ORTHOGONAL GROUP MATRICES OF HYPEROCTAHEDRAL GROUPS

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To the memory of TADASI NAKAYAMA

1. Introduction. The hyperoctahedral group G_n of order $2^n n!$ is generated by permutations and sign changes applied to n digits, $d = 1, 2, \ldots, n$. The 2^n sign changes generate a normal subgroup Σ_n whose factor group G_n/Σ_n is isomorphic with the symmetric group S_n of order $n!$. To each irreducible orthogonal representation $\{\lambda\ ;\ \mu\}$ of G_n corresponds an ordered pair of partitions $[\lambda]$ of l and $[\mu]$ of m, where $l+m=n$. The faithful representation $\{n-1\ ;\ 1\}$ of G_n is the real monomial group R_n of degree n. The representations $\{\lambda\ ;\ 0\}$ of G_n with $l=n$, $m=0$, are isomorphic with corresponding irreducible representations $\{\lambda\}$ of S_n. If the representation $\{\lambda\ ;\ \mu\}$ maps the element g_k of G_n into the real orthogonal matrix $M^{\lambda\mu}(g_k)$ of degree $f^{\lambda\mu}$, we define the *group matrix* of $\{\lambda\ ;\ \mu\}$ to be

$$\mathfrak{M}^{\lambda\mu} = \sum_k g_k^{-1} M^{\lambda\mu}(g_k) \qquad g_k \in G_n \tag{1.1}$$

Our purpose is to determine explicitly for each $\{\lambda\ ;\ \mu\}$ the uv-entry of the group matrix of an irreducible orthogonal representation of G_n, and incidentally those of S_n, in the form

$$\mathfrak{M}^{\lambda\mu}_{uv} = \gamma_v E^{\lambda\mu} \sigma^{\lambda\mu} \gamma_u^{-1} \tag{1.2}$$

by describing in the group ring Γ of G_n a suitable pair of ring elements $E^{\lambda\mu}$ related to permutations of S_n, and $\sigma^{\lambda\mu}$ related to sign changes of Σ_n, and also a set of invertible ring factors γ_v that meet our requirements. Matrices $M^{\lambda 0}(\tau_d)$ for transpositions τ_d of consecutive digits d, $d+1$ are to be those of Young's orthogonal representation $\{\lambda\}$ of S_n [4]. The matrix $M^{\lambda\mu}(\sigma_d)$ for the element σ_d of Σ_n that changes the sign of the digit d is to be a diagonal matrix with vv-entry $+1$ or -1 according as the digit d is assigned to the

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first part $[\lambda]$ or the second part $[\mu]$ in the v^{th} standard tableau $t_v^{\lambda\mu}$ for the diagram $[\lambda\ ;\ \mu]$.

2. Properties of the group matrix. For any group matrix $\mathfrak{M}$ we have the relations [2]

$$g\mathfrak{M} = \sum_k g g_k^{-1} M(g_k) = \sum_k g(g_k g)^{-1} M(g_k g) = \mathfrak{M}M(g) \tag{2.1}$$

$$\mathfrak{M}^T\mathfrak{M}^T = \sum_h \sum_k g_h^{-1} M^T(g_h) g_k^{-1} M^T(g_k) = \sum_{h,k} (g_k g_h)^{-1} M^T(g_k g_h) = {}^oG\mathfrak{M}^T \tag{2.2}$$

where ${}^oG$ denotes the order of G. Since any element g_k of G_n can be factored into transpositions τ_d of consecutive digits $d,\ d+1$, and sign changes σ_d, we require the following conditions on $E^{\lambda\mu}\sigma^{\lambda\mu} = E\sigma$ and γ_v in (1.2). To simplify the formulas we omit the superscripts $\lambda,\ \mu$.

$$\tau_d \gamma_v E\sigma\gamma_t^{-1} = \sum_u \gamma_u E\sigma\gamma_t^{-1} M_{uv}(\tau_d) \tag{2.3}$$

$$\sigma_d \gamma_v E\sigma\gamma_t^{-1} = \gamma_v E\sigma\gamma_t^{-1} M_{vv}(\sigma_d) \tag{2.4}$$

$$\sum_{t=1}^{f} \gamma_u E\sigma\gamma_t^{-1}\gamma_t E\sigma\gamma_v^{-1} = {}^oG(\gamma_u E\sigma\gamma_v^{-1}) \tag{2.5}$$

Conditions (2.3), (2.4), and (2.5) are satisfied if

$$\tau_d\gamma_v = \sum_u \gamma_u M_{uv}(\tau_d) \tag{2.6}$$

$$\sigma_d\gamma_v = \gamma_v M_{vv}(\sigma_d) \tag{2.7}$$

$$E^2 = ({}^oG/f)E = hE \tag{2.8}$$

$$\sigma E = E\sigma,\ \ \sigma^2 = \sigma. \tag{2.9}$$

The requirement that the representation M be orthogonal is satisfied if the coefficient of g_k^{-1} in $\gamma_v E\sigma\gamma_u^{-1}$ is the same as that of g_k in $\gamma_u\sigma E\gamma_v^{-1}$. Hence the ring element $\gamma_v E\sigma\gamma_v^{-1}$ contains g_k and g_k^{-1} with the same coefficient.

3. Standard tableaux and idempotents. In a Young diagram $[\lambda]$ having λ_i nodes in row i and λ'_j nodes in column j, we call $j-i$ the *height* of the *ij*-node, and we define the *ij*-hook h_{ij} by the formula [3]

$$h_{ij} = (\lambda_i - j) + (\lambda'_j - i) + 1. \tag{3.1}$$

In a two part Young diagram we shall add a large w to each height in the second part, and replace λ by μ in hook lengths of the second part. The products of the hook numbers for the nodes in $[\lambda]$, $[\mu]$, and $[\lambda\ ;\ \mu]$ are the hook products h^λ, h^μ, and $h^{\lambda\mu} = h^\lambda h^\mu$, respectively.

For example, for the partition $[\lambda\ ;\ \mu] = [3,\ 2\ ;\ 1^2]$ we have

Diagram $[3,\ 2;\ 1^2]$		Height graph	Hook graph	Hook products	
$[\lambda]$	· · · · ·	$0\ \ 1\ \ 2$ $-1\ \ 0$	$4\ \ 3\ \ 1$ $2\ \ 1$	$h^\lambda = 24$	(3.2)
$[\mu]$	· ·	w $w-1$	2 1	$h^\mu = 2,\ h^{\lambda\mu} = 48.$	

The associated diagram $[\lambda'\ ;\ \mu'] = [2^2 1\ ;\ 2]$, obtained by interchanging rows and columns, has the same hook products, but has heights of opposite signs.

A standard Young tableau $t_v^{\lambda\mu}$ is obtained from a Young diagram by assigning the digits $d = 1, 2, \ldots, n$ to its nodes in any order that increases from left to right in rows and from top to bottom in columns. Since the number f^λ of one part standard tableaux t_v^λ is $f^\lambda = l!/h^\lambda$, the number of two part standard tableaux $t_v^{\lambda\mu} = t_v^\lambda \oplus t_v^\mu$ is

$$f^{\lambda\mu} = \binom{n}{m} f^\lambda f^\mu = n!/h^\lambda h^\mu = n!/h^{\lambda\mu}. \tag{3.3}$$

Hence the required factor $h = {}^oG/f$ in (2.8) is the hook product $h = h^{\lambda\mu}$.

The positive symmetric group $P_v^{\lambda\mu}$ associated with the standard tableau $t_v^{\lambda\mu}$ is the sum of all permutations on n digits that leave each digit in the same row of the diagram as before. The negative symmetric group $N_v^{\lambda\mu}$ associated with $t_v^{\lambda\mu}$ is the sum of the even permutations minus the sum of the odd permutations that leave each digit in the same column as before. The product $P_v^{\lambda\mu}N_v^{\lambda\mu}$ is a weighted double coset that is a sum of permutations g of S_n with coefficients $\delta_v^{\lambda\mu}(g)$ that are 1, -1, or 0.

$$P_v^{\lambda\mu}N_v^{\lambda\mu} = \sum_g \delta_v^{\lambda\mu}(g)g \tag{3.4}$$

This ring element, used in generating Young's semi-normal representation (when $\mu = 0$), is known [4] to be a multiple of an idempotent, and we have the relation

$$(P_v^{\lambda\mu}N_v^{\lambda\mu})^2 = (P_v^{\lambda\mu}N_v^{\lambda\mu})\,h^{\lambda\mu} \tag{3.5}$$

Also, if the group sum $P_1^{\lambda\mu}$ is divided by its order ${}^oP_1^{\lambda\mu}$, then $P_1^{\lambda\mu}/{}^oP_1^{\lambda\mu}$ is an idempotent. We define $E^{\lambda\mu}$ to be the ring element

$$E^{\lambda\mu} = P_1^{\lambda\mu}N_1^{\lambda\mu}P_1^{\lambda\mu}/{}^oP_1^{\lambda\mu} \tag{3.6}$$

in which inverse elements appear with the same coefficient. We see by (3.5)

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and (3.6) that $E^{\lambda\mu}$ satisfies the condition (2.8), since

$$(E^{\lambda\mu})^2 = [P_1^{\lambda\mu} N_1^{\lambda\mu} P_1^{\lambda\mu} N_1^{\lambda\mu}] P_1^{\lambda\mu} / {}^o P_1^{\lambda\mu} = E^{\lambda\mu} h^{\lambda\mu}. \tag{3.7}$$

Next we construct the idempotent $\sigma^{\lambda\mu}$ from the normal subgroup $\sum_n$ of G_n

$$\sigma^{\lambda\mu} = 2^{-n} \prod_{d=1}^{l} (I + \sigma_d) \cdot \prod_{d=l+1}^{n} (I - \sigma_d) \tag{3.8}$$

and the related idempotents to be associated with the other tableaux $t_v^{\lambda\mu}$

$$\sigma_v^{\lambda\mu} = 2^{-n} \prod_{d \in t_v^{\lambda}} (I + \sigma_d) \cdot \prod_{d \in t_v^{\mu}} (I - \sigma_d) \tag{3.9}$$

The idempotent $\sigma^{\lambda\mu}$ commutes with $P_1^{\lambda\mu}$, $N_1^{\lambda\mu}$, and $E^{\lambda\mu}$. Choosing $\gamma_1 = I$, we take the product $E^{\lambda\mu}\sigma^{\lambda\mu}$ to be the 1, 1-entry of the group matrix $\mathfrak{M}$ in (1.2).

4. The ideal basis. The ring elements $\gamma_v E^{\lambda\mu} \sigma^{\lambda\mu}$ from the first row of the group matrix (1.2) should be a left ideal basis in Γ for our irreducible orthogonal representation $\{\lambda\ ;\ \mu\}$ of G_n. The matrix of the transposition $\tau_d = (d, d+1)$ in this representation is to have the vv-entry $\rho_v(\tau_d)$. If the digits $d+1$ and d are assigned to the same part t_v^{λ} or t_v^{μ} of the standard tableau $t_v^{\lambda\mu}$, then the integer $1/\rho_v(\tau_d)$ is equal to the height of the digit $d+1$ minus the height of the digit d. If $d+1$ and d are in different parts of $t_v^{\lambda\mu}$, we let w in (3.2) become infinite, and define the reciprocal of the height difference to be $\rho_v(\tau_d) = 0$.

The $f^{\lambda\mu}$ standard tableaux are assigned a dictionary order, such that if the digits are read from left to right by rows, starting at the top row, the first digit in which two tableaux disagree will be larger for the later tableau. If $t_u^{\lambda\mu}$ precedes $t_v^{\lambda\mu}$ and the two tableaux differ only by the transposition of digits d and $d+1$, we have $\rho_v(\tau_d) = -\rho_u(\tau_d) > 0$, and we take

$$\begin{bmatrix} M_{uu}(\tau_d) & M_{uv}(\tau_d) \\ M_{vu}(\tau_d) & M_{vv}(\tau_d) \end{bmatrix} = \begin{bmatrix} -\rho_v & (1-\rho_v^2)^{1/2} \\ (1-\rho_v^2)^{1/2} & \rho_v \end{bmatrix} = \begin{bmatrix} \rho_u & (1-\rho_u^2)^{1/2} \\ (1-\rho_u^2)^{1/2} & -\rho_u \end{bmatrix} \tag{4.1}$$

as in Young's orthogonal representation [4]. All other off-diagonal entries of $M^{\lambda\mu}(\tau_d)$ are set equal to 0. The resulting real symmetric orthogonal matrix is a direct sum of 1×1 orthogonal matrices $+1$ or -1, and 2×2 orthogonal matrices (4.1). On substituting the matrices (4.1), the condition (2.6) becomes

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$$\begin{aligned}\tau_d \gamma_u &= \rho_u \gamma_u + (1-\rho_u^2)^{1/2}\gamma_v \\ \tau_d \gamma_v &= (1-\rho_v^2)^{1/2}\gamma_u + \rho_v \gamma_v\end{aligned} \qquad \rho_v = -\rho_u > 0. \tag{4.2}$$

These equations yield the following important formula

$$\gamma_v \gamma_u^{-1} = (\tau_d - \rho_u I)(1-\rho_u^2)^{-1/2} = (\tau_d + \rho_v I)(1-\rho_v^2)^{-1/2}. \tag{4.3}$$

The ring factor γ_v for the tableau $t_v^{\lambda\mu}$ is hereby expressed in terms of the ring factor γ_u for an earlier tableau $t_u^{\lambda\mu}$ that differs from $t_v^{\lambda\mu}$ only by the interchange of digits d and $d+1$. Since $\gamma_1 = I$, the factor γ_v can be read off from the tableau $t_v^{\lambda\mu}$, by transposing pairs of consecutive digits until $t_v^{\lambda\mu}$ is reduced by one or more steps to $t_1^{\lambda\mu}$. For example, if

$$t_v^{3,2;1} = \begin{matrix}1\ 3\ 4 \\ 2\ 6 \\ \quad\quad 5\end{matrix} \rightarrow \begin{matrix}1\ 3\ 4 \\ 2\ 5 \\ \quad\quad 6\end{matrix} \rightarrow \begin{matrix}1\ 2\ 4 \\ 3\ 5 \\ \quad\quad 6\end{matrix} \rightarrow \begin{matrix}1\ 2\ 3 \\ 4\ 5 \\ \quad\quad 6\end{matrix} = t_1^{3,2;1} \tag{4.4}$$

then

$$\gamma_v = \frac{\tau_5 + 0\cdot I}{1} \cdot \frac{\tau_2 + I/2}{(3/4)^{1/2}} \cdot \frac{\tau_3 + I/3}{(8/9)^{1/2}} \tag{4.5}$$

The generators τ_d and σ_d of G_n satisfy the relations

$$\sigma_d \tau_d = \tau_d \sigma_{d+1}, \text{ and } \tau_d^2 = \sigma_d^2 = I. \tag{4.6}$$

It follows from (4.6) and (3.9) that

$$\sigma_v^{\lambda\mu}\tau_d = \tau_d \sigma_u^{\lambda\mu}, \text{ and } \sigma_v^{\lambda\mu}(\tau_d + \rho_v I) = (\tau_d + \rho_v I)\,\sigma_u^{\lambda\mu} \tag{4.7}$$

$$\gamma_v^{-1}\sigma_v^{\lambda\mu}\gamma_v = \gamma_u^{-1}\sigma_u^{\lambda\mu}\gamma_u = \sigma^{\lambda\mu}. \tag{4.8}$$

Thus condition (2.4) is satisfied, since

$$\sigma_d \gamma_v E^{\lambda\mu}\sigma^{\lambda\mu} = \sigma_d \sigma_v^{\lambda\mu}\gamma_v E^{\lambda\mu} = \pm \gamma_v E^{\lambda\mu}\sigma^{\lambda\mu} \tag{4.9}$$

where the sign is $+1$ or -1 according as digit d belongs to t_v^λ or t_v^μ.

The construction described above verifies our main result.

Theorem. Every irreducible representation over the complex field of the hyperoctahedral group G_n is similar to a real orthogonal representation $\{\lambda ; \mu\}$, whose group matrix $\mathfrak{M}^{\lambda\mu}$ has the uv-entry (1.2), where $E^{\lambda\mu}$, $\sigma^{\lambda\mu}$, and γ_v are defined respectively by (3.6), (3.8), and (4.3).

For the symmetric group S_n the corresponding irreducible orthogonal group matrices are obtained by taking $\mu = 0$, and replacing $\sigma^{\lambda\mu}$ by I.

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