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NONTRIVIAL SOLUTIONS OF A SEMILINEAR ELLIPTIC PROBLEM VIA VARIATIONAL METHODS

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Using variational methods, we investigate the existence of nontrivial solutions of a nonlinear elliptic boundary value problem at resonance under generalised Ahmad-Lazer-Paul conditions. Some new results are obtained and some results in the literature are improved.

1. INTRODUCTION

In this paper we consider the existence of nontrivial solutions of the following problem on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary

$$(1.1) \qquad \begin{cases} \Delta u + \lambda_k u + g(x,u) = 0 & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $g(x,t)$ is a Caratheodory function such that $g(x,0) = 0$ for almost everhwhere $x \in \Omega$ and λ_k is the $k-th$ $(k \geqslant 2)$ eigenvalue for the elliptic linear operator $-\Delta$ with zero Dirichlet boundary condition.

It is well-known that the operator $-\Delta$ with zero Dirichlet boundary condition has discrete eigenvalues $(0 <)\lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots$ and each eigenspace, is finite dimensional. Denote the eigenspace corresponding to λ_i by E_i and suppose that $E_k = \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$.

We impose the following conditions.

(g1) $\left|g(x,t)\right| \leqslant C|t|^\alpha + b(x)$,

where $b(x) \in L^q$ with $q = (2N/N+2)$ for $N \geqslant 3$, $q = 1$ for $N = 1, 2$ and $C > 0$, $0 \leqslant \alpha < 1$ are constants;

$$(\mathrm{G}\pm) \qquad \frac{\int_\Omega G(x, \sum_{i=1}^m a_i\phi_i)\,dx}{\|a\|^{2\alpha}} \to \pm\infty \text{ as } \|a\| = \left(\sum_{i=1}^m a_i^2\right)^{1/2} \to \infty$$

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where $a = (a_1, a_2, \dots, a_m) \in \mathbb{R}^m$ and $G(x,t) = \int_0^t g(x,s)\,ds$.

The conditions (G $\pm$) with $\alpha = 0$ are proposed by Ahmad-Lazer-Paul ([**1**]) and are in some sense coercivity conditions on the function $G(x,t)$.

Much work has been done on the existence of solutions or multiple solutions to the problem (1.1) since the work in [**12**], either by topological or by variational methods; for example see [**6, 10, 14, 15, 16**] and the references therein. To obtain (nontrivial) solutions, one of the difficulties is that $\big(g(x,t) + \lambda_k t\big)/t$ may approach some λ_i as $t \to \infty$ (resonant at infinity) or as $t \to 0$ (resonant at 0), since when resonance at infinity occurs, it is difficult to obtain the priori estimates needed by the topological methods or obtain the Palais-Smale condition required by the variational methods. The Ahmad-Lazer-Paul conditions have been widely used in the literature to overcome the difficulty. For the strong resonance case, that is, $g(x,t) \to 0$ as $t \to \infty$, where Ahmad-Lazer-Paul conditions fail, [**3, 4, 5**] develop some variational techniques to investigate the nontrivial solutions of (1.1). For the nonresonant or incompletely resonant case, there is also a lot of work in this respect; for example see [**2, 11, 14**]. But it seems that there is not too much work to deal with the middle case where $g(x,t)$ satisfies conditions like (g1). For some related results see [**13**]. In [**9**], we proposed conditions (G $\pm$) to investigate the existence of solutions and proved the following theorem.

THEOREM 1.1. *Suppose that condition pair* (g1), (G_+) *or* (g1), (G_-) *holds. Then equation* (1.1), *where we do not assume that* $g(x,0) = 0$ *for almost everywhere* $x \in \Omega$, *has at least one solution in* $H_0^1(\Omega)$.

The above theorem with $\alpha = 0$ was proved in [**1**]; see also [**15**]. In this paper, we aim to investigate the nontrivial solutions to (1.1) under the conditions (g1) and (G_+) or (G_-) and obtain the following results.

THEOREM 1.2 *Suppose that conditions* (g1) *and* (G_+) *hold. If there exists* $m \leqslant k$ *such that*

$$\limsup_{t \to 0} \frac{g(x,t)}{t} < \lambda_m - \lambda_k \tag{1.2}$$

and

$$\inf_{t \neq 0} \frac{g(x,t)}{t} \geqslant \lambda_{m-1} - \lambda_k \tag{1.3}$$

uniformly for almost everywhere $x \in \Omega$, *then equation* (1.1) *has at least one nontrivial solution in* $H_0^1(\Omega)$.

THEOREM 1.3. *Suppose that conditions* (g1) *and* (G_-) *hold. If there exists* $m \geqslant k$ *such that*

$$\liminf_{t \to 0} \frac{g(x,t)}{t} > \lambda_m - \lambda_k \tag{1.4}$$

and

$$\sup_{t\neq 0} \frac{g(x,t)}{t} \leqslant \lambda_{m+1} - \lambda_k \tag{1.5}$$

uniformly for almost everywhere $x \in \Omega$, *then equation* (1.1) *has at least one nontrivial solution in* $H_0^1(\Omega)$.

It is natural to investigate the case $m > k$ in Theorem 1.2 and the case $m < k$ in Theorem 1.3. The corresponding results are interesting, since the coercivity condition (G_+) or condition (G_-) is not indispensable. In particular, Theorem 1.5 contains one of the main results in [**14**, Theorem 1] as a special case.

THEOREM 1.4. *Suppose that condition* (g1) *and the Palais-Smale condition for* J *at any level* $c < 0$ *hold. If there exists* $m > k$ *such that* (1.4) *and* (1.5) *hold, then equation* (1.1) *has at least one nontrivial solution in* $H_0^1(\Omega)$.

THEOREM 1.5. *Suppose that condition* (g1) *and the Palais-Smale condition for* J *at any level* $c > 0$ *hold. If there exists* $m < k$ *such that such that* (1.2) *and* (1.3) *hold, then equation* (1.1) *has at least one nontrivial solution in* $H_0^1(\Omega)$.

2. Proofs of the Theorems

In the following, the notations $\|\cdot\|$ and $\langle\cdot\rangle$ denote the norm in $H_0^1(\Omega)$ and the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. C denotes a universal constant. For $u \in H_0^1(\Omega)$ and $p > 0$, denote

$$\|u\|_p = \Big(\int_\Omega |u|^p\,dx\Big)^{1/p}$$

and decompose u as $u = \overline{u} + u^0 + \widetilde{u}$, where $\overline{u} \in \sum_{i<k} E_i$, $u^0 \in E_k$ and $\widetilde{u} \in \sum_{i>k} E_i$.

It is well-known that (weak) solutions of (1.1) in $H_0^1(\Omega)$ correspond to the critical points of the C^1 functional in $H_0^1(\Omega)$

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\lambda_k\int_\Omega |u|^2\,dx - \int_\Omega G(x,u)\,dx$$

and

$$\langle J'(u), v\rangle = \int_\Omega \big(\nabla u\nabla v - \lambda_k uv - g(x,u)v\big)\,dx$$

for $u, v \in H_0^1(\Omega)$ (see [**15**]).

If we want to get classical solutions, we need to impose more regularity assumptions on $g(x,t)$, for example that g is locally Lipschitz in $\overline{\Omega} \times \mathbb{R}$; see [**8**] for more details.

LEMMA 2.1. *Under the condition pair* (g1), (G_+) *or* (g1), (G_-), *the functional* J *defined above satisfies the Palais-Smale condition.*

PROOF: We only prove the case where (g1) and (G_+) hold. The other case can be similarly proved.

Suppose that $\{u_n\} \subset H_0^1(\Omega)$ satisfies

$$
\begin{aligned}
&J'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\Omega), \quad \text{as} \quad n \to \infty; \\
&|J(u_n)| \leqslant C.
\end{aligned} \tag{2.1}
$$

Hence

$$
\begin{aligned}
&\langle J'(u_n), -\overline{u}_n \rangle \\
&\quad = \int_\Omega \left(-|\nabla \overline{u}_n|^2 + \lambda_k |\overline{u}|^2 + g(x, u_n)\overline{u}_n\right) dx \\
&\quad \geqslant (\lambda_k - \lambda_{k-1}) \int_\Omega |\overline{u}_n|^2 \, dx - \int_\Omega |\overline{u}_n| \left(C|\overline{u}_n + u_n^0 + \widetilde{u}_n|^\alpha + b\right) dx \\
&\quad \geqslant (\lambda_k - \lambda_{k-1}) \int_\Omega |\overline{u}_n|^2 \, dx - \int_\Omega |\overline{u}_n| \, b \, dx - 3^\alpha C \int_\Omega |\overline{u}_n| \left(|\overline{u}_n|^\alpha + |u_n^0|^\alpha + |\widetilde{u}_n|^\alpha\right) dx \\
&\quad \geqslant (\lambda_k - \lambda_{k-1} - \varepsilon) \int_\Omega |\overline{u}_n|^2 \, dx - C \int_\Omega |\overline{u}_n| |u_n^0|^\alpha \, dx - C \int_\Omega |\overline{u}_n| \, |\widetilde{u}|^\alpha \, dx - C(\varepsilon) \\
&\quad \geqslant (\lambda_k - \lambda_{k-1} - 2\varepsilon) \int_\Omega |\overline{u}_n|^2 \, dx - C(\varepsilon) \int_\Omega |u_n^0|^{2\alpha} \, dx - C(\varepsilon) \int_\Omega |\widetilde{u}_n|^{2\alpha} \, dx - C(\varepsilon)
\end{aligned}
$$

where $C(\varepsilon) > 0$ is a universal constant dependent on the arbitrary $\varepsilon > 0$. Fixing $\varepsilon > 0$ sufficiently small and noting that all norms in $\sum_{i<k} E_i$ are equivalent, we have

$$
\|\overline{u}_n\|^2 \leqslant C\|u_n^0\|_{2\alpha}^{2\alpha} + C\|\widetilde{u}_n\|_{2\alpha}^{2\alpha} + C. \tag{2.2}
$$

By a similar argument we can prove that

$$
\langle J'(u_n), \widetilde{u}_n \rangle \geqslant \left(1 - \frac{\lambda_k}{\lambda_{k+1}} - 2\varepsilon\right) \int_\Omega |\nabla \widetilde{u}_n|^2 \, dx - C(\varepsilon) \int_\Omega |\overline{u}_n|^{2\alpha} \, dx - C(\varepsilon) \int_\Omega |u_n^0|^{2\alpha} \, dx - C(\varepsilon).
$$

Hence we have the inequality

$$
\|\widetilde{u}_n\|^2 \leqslant C\|u_n^0\|_{2\alpha}^{2\alpha} + C\|\overline{u}_n\|_{2\alpha}^{2\alpha} + C. \tag{2.3}
$$

Combining (2.2) and (2.3), we have

$$
\begin{aligned}
\|\widetilde{u}_n\|^2 &\leqslant C\|u_n^0\|_{2\alpha}^{2\alpha} + C\|\overline{u}_n\|^{2\alpha} + C \\
&\leqslant C\|u_n^0\|_{2\alpha}^{2\alpha} + C\|u_n^0\|_{2\alpha}^{2\alpha^2} + C\|\widetilde{u}\|_{2\alpha}^{2\alpha^2} + C.
\end{aligned} \tag{2.4}
$$

By the Hölder inequality and the Sobolev inequality, in view of $2\alpha^2 < 2\alpha < 2$, it follows immediately from (2.4) that

$$
\|\widetilde{u}_n\|^2 \leqslant C\|u_n^0\|_{2\alpha}^{2\alpha} + C\|\widetilde{u}\|_{2\alpha}^{2\alpha^2} + C.
$$

Consequently,

$$
\|\widetilde{u}_n\|^2 \leqslant C\|u_n^0\|_{2\alpha}^{2\alpha} + C. \tag{2.5}
$$

By a similar argument to that in the proof (2.5), we obtain

$$\|\overline{u}_n\|^2 \leqslant C\|u_n^0\|_{2\alpha}^{2\alpha} + C. \tag{2.6}$$

Now we estimate $\int_\Omega \big(G(x,u_n) - G(x,u_n^0)\big)\,dx$.

$$\begin{aligned}
\int_\Omega \big(G(x,u_n) - G(x,u_n^0)\big)\,dx &= \int_\Omega dx \int_0^1 g\big(x, u_n^0 + s(\widetilde{u}_n + \overline{u}_n)\big)(\widetilde{u}_n + \overline{u}_n)\,ds \\
&\leqslant \int_\Omega dx \int_0^1 \big(|\widetilde{u}_n| + |\overline{u}_n|\big)\Big(C\big|u_n^0 + s(\widetilde{u}_n + \overline{u}_n)\big|^\alpha + b\Big)\,ds \\
&\leqslant C\int_\Omega \Big(|\widetilde{u}_n|\,|u_n^0|^\alpha + |\widetilde{u}_n|^{1+\alpha} + |\widetilde{u}_n|\,|\overline{u}_n|^\alpha + b|\widetilde{u}_n|\Big)\,dx \\
&\qquad + \int_\Omega \big(|\overline{u}_n||u_n^0|^\alpha + |\overline{u}_n|\,|\widetilde{u}_n|^\alpha + |\overline{u}_n|^{1+\alpha} + b|\overline{u}_n|\big)\,dx.
\end{aligned}$$

By (2.5) and (2.6) and a simple calculation, we can obtain

$$\int_\Omega \big(G(x,u_n) - G(x,u_n^0)\big)\,dx \leqslant C\|u_n^0\|_{2\alpha}^{2\alpha} + C. \tag{2.7}$$

Obviously, by (2.1) and the definition of J,

$$-C \leqslant \frac{1}{2}\int_\Omega |\nabla \widetilde{u}_n|^2\,dx - \int_\Omega \big(G(x,u_n) - G(x,u_n^0)\big)\,dx - \int_\Omega G(x,u_n^0)\,dx.$$

Moreover, by (2.3) and (2.7), we have

$$-C \leqslant C\|u_n^0\|_{2\alpha}^{2\alpha} + C - \int_\Omega G(x,u_n^0)\,dx.$$

Write $u_n^0 = \sum\limits_{i=1}^{m} a_i^n \phi_i$. The above inequality is converted to

$$-C \leqslant C\Big(\sum_{i=1}^{m}(a_i^n)^2\Big)^\alpha + C - \int_\Omega G\Big(x, \sum_{i=1}^{m} a_i^n\phi_i\Big)\,dx.$$

Hence $\Big\{\sum\limits_{i=1}^{m}(a_i^n)^2\Big\}$ is bounded by the condition (G_+). Furthermore, $\{u_n\}$ is bounded in $H_0^1(\Omega)$ by (2.5) and (2.6). A standard argument implies that J satisfies the Palais-Smale condition in $H_0^1(\Omega)$ (see [**15**]). ∎

Proof of Theorem 1.2: We only need to prove the existence of nontrivial solutions of J in $H_0^1(\Omega)$. Write $H_0^1(\Omega) = \sum\limits_{i<m} E_i \oplus \sum\limits_{i\geqslant m} E_i$. In order to use [**15**, Theorem 5.3], in view of Lemma 2.1, we need to verify the following conditions:

(i) there are $\rho, d > 0$ such that $J \geqslant d$ on $\Big\{u \in \sum\limits_{i\geqslant m} E_i \mid \|u\| = \rho\Big\}$;

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(ii) there are $e \in \sum_{i \geqslant m} E_i$ with $\|e\| = 1, R > \rho$, and $\varepsilon < d$ such that if $Q = \left\{u \in \sum_{i<m} E_i \mid \|u\| \leqslant R\right\} \oplus \{te : 0 < t < R\}$, then $J \leqslant \varepsilon$ on ∂Q, where ∂Q denotes the boundary of Q in $\sum_{i<m} E_i \oplus \mathbb{R}e$.

By (g1) and (1.2), the condition (i) can be proved by the argument in the proof of [**14**, Theorem 1]. By (1.3), it is obvious that $J \leqslant 0$ on $\sum_{i \leqslant m-1} E_i$. Hence, in order to obtain (ii), we only need to prove

$$\lim_{\substack{\|u\| \to \infty \\ u \in \sum_{i \leqslant m} E_i}} J(u) = -\infty, \tag{2.8}$$

since then e can be taken as any element in E_m with $\|e\| = 1$, R any number sufficiently large and $\varepsilon < d$ any number sufficiently small.

In fact, for $u \in \sum_{i \leqslant m} E_i$, we have $u = \overline{u} + u^0$, since $m \leqslant k$. Suppose that $m = k$. Then

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega \left(|\nabla \overline{u}|^2 - \lambda_k |\overline{u}|^2\right) dx - \int_\Omega \left(G(x,u) - G(x,u^0)\right) dx - \int_\Omega G(x,u^0)\, dx \\ &\leqslant \frac{1}{2}(\lambda_{k-1} - \lambda_k) \int_\Omega |\overline{u}|^2\, dx - \int_\Omega dx \int_0^1 g(x, u^0 + s\overline{u})\overline{u}\, ds - \int_\Omega G(x,u^0)\, dx \\ &\leqslant \frac{1}{2}(\lambda_{k-1} - \lambda_k) \int_\Omega |\overline{u}|^2\, dx + C\int_\Omega |\overline{u}|\left(|u^0|^\alpha + |\overline{u}|^\alpha + b\right) dx - \int_\Omega G(x,u^0)\, dx \\ &\leqslant \frac{1}{2}(\lambda_{k-1} - \lambda_k + \varepsilon) \int_\Omega |\overline{u}|^2\, dx + C\int_\Omega |\overline{u}|\, |u^0|^\alpha\, dx - \int_\Omega G(x,u^0)\, dx + C(\varepsilon) \quad (\forall \varepsilon > 0) \\ &\leqslant \frac{1}{2}(\lambda_{k-1} - \lambda_k + 2\varepsilon) \int_\Omega |\overline{u}|^2\, dx + C(\varepsilon)\int_\Omega |u^0|^{2\alpha}\, dx - \int_\Omega G(x,u^0)\, dx + C(\varepsilon). \qquad (2.9) \end{aligned}$$

Choosing $0 < \varepsilon < (\lambda_k - \lambda_{k-1})/2$ and using the condition (G_+), we obtain (2.8).

If $m < k$, then $u = \overline{u}$. The proof of (2.8) is much easier. The theorem is proved. ∎

Proof of Theorem 1.3: Under the conditions of the theorem, we can prove

(i) there are $\rho, d > 0$ such that $J \leqslant -d$ on $\left\{u \in \sum_{i \leqslant m} E_i \mid \|u\| = \rho\right\}$;

(ii) $J \geqslant 0$ on $\sum_{i \geqslant m+1} E_i$;

(iii) $J(u) \to +\infty$ as $u \in \sum_{i \geqslant m} E_i$ and $\|u\| \to \infty$.

Then, for $I = -J$, we use [**15**, Theorem 5.29] and obtain a positive(nonzero) critical value for I. This completes the proof. ∎

Proofs of Theorems 1.4–1.5: Under the conditions of Theorem 1.4, we obtain a negative critical value for J by obtaining (i)–(iii) in the proof of Theorem 1.3, where in the proof of (iii) we notice $m > k$ and use a similar (and simpler) argument to the proof of (2.9). The proof of Theorem 1.5 can be given as that of Theorem 1.2. ∎

Lemma 2.2. *Suppose that $g(x,t)$ satisfies* (g1) *and there exist $a(x) \in L^1(\Omega)$ $a(x) \geqslant 0$ with $\int_\Omega a(x)\, dx > 0$ and $b(x) \in L^1(\Omega)$ such that*

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$$\text{(g2)} \quad a(x) \leqslant \liminf_{t\to\infty} \frac{g(x,t)t}{|t|^{1+\alpha}} \leqslant \limsup_{t\to\infty} \frac{g(x,t)t}{|t|^{1+\alpha}} \leqslant b(x)$$

uniformly for almost everywhere $x \in \Omega$. Then the condition (G_+) *holds.*

PROOF: It is easy to get an $\varepsilon > 0$ such that

$$\int_\Omega a(x)\big|v(x)\big|^{1+\alpha}\,dx \geqslant \varepsilon \int_\Omega \big|v(x)\big|^{1+\alpha}\,dx$$

for all $v \in E_k$, $v \neq \theta$. For this $\varepsilon > 0$, $\exists T > 0$ such that $g(x,t)t \geqslant \big(a(x) - \varepsilon\big)\,|t|^{1+\alpha}$ for $|t| > T$. Hence

$$\begin{aligned} G(x,t) - G(x,0) &= \int_0^1 g(x,ts)t\,ds \\ &= \int_{|ts|>T} \frac{1}{s} g(x,ts)\,ts\,ds + \int_{|ts|\leqslant T} \frac{1}{s} g(x,ts)\,ts\,ds \\ &\geqslant \int_{|ts|>T} \frac{1}{s}\big(a(x) - \varepsilon\big)|ts|^{1+\alpha}\,ds - |t| \int_{|ts|\leqslant T} h_\varepsilon(x)\,ds \\ &\geqslant \frac{1}{1+\alpha}\big(a(x)-\varepsilon\big)|t|^{1+\alpha} - h_\varepsilon(x)|t| \end{aligned} \tag{2.10}$$

for some $h_\varepsilon \in L^q(\Omega)$.

For any sequence $\{u_n\} \in E_k$ with $\|u_n\| \to \infty$, set $v_n = u_n/\|u_n\|$. Without loss of generality, we assume that $v_n \to v$ in $C(\overline{\Omega})$ and $\big|v_n(x)\big| \leqslant C$ for almost everywhere $x \in \Omega$ and $n \geqslant 1$, where $v \neq \theta$. Therefore, by Fatou lemma and the inequality (2.10),

$$\begin{aligned} &\liminf_{n\to\infty} \|u_n\|^{-(\alpha+1)} \int_\Omega G(x,u_n)\,dx \\ &\quad\geqslant \liminf_{n\to\infty} \|u_n\|^{-(\alpha+1)} \int_\Omega \Big(\frac{1}{1+\alpha}\big(a(x)-\varepsilon\big)|u_n|^{1+\alpha} - h_\varepsilon(x)|u_n|\Big)\,dx \\ &\quad\geqslant \frac{1}{\alpha+1} \int_\Omega \big(a(x)-\varepsilon\big)\big|v(x)\big|^{1+\alpha}\,dx > 0. \end{aligned}$$

Consequently, by $\alpha + 1 > 2\alpha$, we have

$$\lim_{n\to\infty} \|u_n\|^{-2\alpha} \int_\Omega G(x,u_n)\,dx = +\infty.$$

This completes the proof. ∎

LEMMA 2.3 *Suppose that $g(x,t)$ satisfies* (g1) *and there exist $a(x) \in L^1(\Omega)$ and $b(x) \in L^1(\Omega)$, $b(x) \leqslant 0$ with $\int_\Omega b(x)\,dx < 0$ such that*

$$\text{(g3)} \quad a(x) \leqslant \liminf_{t\to\infty} \frac{g(x,t)t}{|t|^{1+\alpha}} \leqslant \limsup_{t\to\infty} \frac{g(x,t)t}{|t|^{1+\alpha}} \leqslant b(x)$$

uniformly for almost everywhere $x \in \Omega$. Then the condition (G_-) *holds.*

PROOF: The proof is similar to that of Lemma 2.2. ∎

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Corollary 2.1 *Suppose that the assumptions in Theorem 1.2 hold with* (G_+) *replaced by* (g2). *Then* (1.1) *has at least one nontrivial solution in* $H_0^1(\Omega)$.

Remark 2.1. The above corollary is essentially proved in [**16**, Theorem 1]. Existence of nontrivial solutions of (1.1) under (g2) and some other conditions is also investigated in [**13**] by Morse theory where more regularity conditions on $g(x,t)$ and different conditions near $t = 0$ are needed.

The condition (1.2) is a one-sided nonresonant condition at the origin with respect to the eigenvalue λ_m. Using the ideas in [**7, 14**], we can relax it. First, we make some preparations.

A measurable subset E of $\mathbb{R}$ is said to have positive density at $+0(-0)$ if

$$\liminf_{r\to+0} \frac{\operatorname{meas}(E\cap[0,r])}{\operatorname{meas}([0,r])} > 0 \quad \Big(\liminf_{r\to-0} \frac{\operatorname{meas}(E\cap[r,0])}{\operatorname{meas}([r,0])} > 0\Big).$$

We say that a measurable subset A of a measurable set B is a full subset of B if $B\backslash A$ has measurable zero. For $A\subset\Omega$ and $r>0$, write

$$E(A,r) = \bigcap_{x\in A}\Big\{t\in\mathbb{R}\backslash\{0\} : \frac{G(x,t)}{2t^2} \leqslant r\Big\}.$$

Now we present the following improvement of Theorem 1.2. Other theorems in this paper can be similarly improved.

Theorem 2.1. *Suppose that conditions* (g1) *and* (G_+) *hold. Assume there exists* $m \leqslant k$ *such that*

$$\limsup_{t\to0} \frac{g(x,t)}{t} \leqslant \lambda_m - \lambda_k$$

and

$$\inf_{t\neq0} \frac{g(x,t)}{t} \geqslant \lambda_{m-1} - \lambda_k$$

uniformly for almost everywhere $x\in\Omega$. *If there exists a full subset* Ω' *of* Ω *and* $\eta>0$ *such that* $E(\Omega',\lambda_m-\lambda_k-\eta)$ *has positive density at* $+0$ *or* (-0), *then equation* (1.1) *has at least one nontrivial solution in* $H_0^1(\Omega)$.

Proof: The proof can be given combining the arguments in the proof of Theorem 1.2 in this paper and those in the proof of [**14**, Theorem 2]. ∎

Remark 2.2: Finally, we point out that since there are no coercive conditions on $g(x,t)$ in Theorems 1.4 and 1.5, by weakening the Palais-Smale condition to the (C) condition (for example see [**3**]), these theorems may be applied to investigate the strong resonance case for the problem (1.1). Further research may appear elsewhere.

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