TOPOLOGY OF COMPLEX MANIFOLDS

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1. The aim of this paper is to collect some interesting unsolved problems in the topology of compact complex manifolds which are scattered in literature; needless to remark, some of them are known to specialists. Besides, the reader may consult the well-known list of problems by F. Hirzebruch [5] which has stimulated remarkable progress in this direction. But the topology of complex manifolds has always baffled workers in this field and is still far from satisfactory. Of course, there is remarkable progress in the case of homogeneous compact complex manifolds. We start with the following result.

LEMMA. Let V be a compact almost-complex surface whose second Betti member b_2[*] is zero; then its Euler-Poincaré characteristic E is zero.

Let I(V) denote the index of V and let c_i its i-th chern class. The index formula of Thom shows that $3I(V) = c_1^2 - 2c_2$. Since $b_2 = 0$, we have $c_1 = 0$ and $I(V) = 0$; consequently $E = 0$.

Remark. This gives a new proof of Ehresmann's result that the sphere S^4 has no almost-complex structure. We recall that Borel and Serre have shown that the sphere S^{2n}

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* b_i denotes the i^{th} Betti number of V.

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has no almost-complex structure for $n \geq 4$. This part follows easily from the following divisibility theorem of Bott: the chern number c_n of a complex vector bundle over the sphere S^{2n} is divisible by $(n-1)!$.

It is well-known that the sphere S^6 has a non-integrable almost complex structure induced by the cayley numbers; in fact, any product manifold $M \times S^4$ where M is any closed, orientable surface has an almost-complex structure (cf. [2]). All almost-complex structures on S^6 inducing the same orientation are homotopic [3]. A. Adler has recently announced (to be published) that the sphere S^6 has no complex structure at all; his method makes use of the fact that S^6 is a sphere. Thus there is an immediate necessity to find some new results which include Adler's theorem. Probably a reasonable conjecture seems to be the following (cf. problem 13 [5]).

PROBLEM 1 (Wang). Let V be a compact complex manifold whose second Betti number is zero. Does the Euler-Poincaré characteristic of V vanish?

Note that the "complex structure" cannot be replaced by a non-integrable almost-complex structure as S^6 shows.

In the case of complex surfaces, Kodaira has shown that a compact complex surface with $c_1^2 > 0$ is always algebraic. This result (beside others) has played an important role in the classification of surfaces by Kodaira [7]. So far, the following conjecture of Severi remains unsolved.

PROBLEM 2. Let V be a simply-connected algebraic surface whose geometric genus p_g $(= h^{2,0})$* vanishes. Is V rational?

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* $h^{2,0}$ denotes the number of linearly independent holomorphic 2-forms on V.

An affirmative answer to this problem together with the above result of Kodaira and a theorem of Nagata-Andreotti [8] solves the problem 25 of Hirzebruch [5]. We remark that the problems 26, 27, 28 and 30 [5] are also connected with problem 2.

In the classification of surfaces, Kodaira [7] has come across the class of compact complex surfaces V with $b_1 = 1$. If V has no exceptional curves of the first kind, it is known (cf. [7], [10]) that: i) if V has no non constant meromorphic functions, then $p_g = 0$; ii) if V has a non-constant meromorphic function, then V is a holomorphic principal fibre bundle over S_2 with fibre a torus; thus the second Betti number of V is zero. However, we have the following [10]:

PROPOSITION. Let V be a compact complex surface without exceptional curves of the first kind whose fundamental group is infinite cyclic. If V has a non-constant meromorphic function, then V is a Hopf manifold.

PROBLEM 3. (Kodaira). Classify all compact complex manifolds with $b_1 = 1$ and having no exceptional curves of the first kind.

2. Another important problem in the differential geometry of compact complex manifolds V is the relation between the curvature of a Kähler metric on V and its Topology. There is some progress in this direction in the case of pinched Kähler manifolds. A striking result is the following [1b]: if $\dim_c V = 2$, and if V has strictly positive or negative holomorphic sectional curvature, then its Euler-Poincaré characteristic E is positive. Another interesting result for manifolds of any dimension is the following ([1a]): if V is a compact Kähler manifold of strictly positive curvature, then its second Betti number is 1; thus $h^{2,0} = 0$ and hence V is algebraic by a theorem of Kodaira.

PROBLEM 4 (Frankel). Let V be a compact Kähler manifold of strictly positive holomorphic sectional curvature. Is V isomorphic to the complex projective space?

If $\dim_c V = 2$, it is so after Frankel-Andreotti.

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Perhaps another class of compact Kähler manifolds which is of some interest are those whose Ricci curvature is strictly positive or negative; such manifolds are algebraic by a theorem of Kodaira. If V is of strictly positive Ricci curvature and if $\dim_c V = 2$, the vanishing theorem of Kodaira shows that $H^i(V, \Omega^0(iK)) = 0$ for $i \neq \dim_c V$ where K denotes the canonical line bundle; thus $p_g = 0$ and $P_2 = \dim H^2(V, \Omega^0(2K)) = 0$. On the other hand, such a manifold is always simply-connected (cf. p. 483 [9]): hence V is rational by a classical result of Castelnuova-Enriques.

Thus we have the following:

PROBLEM 5. Let V be a compact Kähler manifold of strictly positive Ricci curvature. Is V rational?

Let V be a compact Kähler manifold with negative Ricci curvature; then V is algebraic.

PROBLEM 6. Is the universal covering $\tilde{V}$ of V a hermitian symmetric domain?

It is so, if $\tilde{V}$ is homogeneous by a theorem of Hano [3].

A well-known result of Igusa (cf. p. 675 [5]) shows that if V is uniformisable by a bounded domain (i.e. if $\tilde{V}$ is a bounded domain), then V is minimal in the sense that any meromorphic mapping from a complex manifold into V is necessarily holomorphic.

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