le \ell < m \le 3, \tag{1.7}$$

$$\frac{P(e_\ell)}{P(e_m)} \neq \frac{P(e_m)}{P(e_n)} \quad \text{for any permutation } (\ell, m, n) \text{ of } (1,2,3). \tag{1.8}$$

Then, $P(w)$ is a uniqueness polynomial.

Lastly, we give some sufficient conditions for uniqueness polynomial for the case $k = 2$, which is not treated in [1].

§2. Uniqueness polynomials in a broad sense

Let $P(w)$ be a monic polynomial of degree q (> 0) without multiple zero, and let its derivative be given by

$$P'(w) = q(w - e_1)^{q_1}(w - e_2)^{q_2} \dots (w - e_k)^{q_k}, \tag{2.1}$$

where $e_1, \dots, e_k$ are mutually distinct and $q_1 + q_2 + \dots + q_k = q - 1$.

In the followings, we assume $k \geq 2$, because $P(w)$ cannot be a uniqueness polynomial in a broad sense for the case $k = 1$ (cf., [1, p. 1183]). Furthermore, by technical reasons we assume the following:

$$P(e_\ell) \neq P(e_m) \quad \text{for } 1 \le \ell < m \le k. \tag{H}$$

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Consider the polynomial

$$Q(z,w) := \frac{P(z)-P(w)}{z-w}$$

in two variables z and w, and the associated homogeneous polynomial

$$Q^*(u_0,u_1,u_2) := u_0^d Q\left(\frac{u_1}{u_0},\frac{u_2}{u_0}\right)$$

of degree d in three variables u_0,u_1,u_2, where $d := q-1$. By using this, we define the algebraic curve

$$C : Q^*(u_0,u_1,u_2) = 0, \quad (u_0 : u_1 : u_2) \in P^2(\mathbf{C}), \tag{2.2}$$

where $(u_0 : u_1 : u_2)$ denote the homogeneous coordinates on $P^2(\mathbf{C})$.

PROPOSITION 2.3. *The algebraic curve C has ordinary singularities with multiplicities q_ℓ at the points $P_\ell := (1 : e_\ell : e_\ell)$ $(1 \le \ell \le k)$, and has regular points at all other points.*

Proof. Set $L_\infty := \{u_0 = 0\}$. We first investigate points in $C \cap L_\infty$. By the assumption, $P(w)$ can be written as

$$P(w) = w^{d+1} + \text{terms of lower degree}$$

and so we have

$$Q^*(u_0,u_1,u_2) = (u_1^d + u_1^{d-1}u_2 + \cdots + u_2^d) + u_0 R(u_0,u_1,u_2),$$

where $R(u_0,u_1,u_2)$ is a homogeneous polynomial of degree $d-1$. It is easily seen that the first term is factorized into mutually distinct d linear functions $u_1 - \zeta^\ell u_2$ $(\ell = 1,2,\ldots,d)$, where ζ denotes a primitive $(d+1)$-st root of unity. This shows that $C \cap L_\infty$ consists of mutually distinct d points $Q_\ell := (0 : \zeta^\ell : 1)$ $(\ell = 1,2,\ldots,d)$ and each Q_ℓ is a regular point of C.

We next investigate the singularities of $C \setminus L_\infty$. We may use inhomogeneous coodinates z,w. Let $P_0 = (z_0,w_0)$ $(= (1 : z_0 : w_0))$ be a singularity of C, namely, let P_0 satisfy the condition

$$Q(z_0,w_0) = Q_z(z_0,w_0) = Q_w(z_0,w_0) = 0.$$

Then, by differentiating the identity

$$P(z) - P(w) = (z-w)Q(z,w),$$

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we have

$$q(z_0-e_1)^{q_1}(z_0-e_2)^{q_2}\cdots(z_0-e_k)^{q_k}=(z_0-w_0)Q_z(z_0,w_0)+Q(z_0,w_0)=0.$$

This implies that $z_0=e_\ell$ for some ℓ $(1\leq\ell\leq k)$. By the same reason, we see $w_0=e_m$ for some m. It then follows that

$$P(e_\ell)-P(e_m)=(z_0-w_0)Q(z_0,w_0)=0.$$

By virtue of the assumption (H), we can conclude $\ell=m$. Therefore, C has no singulalrities outside P_ℓ's.

We next investigate shapes of C around each point P_ℓ. Without loss of generality, we may assume $\ell=1$ and $e_1=0$ after suitable translations of coordinates. Then, by the assumption (2.1), we can write

$$P(w)-P(e_1)=cw^{q_1+1}+\text{terms of higher degree}$$

with a nonzero constant c, and so

$$Q(z,w)=c(z^{q_1}+z^{q_1-1}w+\cdots+w^{q_1})+\text{terms of higher degree.}$$

The first term in this expansion can be factorized into the product of mutually distinct linear forms $z-\eta^\ell w$ $(\ell=1,2,\ldots,q_1)$ in z and w, where η denotes a primitive (q_1+1)-st root of unity. This shows that P_1 is an ordinary singularity of C with multiplicity q_1 (cf., [2, p. 66]). The proof of Proposition 2.3 is completed. □

PROPOSITION 2.4. *The curve C is irreducible.*

Proof. Suppose that C is reducible and so $Q(z,w)$ can be written as

$$Q(z,w)=Q_1(z,w)Q_2(z,w)$$

with nonconstant polynomials Q_1 and Q_2. Consider the curves

$$C_i: Q_i^*(u_0,u_1,u_2):=u_0^{d_i}Q_i\left(\frac{u_1}{u_0},\frac{u_2}{u_0}\right)=0,\quad (i=1,2)$$

in $P^2(\mathbf{C})$, where each d_i denotes the degree of C_i. We then have

$$C_1\cap C_2\subseteqq\{P_1,P_2,\ldots,P_k\},$$

because C has a singularity at every point in $C_1 \cap C_2$. Since $C_1 \cap C_2$ is discrete, C_1 and C_2 have no common irreducible component. For each ℓ, there is a neighborhood U of P_ℓ such that $U \cap C$ has mutually distinct q_ℓ irreducible components by virtue of Hensel's lemma (cf., [6, p. 68]). Some of them are included in C_1 and the others are included in C_2. These guarantee that C_i has at worst ordinary singularities at some of the points P_ℓ's and regular points elsewhere. Assume that C_1 and C_2 have ordinary singularities of multiplicities r_ℓ and s_ℓ $(0 \leq r_\ell, s_\ell \leq q_\ell)$ at each P_ℓ respectively, where an ordinary singularity of multiplicity 0 means that the curve does not contain P_ℓ. We then have

$$q_\ell = r_\ell + s_\ell \quad (\ell = 1, 2, \ldots, k). \tag{2.5}$$

Moreover, we can show

$$d_1 = r_1 + r_2 + \cdots + r_k, \quad d_2 = s_1 + s_2 + \cdots + s_k. \tag{2.6}$$

To see this, we consider the diagonal line

$$L_\Delta : u_1 - u_2 = 0$$

in $P^2(\mathbf{C})$. Since

$$Q(z, z) = \lim_{w \to z} Q(z, w) = \lim_{w \to z} \frac{P(w) - P(z)}{w - z} = P'(z),$$

we have $C_1 \cap L_\Delta \subseteqq \{P_1, P_2, \ldots, P_k\}$. The tangent lines

$$z - e_\ell - \eta^\ell (w - e_\ell) = 0$$

of C at P_ℓ do not coincide with L_Δ, and so the intersection number of C_1 and L_Δ at P_ℓ is r_ℓ. By the classical Bezout's theorem (cf., [2, p. 112]), we get

$$d_1 = r_1 + r_2 + \cdots + r_k.$$

Similarly, we have $d_2 = s_1 + s_2 + \cdots + s_k$.

On the other hand, the intersection number of C_1 and C_2 at each point P_ℓ is $r_\ell s_\ell$. Applying Bezout's theorem again, we obtain

$$d_1 d_2 = r_1 s_1 + r_2 s_2 + \cdots + r_k s_k.$$

Therefore,

$$\sum_{\ell, m} r_\ell s_m - \sum_\ell r_\ell s_\ell = \sum_{\ell \neq m} r_\ell s_m = 0.$$

Since r_ℓ and s_ℓ are nonnegative integers, we have necessarily $r_\ell s_m = 0$ for all mutually distinct ℓ and m. Changing indices if necessary, we may assume $r_1 \neq 0$, because $d_1 = \sum r_\ell > 0$. Then, $s_\ell = 0$ for $\ell = 2, 3, \ldots, k$. On the other hand, since $d_2 = \sum s_\ell > 0$, we see $s_1 \neq 0$. This implies that $r_\ell = 0$ for $\ell = 2, \ldots, k$, because $s_1 r_\ell = 0$ for $\ell = 2, \ldots, k$. By (2.5), this shows that $k = 1$, which contradicts the assumption $k \geq 2$. Proposition 2.4 is completely proved. □

With each irreducible algebraic curve V in $P^2(\mathbf{C})$ we can associate the normalization $(\widetilde{V}, \mu)$ of V, namely, a compact Riemann surface $\widetilde{V}$ and a holomorphic mapping μ of $\widetilde{V}$ onto V which is injective outside the inverse image of the singular locus of V. By definition, the genus $g(V)$ of V means the genus of the compact Riemann surface $\widetilde{V}$.

PROPOSITION 2.7. *The genus of the curve C defined as above is given by*

$$g(C) = \frac{(d-1)(d-2)}{2} - \sum_{\ell=1}^{k} \frac{q_\ell(q_\ell - 1)}{2}.$$

This is an easy result of Propositions 2.3, 2.4 and the classical Plücker's genus formula (cf., [2, p. 199]).

THEOREM 2.8. *Let $P(w)$ be a monic polynomial whose derivative has k distinct zeros $e_1, e_2, \ldots, e_k$ with multiplicities $q_1, q_2, \ldots, q_k$, respectivley. Assume that*

$$P(e_\ell) \neq P(e_m), \quad (1 \leq \ell < m \leq k).$$

If $k \geq 4$, then $P(w)$ is a uniqueness polynomial in a broad sense.

Moreover, $P(w)$ is a uniqueness polynomial in a broad sense when and only when

$$\max(q_1, q_2, q_3) \geq 2$$

for the case $k = 3$, and when and only when

$$\min(q_1, q_2) \geq 2 \quad \textit{and} \quad q_1 + q_2 \geq 5$$

for the case $k = 2$.

Remark. (1) In [1], the author proved Theorem 2.8 for the case $k \geq 4$ and the 'when' part for the case $k = 3$ under the additional assumption $(e_1, e_2, e_3, \infty) = -1$ by function-theoretic method.

(2) For the case $k = 2$, there is no harm in assuming that $e_1 = 0$ and $e_2 = 1$ after a suitable linear change of coordinate on $\mathbf{C}$. In this case, $P(w)$ is nothing but the polynomial studied by Frank and Reinders in [3] after a suitable multiplication of a nonzero constant. In this particular case, the condition (H) is automatically satisfied, because

$$(-1)^{q_2}(P(1) - P(0)) = \int_0^1 qx^{q_1}(1-x)^{q_2}\,dx > 0.$$

In [3], Frank and Reinders proved Theorem 2.8 for a particular case where $k = 2$, $\min(q_1, q_2) = 2$ and $q_1 + q_2 \geq 6$.

Proof. Suppose that $P(w)$ is not a uniqueness polynomial in a broad sense. By definition, there exist two distinct nonconstant meromorphic functions f and g satisfying the condition $P(f) = P(g)$. We can write $f = f_1/f_0$ and $g = f_2/f_0$ with suitably chosen entire functions f_0, f_1, f_2 without common zeros. Consider a holomorphic map

$$\Phi := (f_0 : f_1 : f_2) : \mathbf{C} \longrightarrow P^2(\mathbf{C}).$$

We denote by E the union of the sets of all poles of f, of all poles of g and of all points z with $f(z) = g(z)$. By the assumption, E is a discrete subset of $\mathbf{C}$, and we have

$$\Phi(\mathbf{C} \setminus E) \subseteqq \left\{(z, w) \in P^2(\mathbf{C}) \setminus L_\infty \,;\, Q(z, w) := \frac{P(z) - P(w)}{z - w} = 0\right\}.$$

Therefore, by the continuity of Φ the image $\Phi(\mathbf{C})$ is included in the algebraic curve C defined by (2.2). Take the normalization $(\widetilde{C}, \mu)$ of C. Then, there is a nonconstant holomorphic map $\widetilde{\Phi}$ of $\mathbf{C}$ into $\widetilde{C}$ with $\mu \cdot \widetilde{\Phi} = \Phi$. For our purpose, it suffices to seek the condition for the genus $g(\widetilde{C})$ $(= g(C))$ of the compact Riemann surface $\widetilde{C}$ is greater than one. In fact, in this case, we have an absurd conclusion that the map $\widetilde{\Phi}$, and so Φ, is a constant by virtue of the classical Picard's theorem, which asserts that every holomorphic map of $\mathbf{C}$ into a compact Riemann surface of genus greater than one is necessarily a constant. On the other hand, if $g(\widetilde{C})$ is not larger than one, then $\widetilde{C}$ is a torus or the Riemann sphere. Therefore, there exists a nonconstant holomorphic map $\widetilde{\Psi}$ of $\mathbf{C}$ into $\widetilde{C}$. Consider the map $\Psi := \mu \cdot \widetilde{\Psi}$, which can be regarded as a holomorphic map of $\mathbf{C}$ into $P^2(\mathbf{C})$. We write $\Psi = (f_0^* : f_1^* : f_2^*)$ with nonzero holomorphic functions which have no common zeros. It is easily seen that $f^* := f_1^*/f_0^*$ and $g^* := f_2^*/f_0^*$ are nonconstant distinct meromorphic functions satisfying the condition $P(f^*) = P(g^*)$. The polynomial $P(w)$ cannot be a uniqueness polynomial in a broad sense. On the other hand, according to Proposition 2.7 the genus of C is given by

$$g(C) = \frac{(d-1)(d-2)}{2} - \sum_{\ell=1}^{k} \frac{q_\ell(q_\ell - 1)}{2} = \sum_{1\le\ell<m\le k} q_\ell q_m - \sum_{\ell=1}^{k} q_\ell + 1 \ (\ge 0).$$

Therefore, $P(w)$ is a uniqueness polynomial in a broad sense if and only if it satisfies the condition (1.5) as mentioned in Section 1.

For the case $k \ge 4$, it is easily seen that

$$g(C) = q_1\left(\sum_{\ell=2}^{k} q_\ell - 1\right) + \left(\sum_{2\le\ell<m\le k} q_\ell q_m - \sum_{\ell=2}^{k} q_\ell + 1\right) \ge k - 2 \ge 2.$$

For the case $k = 3$, under the assumption that at least one of q_ℓ's is larger than one, say $q_3 \ge 2$, we have

$$g(C) = q_1(q_2 + q_3 - 1) + (q_2 - 1)(q_3 - 1) \ge 2.$$

Moreover, for the case $k = 2$, under the assumption $\min(q_1, q_2) \ge 2$ and $q_1 + q_2 \ge 5$, we have

$$g(C) = (q_1 - 1)(q_2 - 1) \ge 2.$$

Conversely, for the case $k = 3$, if $q_1 = q_2 = q_3 = 1$, we have $g(C) = 1$. For the case $k = 2$, $q_1 = 1$, $q_2 = 1$ or $q_1 + q_2 \le 4$, then $g(C) \le 1$. The proof of Theorem 2.8 is completed. □

§3. Uniqueness polynomials

As in the previous section, we consider a monic polynomial $P(w)$ without multiple zero whose derivative has mutually distinct k (> 1) zeros $e_1, e_2, \ldots, e_k$ with multiplicities $q_1, q_2, \ldots, q_k$ respectively, and assume that $P(w)$ satisfies the condition (H).

In the previous paper [1], the author proved the following:

THEOREM 3.1. *Assume that $k \ge 4$. If $P(w)$ is not a uniqueness polynomial, then there is a permutation $(i_1, i_2, \ldots, i_k)$ of $(1, 2, \ldots, k)$ such that*

$$\frac{P(e_{i_1})}{P(e_1)} = \frac{P(e_{i_2})}{P(e_2)} = \cdots = \frac{P(e_{i_k})}{P(e_k)} \ne 1.$$

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We note that Theorem 1.3 mentioned in Section 1 is an immediate consequence of Theorem 3.1.

We now investigate the polynomial with $k = 3$. Changing indices if necessary, we assume that $q_1 \le q_2 \le q_3$.

THEOREM 3.2. *Assume that $P(w)$ with $k = 3$ is not a uniqueness polynomial.*

(1) *If $q_1 \ge 2$, then $P(w)$ satisfies the condition*

$$\text{(C1)} \qquad \frac{P(e_{i_1})}{P(e_1)} = \frac{P(e_{i_2})}{P(e_2)} = \frac{P(e_{i_3})}{P(e_3)} \ne 1.$$

for some permutation (i_1, i_2, i_3) of the indices $(1, 2, 3)$.

(2) *If $q_1 = 1$ and $2 \le q_2 \le q_3$, then $P(w)$ satisfies the condition* (C1) *or*

$$\text{(C2)} \qquad P(e_2) + P(e_3) = 0$$

(3) *If $q_1 = q_2 = 1$ and $q_3 \ge 2$, then $P(w)$ satisfies the condition* (C1) *or*

$$\text{(C3)} \qquad P(e_1) + P(e_3) = 0, \;\; P(e_2) + P(e_3) = 0 \;\; or \;\; P(e_1)P(e_2) = P(e_3)^2$$

For the proof of Theorem 3.2, we assume that there are distinct non-constant meromorphic functions f and g and a nonzero constant c such that $P(f) = cP(g)$. For all cases of Theorem 3.2, the assumptions of Theorem 2.8 are satisfied and so $P(w)$ is a uniqueness polynomial in a broad sense. Therefore, we have necessarily $c \ne 1$. As in the previous paper ([1]), we set

$$\Lambda := \{(\ell, m)\ ;\ P(e_\ell) = cP(e_m)\}.$$

We give the following lemma, which is an improvement of [1, Lemma 5.3].

LEMMA 3.3. *Assume that $k = 3$ and $q_{\ell_0} \ge 2$ for some ℓ_0. Then, there are some indices m and m' such that $(\ell_0, m) \in \Lambda$ and $(m', \ell_0) \in \Lambda$.*

Proof. This is proved by the same argument as in the proof of Lemma 5.3 of [1] with some simple modifications. For reader's convenience, we state the outline of the proof. We assume that $(\ell_0, m) \notin \Lambda$ for any m. For each point z_0 with $f(z_0) = e_{\ell_0}$, we see $g(z_0) \ne e_m$ for any m. Since $P'(f)f' = cP'(g)g'$, we have necessarily $g'(z_0) = 0$. This implies that

$N(r, \nu_f^{e_{\ell_0}}) \le N(r, \nu_{g'}^*|_{f=e_{\ell_0}})$. Here, $N(r, \nu_f^{e_{\ell_0}})$ and $N(r, \nu_{g'}^*|_{f=e_{\ell_0}})$ denote the counting functions of zeros of $f - e_{\ell_0}$ counted with multiplicities and of zeros z of g' counted with multiplicities such that $f(z) = e_{\ell_0}$ and $g(z) \ne e_m$ for any m, respectively. Assume that there are constants c_0 ($\ne 0$) and c_1 with $g = c_0 f + c_1$. Then, the assumption $P(f) = cP(g)$ implies

$$\begin{aligned}&(f - e_1)^{q_1}(f - e_2)^{q_2}(f - e_3)^{q_3}\\ &\quad = cc_0(c_0 f + c_1 - e_1)^{q_1}(c_0 f + c_1 - e_2)^{q_2}(c_0 f + c_1 - e_3)^{q_3}.\end{aligned}$$

Since f is not a constant, this is regarded as an identity of polynomials with indeterminate f. Using the unique factorization theorem as in [1, p. 1191], we can easily show that, for every ℓ, there is some m with $(\ell, m) \in \Lambda$, which contradicts the assumption. Hence, there does not exist such constants c_0 and c_1. As in [1, p. 1184], we set $k_0 = \#\Lambda$. By the assumption, we see $k_0 \le 2$, and so we can apply Lemma 3.8 of [1] to obtain $N(r, \nu_{g'}^*|_{f=e_{\ell_0}}) = S(r, f) + S(r, g)$. Therefore, $N(r, \nu_f^{e_{\ell_0}}) = S(r, f) + S(r, g)$.

Consider the polynomial $Q(w) := P(w) - P(e_{\ell_0})$ and $Q^*(w) := cP(w) - P(e_{\ell_0})$. We denote all distinct zeros of $Q(w)$ and $Q^*(w)$ by α_1 ($= e_{\ell_0}$), $\alpha_2, \ldots, \alpha_m$ and $\beta_1, \beta_2, \ldots, \beta_n$, respectively. Since Q has a zero of multiplicity $q_{\ell_0} + 1$ at α_1, we have $m \le q - q_{\ell_0} \le q - 2$. Moreover, each β_j ($1 \le j \le n$) is not equal to e_m for any m, because $Q^*(e_m) = 0$ implies $(\ell_0, m) \in \Lambda$. This shows that all β_j's are simple zeros of $Q^*(w)$ and so $n = q$. On the other hand, if $g = \beta_j$ for some j at a point z_0, then $P(f(z_0)) = cP(g(z_0)) = cP(\beta_j) = P(e_{\ell_0})$ and so $f(z_0) = \alpha_i$ for some i. By the second main theorem in value distribution theory, we obtain

$$\begin{aligned}(q-2)T(r,g) &\le \sum_{j=1}^{q} N(r, \bar{\nu}_g^{\beta_j}) + S(r,g)\\ &\le N(r, \bar{\nu}_f^{e_{\ell_0}}) + \sum_{i=2}^{m} N(r, \bar{\nu}_f^{\alpha_i}) + S(r,g)\\ &\le (m-1)T(r,f) + S(r,g),\end{aligned}$$

where $N(r, \bar{\nu}_g^{\beta_j})$ denotes the counting functions of the points z with $g(z) = \beta_j$ counted without multiplicities. This gives an absurd conclusion $q - 2 \le m - 1 \le q - 3$. Therefore, there is some m with $(\ell_0, m) \in \Lambda$. The proof of the existence of m' with (m', ℓ_0) is similar. Thus, we get Lemma 3.3.

Now, we start to inquire into the assertion (1) of Theorem 3.2, namely, the case $\min(q_1, q_2, q_3) \ge 2$. By Lemma 3.3 there are indices i_1, i_2, i_3 such that $(\ell, i_\ell) \in \Lambda$ $(\ell = 1, 2, 3)$. In this situation, it is easily seen that these i_1, i_2, i_3 are mutually distinct by Lemma 3.5 of [1]. As its consequence, we have the desired conclusion for the case (1).

We next inquire into the assertion (2), namely, the case $q_1 = 1$ and $2 \leq q_2 \leq q_3$. In this case, there are indices i_2, i_3 and j_2, j_3 such that

$$(2, i_2) \in \Lambda, \quad (3, i_3) \in \Lambda, \quad (j_2, 2) \in \Lambda, \quad (j_3, 3) \in \Lambda.$$

If $\min(i_2, i_3) \geq 2$, then we have necessarily $i_2 = 3$ and $i_3 = 2$ by Lemma 3.5 of [1] because $c \neq 1$. Therefore, we get

$$c = \frac{P(e_2)}{P(e_3)} = \frac{P(e_3)}{P(e_2)},$$

and so $P(e_2)^2 = P(e_3)^2$. Since $P(e_2) \neq P(e_3)$ by the assumption (H), we have the conclusion (C2). It remains to consider the case $i_2 = 1$ or $i_3 = 1$. Changing indices if necessary, we assume that $i_2 = 1$, namely, $(2, 1) \in \Lambda$. This implies that $i_3 = 2$, namely, $(3, 2) \in \Lambda$, because $i_3 \neq 1, 3$ by Lemma 3.5 of [1] and the fact $c \neq 1$. Moreover, we have $(1, 3) \in \Lambda$ by the same reason. Therefore, we have (C1).

Lastly, we inquire into the assertion (3), namely, the case $q_1 = q_2 = 1$ and $q_3 \geq 2$. In this case, there are indices i and j such that $(3, i) \in \Lambda$ and $(j, 3) \in \Lambda$. Then, we may assume $i = 1$ and so $(3, 1) \in \Lambda$ by exchanging the role of indices 1 and 2 if necessary. If $j = 1$, then we have $P(e_1) + P(e_3) = 0$ and, if $j = 2$, then we have $P(e_1)P(e_2) = P(e_3)^2$. The proof of Theorem 3.2 is completed. □

We note here that Theorem 1.6 mentioned in Section 1 is an easy consequence of Theorem 3.2.

For the case $k = 2$, we can prove the following:

Theorem 3.4. *Assume that the derivative $P'(w)$ has two distinct zeros e_1 and e_2 with multiplicities q_1 and q_2 respectively and assume that $q_1 \leq q_2$. If it satisfies one of the conditions*

(1) $q_1 \geq 3$ and $P(e_1) + P(e_2) \neq 0$,

(2) $q_1 \geq 2$ and $q_2 \geq q_1 + 3$,

then $P(w)$ is a uniqueness polynomial.

Proof. Assume that $P(w)$ is not a uniqueness polynomial. Then, there are nonconstant distinct meromorphic functions f, g and a nonzero constant c such that $P(f) = cP(g)$. By virtue of Theorem 2.8 we have $c \neq 1$.

We first show the following:

LEMMA 3.5. *If* $c \neq P(e_2)/P(e_1)$, *then* $q_2 \leq 2$.

Proof. As in the proof of Lemma 3.3, we consider the polynomials $Q(w) := P(w) - P(e_2)$ and $Q^*(w) := cP(w) - P(e_2)$, and denote all zeros of $Q(w)$ and $Q^*(w)$ by $\alpha_1\ (= e_2), \alpha_2, \ldots, \alpha_m$ and $\beta_1, \beta_2, \ldots, \beta_n$, respectively. Then, α_1 is a zero of $Q(w)$ with multiplicity $q_2 + 1$ and α_i are simple zeros of it for $i = 2, 3, \ldots, m$. Moreover, by the assumption, all β_j $(1 \leq j \leq n)$ are simple zeros of $Q^*(w)$. Therefore, $m = q - q_2 = q_1 + 1$ and $n = q$. We now apply the second main theorem to the function g and q values β_j's to obtain

$$(q-2)T(r,g) \leq \sum_{j=1}^{q} N(r, \bar{\nu}_g^{\beta_j}) + S(r,g),$$

For every point z_0 with $g(z_0) = \beta_j$, we have $P(f(z_0)) = cP(g(z_0)) = cP(\beta_j) = P(e_2)$ and so $f(z_0)$ is equal to one of the values $\alpha_1, \alpha_2, \ldots, \alpha_m$. Noting that $T(r,f) = T(r,g) + O(1)$ by Lemma 3.2 of [1], we obtain

$$\begin{aligned}(q-2)T(r,g) &\leq \sum_{j=1}^{m} N(r, \bar{\nu}_f^{\alpha_j}) + S(r,f) \\ &\leq mT(r,f) + S(r,f) \\ &\leq (q_1+1)T(r,g) + S(r,g).\end{aligned}$$

This concludes that $q - 2 = q_1 + q_2 + 1 - 2 \leq q_1 + 1$, whence $q_2 \leq 2$. □

We continue the proof of Theorem 3.4. Under the assumption of (1), we have either $c \neq P(e_2)/P(e_1)$ or $c \neq P(e_1)/P(e_2)$, because otherwise

$$c^2 = \frac{P(e_2)}{P(e_1)} \frac{P(e_1)}{P(e_2)} = 1,$$

which contradicts to the assumption $P(e_1) + P(e_2) \neq 0$. Therefore, $q_1 \leq 2$ or $q_2 \leq 2$ as a consequence of Lemma 3.5. Thus, we have the assertion (1).

The proof of the assertion (2) is given by the the same argument as in [3, 191]. For readers' convenience, we repeat it here. By virtue of Lemma 3.5, it suffices to consider the only case $c = P(e_2)/P(e_1)$. By the same argument as in the proof of Lemma 3.5, $Q(w) := P(w) - P(e_2)$ has mutually distinct $m := q_1 + 1$ zeros $\alpha_1, \ldots, \alpha_m$ and $Q^*(w) := cP(w) - P(e_2)$ has mutually distinct $n := q_2 + 1$ zeros $\beta_1, \ldots, \beta_n$. In this case, if $g(z_0) = \beta_j$ for some

$z_0 \in \mathbf{C}$ and some j, then $f(z_0) = \alpha_i$ for some i. Therefore, we have

$$\begin{aligned}((q_2+1)-2)T(r,g) &\leq \sum_{j=1}^{m} N(r, \bar{\nu}_g^{\beta_j}) + S(r,g) \\ &\leq \sum_{i=1}^{m} N(r, \bar{\nu}_f^{\alpha_i}) + S(r,g) \\ &\leq (q_1+1)T(r,g) + S(r,g).\end{aligned}$$

This concludes $q_2 - 1 \leq q_1 + 1$, which contradicts the assumption. The proof of Theorem 3.4 is completed. □

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