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A 1-ALG SIMPLE CLOSED CURVE IN E^3 IS TAME

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0. Introduction. Let J be a simple closed curve in a 3-manifold M^3. We say $M - J$ is 1-ALG *at* $p \in J$ (or has locally abelian fundamental group at p) if and only if for each sufficiently small open set U containing p, there is an open set V such that $p \in V \subset U$ and each loop in $V - J$ which bounds in $U - J$ is contractible to a point in $U - J$. Our main result is

MAIN THEOREM. *If J is a simple closed curve embedded in a 3-manifold M^3 so that $M^3 - J$ is* 1-ALG *at each point of J, then J is tame.*

The case where M^3 is non-orientable can be reduced to the orientable case by looking at the orientable double covering space of M^3. Because any simple closed curve in an orientable 3-manifold M^3 lies in a cube-with-handles in M^3 (see, for example, [**1**, Theorem 1]), some neighbourhood of such a curve can be embedded in E^3. Thus, it suffices to prove the theorem in the case that M^3 is E^3. Throughout the remainder of this paper, we will assume that J is a simple closed curve in E^3 and that $E^3 - J$ is 1-ALG at each point of J.

THEOREM 1. *$E^3 - J$ is* 1-ALG *at each point of J if and only if $E^3 - J$ is* 1-ULC *for homologically trivial loops* (i.e., *for each $\epsilon > 0$, there is a $\delta > 0$ such that any δ-loop in $E^3 - J$ which bounds in $E^3 - J$ is contractible to a point in an ϵ-subset of $E^3 - J$*).

Proof. Suppose that $E^3 - J$ is 1-ALG at each point $p \in J$. By [**6**, Corollary X.4.8], for any simple closed curve J in E^3, $E^3 - J$ is 1-ulc for homologically trivial cycles. Thus, for each sufficiently small open set U containing $p \in J$, there is an open set V' with $p \in V' \subset U$ such that each loop in $V' - J$ which bounds in $E^3 - J$ also bounds in $U - J$. As $E^3 - J$ is 1-ALG at $p \in J$, there is an open set $V \subset V'$ with $p \in V \subset U$ such that any loop in $V - J$ which bounds in $U - J$ is contractible to a point in $U - J$. Therefore, if l is a loop in $V - J$ which bounds in $E^3 - J$, then l bounds in $U - J$ and hence is contractible to a point in $U - J$.

The converse is obvious.

As a consequence of Theorem 1, we need only check to see whether a small loop links J in order to know if it can be shrunk to a point missing J – we do not have to show that it bounds in some preassigned open subset of $E^3 - J$.

Note that for every $\epsilon > 0$, there is a $\delta > 0$ such that if J_1 and J_2 are two simple closed curves in $E^3 - J$, each with unsigned linking number 1 with J,

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and diameter $(J_1 \cup J_2) < \delta$, then $J_1 \cup J_2$ bounds a singular annulus in $E^3 - J$ of diameter less than ϵ.

1. Canonical neighbourhoods. The results of this section hold for any simple closed curve J in an orientable 3-manifold.

Let N be a cube-with-handles which contains n disjoint polyhedral properly embedded disks $D_1, D_2, \ldots, D_n$ whose union separates N into n cubes-with-handles $N_1, N_2, \ldots, N_n$ such that $N_i \cap N_{i+1} = D_i$, and $N_i \cap N_j = \emptyset$ if $i \neq j - 1, j, j + 1$ (where the subscripts are taken mod n). The disks $D_1, \ldots, D_n$ are called *sectioning* disks of N, and $N_1, \ldots, N_n$ are called *sections*. Then N is said to be a *canonical neighbourhood* of J if

(1) $J \subset \operatorname{Int} N$;

(2) for each sectioning disk D_i of N, $D_i \cap J$ is contained in a subarc of J which intersects no other sectioning disk of N;

(3) J is homotopic in $\operatorname{Int} N$ to a polyhedral simple closed curve which pierces each sectioning disk D_i exactly once.

We say that N is a *canonical ϵ-neighbourhood* of J if for each i, diam $N_i < \epsilon$, and is a *solid torus canonical neighbourhood* of J if each section is a 3-cell.

A *chain of sections* of a canonical neighbourhood N is a collection $N_i, N_{i+1}, \ldots, N_{i+k}$ of sections (where the subscripts are mod n, the total number of sections of N) of N.

LEMMA 2. *For any $\epsilon > 0$, J has a canonical ϵ-neighbourhood.*

Proof. The neighbourhood constructed in Theorem 1 of [**1**] has all the required properties except for (2). However, if we take a neighbourhood of [**1**] whose sections have diameter less than $\epsilon/3$, and delete at least every other sectioning disk, then the resulting neighbourhood will have property (2).

LEMMA 3. *Let N be a canonical neighbourhood of a simple closed curve J. For any $\epsilon > 0$, there is a canonical ϵ-neighbourhood N' of J in* Int N *and an ϵ-homeomorphism h of N onto itself such that*

(1) *for any sectioning disk D_i of N, each component of $h(D_i) \cap N'$ is contained in one section of N';*

(2) *h is the identity on ∂N and outside of an ϵ-neighbourhood of $J \cap (\bigcup D_i)$;*

(3) *for each sectioning disk D_i of N, there is a chain η_i of sections of N' so that $h(D_i) \cap N'$ is contained in η_i, and η_i intersects the image under h of no other sectioning disk of N. Furthermore, $\eta_i \cap \eta_j = \emptyset$ if $i \neq j$.*

2. Solid torus neighbourhoods. Fix a canonical neighbourhood N^0 of J, and on this neighbourhood fix a meridian m_0 which is the boundary of a sectioning disk of N^0. If l is an oriented simple closed curve in $\operatorname{Int}(N^0) - J$, we will speak of $\operatorname{lk}(l, m_0)$ and $\operatorname{lk}(l, J)$, the linking numbers of l with respect to m_0 and J respectively. Note that if N is a second canonical neighbourhood of J in N^0 with $l \subset \operatorname{Int} N$, and m is the boundary of a sectioning disk of N, then $\operatorname{lk}(l, m) = \pm \operatorname{lk}(l, m_0)$.

LEMMA 4. *For every open set U with $J \subset U \subset N^0$, there is an open set V with $J \subset V \subset U$ such that if l is a simple closed curve in $V - J$ with* $\mathrm{lk}(l, J) = 0$ *and* $\mathrm{lk}(l, m_0) = 0$, *then l is homotopic to zero in $U - J$.*

Proof. Choose an $\epsilon > 0$ and a canonical ϵ-neighbourhood V such that any nonlinking ϵ simple closed curve in $V - J$ can be shrunk to a point in $U - J$. It is sufficient to consider polygonal simple closed curves l in

$$(\mathrm{Int}\, V) - J \text{ with } \mathrm{lk}(l, m_0) = \mathrm{lk}(l, J) = 0,$$

and with l in general position with respect to the sectioning disks of V. If p, q are points of l at which l pierces some sectioning disk D of V in opposite directions, then p, q can be joined by an arc α in $D - J$. If α_1, α_2 are the two arcs of $l - \{p, q\}$ then l is homotopic to the sum of the two simple closed curves $\alpha_1 \cup \alpha$ and $\alpha_2 \cup \alpha$, where $\mathrm{lk}(\alpha_1 \cup \alpha, m_0) = \mathrm{lk}(\alpha_2 \cup \alpha, m_0) = 0$. By proper choice of α, we will have, in addition, that $\mathrm{lk}(\alpha_2 \cup \alpha, J) = \mathrm{lk}(\alpha_1 \cup \alpha, J) = 0$. Pushing $\alpha_1 \cup \alpha$ and $\alpha_2 \cup \alpha$ off D, l is replaced by a collection of simple closed curves having the additional property of intersecting the union of the sectioning disks of V two fewer times. After a finite number of steps this procedure yields a collection of simple closed curves each of which lies in a section of V and whose sum is homotopic to l in $V - J$. As each of these bounds a singular disk in $U - J$, $l \simeq 0$ in $U - J$.

LEMMA 5. *Let U be an open subset of S^3 with $J \subset U$ and $U \cap m_0 = \emptyset$. Then the inclusion induces an epimorphism of $H_1(U - J)$ onto*

$$H_1(S^3 - J - m_0) = Z \oplus Z.$$

Proof. The inclusion of the excisive couple $(U, S^3 - J)$ of subsets of S^3 into the excisive couple $(S^3 - m_0, S^3 - J)$ of subsets of S^3 induces a map from the Mayer-Vietoris sequence of $(U, S^3 - J)$ to the Mayer-Vietoris sequence of $(S^3 - m_0, S^3 - J)$ yielding the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & H_1(U - J) & \xrightarrow{\approx} & H_1(U) & \oplus H_1(S^3 - J) & \longrightarrow 0 \\
& \downarrow i & & & \downarrow j & \\
0 \longrightarrow & H_1(S_3 - J - m_0) & \xrightarrow{\approx} & H_1(S^3 - m_0) & \oplus H_1(S^3 - J) & \longrightarrow 0.
\end{array}$$

The map j is the sum of the maps

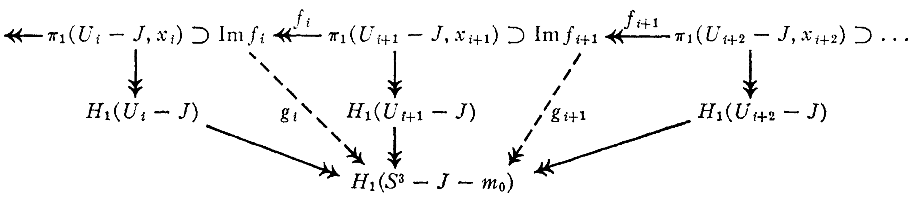
$$H_1(U) \rightarrow H_1(S^3 - m_0) \quad \text{and} \quad H_1(S^3 - J) \rightarrow H_1(S^3 - J),$$

both induced by inclusion. Clearly the second of these maps is onto. The first of these is also onto, because J, having linking number of 1 with respect to m_0, is a generator of $H_1(S^3 - m_0)$ and lies in U. Thus j is an epimorphism, and it follows from the diagram that i is an epimorphism of $H_1(U - J)$ onto

$H_1(S^3 - J - m_0)$. Moreover, $H_1(S^3 - J - m_0) = Z \oplus Z$ because it is isomorphic to $H_1(S^3 - m_0) \oplus H_1(S^3 - J)$.

LEMMA 6. *$S^3 - J$ has stable end ϵ with $\pi_1(\epsilon) = Z \oplus Z$* (for definitions, see [3]).

Proof. Choose a sequence $U_1, U_2, \ldots$ of connected neighbourhoods of J lying in $S^3 - m_0$ with U_{i+1} lying in the open set V given by Lemma 4 for $U = U_i$ and with $J = \bigcap U_i$. Choose a point $x_i \in U_i - J$ and a path α_i in $U_i - J$ from x_i to x_{i+1}. Define $f_i : \pi_1(U_{i+1} - J, x_{i+1}) \to \pi_1(U_i - J, x_i)$ to be the inclusion followed by the homomorphism induced by α_i. Consider the following commutative diagram



where each of the (inclusion) maps to $H_1(S^3 - J - m_0)$ is onto by Lemma 5, each of the maps $\pi_1(U_j - J, x_j) \to H_1(U_j - J)$ is onto and each

$$\pi_1(U_{j+1} - J, x_j) \to \operatorname{Im} f_j = \operatorname{Image} f_j$$

is onto. Thus each of the maps g_i (dotted arrows) which is the composition of the maps

$$\operatorname{Im} f_j \subset \pi_1(U_j - J, x_j) \to H_1(U_j - J) \to H_1(S^3 - J - m_0)$$

is onto. To show each g_i is an isomorphism choose $x \in \operatorname{Im} f_i$ in the kernel of g_i. There is a loop l in $\pi_1(U_{i+1} - J, x_{i+1})$ such that $f_i(l) = x$. As $g_i f_i(l) = 0$, l is homologous to zero in $S^3 - J - m_0$ so that $\operatorname{lk}(l, J) = \operatorname{lk}(l, m_0) = 0$. By Lemma 4, $l \simeq 0$ in $\pi_1(U_i - J, x_i)$. It follows that $x = f_i(l) \simeq 0$ in $\operatorname{Im} f_i$. Thus each g_i is an isomorphism of $\operatorname{Im} f_i$ onto $Z \oplus Z$, whence

$$f_i : \operatorname{Im} f_{i+1} \to \operatorname{Im} f_i$$

is also an isomorphism.

We have shown that the sequence

$$\pi_1(U_1 - J, x_1) \xleftarrow{f_1} \pi_1(U_2 - J, x_2) \xleftarrow{f_2} \ldots$$

induces isomorphisms on the sequence

$$\operatorname{Im} f_1 \xleftarrow{f_1} \operatorname{Im} f_2 \xleftarrow{f_2} \ldots$$

so that ϵ, the end of $S^3 - J$ is stable and

$$\pi_1(\epsilon) = \varprojlim\{\pi_1(U_i - J, x_i), f_i\} = \varprojlim\{\operatorname{Im} f_i, f_i\}$$

$$= H_1(S^3 - J - m_0) = Z \oplus Z.$$

We state the following easy to prove lemma without proof.

LEMMA 7. *Let O and O' be the complementary domains of a polyhedral torus in S^3 and suppose that O' contains an unknotted simple closed curve which is not homologous to zero in O'. Then* $\mathrm{Cl}(O)$ *is a solid torus.*

THEOREM 8. *J is definable by solid tori.*

Proof. It is clear from Lemma 6 that $S^3 - J$ satisfies the hypotheses of Theorem 1 of [**3**]. Thus there is a 2-manifold $S \subset S^3$ and a neighbourhood O of J such that $O - J \approx S \times [0, \infty)$. As $\pi_1(\epsilon) = Z \oplus Z$, where ϵ is the end of $S^3 - J$, S must be a torus. Define $O_t = \{S \times [t, \infty)\} \cup J$. As we may assume that $m_0 \subset S^3 - O$, the previous lemma tells us that O_t is a solid torus for each t. Because $J = \bigcap \{O_t : t = 1, 2, \ldots\}$, J is definable by solid tori.

Because we have not used the full strength of the 1-ALG condition, we have proved the following theorem:

THEOREM 9. *Let J be a simple closed curve in an orientable* 3*-manifold and let J satisfy the following condition: For every sufficiently small open set U with $J \subset U$, there is an open set V with $J \subset V \subset U$ such that any loop in $V - J$ which is homologous to zero in $U - J$ is also homotopic to zero in $U - J$. Then J has arbitrarily close neighbourhoods whose closures are solid tori.*

Remark. If a simple closed curve on the boundary of one of the solid tori of Theorem 8 is homologous to zero in $S^3 - m_0 - J$, then it bounds a disk on the boundary of the solid torus.

3. Cutting off feelers and foldbacks.

LEMMA 10. *Let $\epsilon > 0$. Then there is a canonical ϵ-neighbourhood N of J, and a solid torus neighbourhood T of J, with $T \subset$* Int *N, so that, if D is a sectioning disk of N, $\partial T \cap D$ is a finite collection of simple closed curves each of which link J (and hence are meridional on T).*

Remark. This lemma says that we can "cut the feelers" off T.

Proof. Let N be a canonical $(\epsilon/8)$-neighbourhood of J. We can suppose that the number of sections of N is divisible by 4 and that the sectioning disks of N intersect J in a 0-dimensional set. Using Theorems 1 and 8, we find a solid torus neighbourhood T of J with ∂T in general position with respect to the sectioning disks of N and with T so close to J that, for any sectioning disk D of N, and for any simple closed curve l of $\partial T \cap D$ which bounds a disk on ∂T, l bounds a singular disk in $(\text{Int } N) - J$ which intersects no other sectioning disk of N.

We now fix a sectioning disk D_i. From this point on, we consider our subscripts on sectioning disks to be mod n, where n is the number of sections of N. There are pairwise disjoint disks $E_1, E_2, \ldots, E_m$ in ∂T with $\partial E_j \subset D_i$ so any simple closed curve of $\partial T \cap D_i$ which does not link J lies in some E_j. Let E_j' be the closure of the component of $E_j - D_{i-1} - D_{i+1}$ which contains ∂E_j. Then E_j' is a punctured disk, and we can fill in the holes of E_j' with singular disks which do not hit D_i, J, and the remaining D_k's. Thus we obtain a singular Dehn disk with the same boundary as E_j and which lies in four sections of N. We apply Dehn's lemma to obtain nonsingular disks $E_1'', \ldots, E_m''$ with the same properties. Using a disk trading argument, we can assume that these disks are pairwise disjoint.

By a general position argument, we can assume that $\partial E_j'' \subset \partial T$ while $\text{Int } E_j'' \cap \partial T$ is a finite collection of simple closed curves. Each of these simple closed curves bounds a disk on ∂T. Then, using a disk-trading argument, we can cut ∂T off $\bigcup \text{Int } E_j''$. Then, if $\partial E_j''$ still lies on ∂T, we replace the disk it bounds on ∂T with E_j''. We now have that each simple closed curve of $\partial T \cap D_i$ which bounds a disk on ∂T, bounds a disk on ∂T which lies in four sections of N. Now, we use another disk-trading argument to cut D_i off ∂T to obtain a new sectioning disk D_i' which intersects ∂T only in curves that link J. Then D_i' lies in four sections of N.

Let D_j be any sectioning disk of N except D_{i-1}, D_i, or D_{i+1}. In our modifications of T, we may have changed $\partial T \cap D_j$. However, with the new T, $\partial T \cap D_j$ will be a subset of what it was with the old T. Thus, we still have that for any simple closed curve l of $\partial T \cap D_j$ which bounds a disk on ∂T, l bounds a singular disk in $(\text{Int } N) - J$ which intersects no other sectioning disk of N.

We now go to the sectioning disk D_{i+4} and repeat the above process to get a disk D_{i+4}' and a new solid torus, still called T. In this way we can find a new sequence of sectioning disks $D_i', D_{i+4}', D_{i+8}', \ldots, D_{i-4}'$ of N, so that N, with the new sectioning disks and sections, has the required properties.

LEMMA 11. *Let $\epsilon > 0$. Then there is a canonical ϵ-neighbourhood N of J, and a solid torus neighbourhood T of J, with $T \subset \text{Int } N$, so that, for any sectioning disk D of N, any two simple closed curves of $\partial T \cap D$ bound an annulus on ∂T which links J and which intersects no other sectioning disk of N.*

Remark. This theorem cuts the long foldbacks off ∂T.

Proof. Let N be a canonical $(\epsilon/8)$-neighbourhood of J. Let η be less than the distance from J to ∂N and less than the minimum distance between the sectioning disks of N. Let δ be chosen for $\eta/4$ using the 1-ULC condition for homologically trivial loops as specified in Theorem 1. Let N' be a canonical δ-neighbourhood of J and let T be a solid torus neighbourhood of J in $\text{Int } N'$

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so that, for each sectioning disk D' of N', each component of $\partial T \cap D'$ is a simple closed curve which links J. Using Lemma 3, after a δ-adjustment of the sectioning disks of N, we can assume that N' intersects the sectioning disks of N as specified in Lemma 3. By a disk-trading argument similar to that done in the proof of Lemma 10, we can also assume that for each sectioning disk D of N, D has been adjusted so that $\partial T \cap D$ consists of simple closed curves which link J. The sectioning disks of N now lie homeomorphically within 2δ of where they originally lay. Since $\delta < \eta/4$, the minimum distance between the sectioning disks is still greater than $\eta/2$.

We now have the condition on T which we will use in the remainder of the proof; namely, for any sectioning disk D of N, any two simple closed curves of $\partial T \cap D$ which lie in one section of N' bound a singular annulus missing J which lies in the two adjacent sections of N. (See the remark at the end of Section 0.)

Without loss of generality we can assume that the number of sections of N is divisible by four. We now fix a sectioning disk D_i of N. We can consider ∂T as the union of two annuli, C and A, so that $\partial A = \partial C \subset D_i$ and $C \cap D_i = \partial C$. Furthermore, C and A can be chosen so that any simple closed curve consisting of two arcs, one in C spanning between the boundary components of C, and one in D_i, must link m_0. (For the definition of m_0 see the beginning of Section 2.) The corresponding simple closed curve in $A \cup D_i$ would not link m_0. Let N_j' be a section of N' so that D_{i-1} separates the end sectioning disks of N_j', and let N_k' be a section of N' so that D_{i+1} separates the end sectioning disks of N_k'. We wish to replace A by an annulus which lies in four sections of N. If A does not satisfy this condition, then let A_1^* and A_2^* be the disjoint minimal subannuli of A with $\partial A \subset \partial A_1^* \cup \partial A_2^*$ and with

$$\partial A_j^* - \partial A \subset (N_j' \cap D_{i-1}) \cup (N_k' \cap D_{i+1}), \quad j = 1, 2.$$

Then $A_1^* \cup A_2^*$ must be contained in the chain of sections of N' from N_j' to N_k' which lies in the chain of four sections of N around D_i. Thus, $A_1^* \cup A_2^*$ also lies in this chain of four sections of N.

Case 1. A_1^* and A_2^* both have a boundary component in $D_{i+1} \cap N_j'$: in this case, there must be a singular annulus missing J joining the two boundary components of $A_1^* \cup A_2^*$ which lie in D_{i+1}. This singular annulus can be chosen to miss D_i and D_{i+2}. Piecing together this singular annulus with A_1^* and A_2^*, we obtain a singular annulus missing J, D_{i-1}, and D_{i+2}, with the same boundary as A, and with no singularities in a neighbourhood of the boundary. Using Dehn's lemma as stated in Theorem 1.1 of [**5**] we can find either: (1) a nonsingular annulus A' lying in four sections of N, missing J, and with $\partial A' = \partial A$; or (2) a nonsingular disk missing J whose boundary is contained in ∂A. However, (2) is impossible since each component of ∂A links J.

Case 2. A_1^* and A_2^* both have one boundary component lying in D_{i-1}: this is similar to Case 1.

Case 3. A_1^* has a boundary component in D_{i-1} and A_2^* has a boundary component in D_{i+1} (or vice versa): in this case we can find a subannulus A_3^* in $A - A_1^* - A_2^*$ with one boundary component in $D_{i-1} \cap N_j'$ and one boundary component in $D_{i+1} \cap N_k'$, and lying in four sections of N. We can then join the boundary components of A_1^* and A_3^* which lie in D_{i-1} with a singular annulus missing J, D_{i-2}, and D_i. Similarly, we can join the boundary components of A_2^* and A_3^* which lie in D_{i+1} with a singular annulus missing J, D_i, and D_{i+2}. Piecing together these two singular annuli with A_1^*, A_3^* and A_2^*, we get a singular annulus lying in four sections of N, with the same boundary as A, and with no singularities in some neighbourhood of the boundary. By applying Dehn's lemma, we can replace this singular annulus with a nonsingular annulus A' missing J, and with $\partial A = \partial A'$.

In all three cases we have constructed a nonsingular annulus A' so that

$$\partial A' = \partial A \subset D_i \quad \text{and} \quad A' \cap (D_{i-2} \cup D_{i+2}) = \emptyset.$$

Using general position, we can assume that each component of $(\text{Int } A') \cap (\text{Int } C')$ is a simple closed curve. If one of these simple closed curves bounds a disk on A', we can find an innermost such simple closed curve on A'. We replace the disk this simple closed curve bounds on C with the disk it bounds on A' and then push the disk off A'. In this way, we can assume that each simple closed curve of $A' \cap C$ links J and is nontrivial on both A' and C.

Choose an arc α which spans from one boundary component of C to the other and intersects each simple closed curve of $C \cap A'$ once. By our choice of C, α crosses each sectioning disk of N except D_i algebraically once. We can choose a subannulus C' of C so that $C' \cap A' = \partial C'$ and so that the subarc of α which spans C' intersects each sectioning disk of N except possibly for D_{i-1}, D_i and D_{i+1} algebraically once. Then $\partial C'$ bounds a subannulus A'' of A'. Together, C' and A' make up a torus which we claim bounds a solid torus which contains J. To prove this claim, we consider $C' \cap D_{i+2}$. By our construction of C and C', we have that

$$C' \cap D_{i+2} \subset C \cap D_{i+2} \subset \partial T \cap D_{i+2}.$$

Hence, each component of $C' \cap D_{i+2}$ is a simple closed curve which links J. We choose a component of $C' \cap D_{i+2}$ which is innermost on D_{i+2}; this is a simple closed curve on the torus $C' \cup A''$ which links J and which bounds a disk whose interior misses $C' \cap A''$. Thus, $C' \cup A''$ bounds a solid torus which we will now call T. Since $\partial T \cap D_i \subset A'' \subset A'$, any two simple closed curves of $\partial T \cap D_i$ bound an annulus which links J and which is contained in four sections of N.

We now repeat this process using D_{i+4} in place of D_i. After modifying T for every fourth sectioning disk of N, we delete all but every fourth sectioning disk of N and combine sections.

THEOREM 12. *For any $\epsilon > 0$, J has a solid torus canonical ϵ-neighbourhood.*

Proof. Let N and T be the neighbourhoods of J as described in Lemma 11 for $\epsilon/3$. For each sectioning disk D_i of N, choose a simple closed curve of $\partial T \cap D_i$ which is innermost on D_i. Since this simple closed curve links J, the disk D_i' which it bounds in $\text{Int}\, D_i$ must be a meridional disk for T. Then we let $D_1', D_2', \ldots, D_n'$ be sectioning disks for T. These sectioning disks divide T into sections, each with diameter less than ϵ.

4. Constructing a piercing disk.

LEMMA 13. *For any $\epsilon > 0$, there is a solid torus canonical ϵ-neighbourhood T of J with the following property:*

If D_i is a sectioning disk of T and J_1, J_2 are two simple closed curves in $D_i - J$, each of which has linking number 1 with J, then $J_1 \cup J_2$ bounds a singular annulus in $S^3 - J$ which does not intersect any section of T except T_i and T_{i+1}.

Proof. Let N be a canonical $(\epsilon/8)$-neighbourhood of J, and suppose that the number of sections of N is divisible by 4. Let η be less than the minimum distance between any two non-adjacent sections of N. Using the 1-ULC condition for homologically trivial loops as defined in Theorem 1, pick a $\delta > 0$ so that any loop of diameter less than δ which does not link J bounds a singular disk missing J of diameter less than $\eta/2$. Using Lemma 3 and Theorem 12, we can find a solid torus canonical $\delta/2$-neighbourhood T of J and a $\delta/2$-homeomorphism which adjusts the sectioning disks of N so that T lies in N as specified in Lemma 3 with N' replaced by T. Then the minimum distance between non-adjacent sections of N is still greater than $\eta/2$ after the sectioning disks were adjusted.

In every fourth section of N, choose one sectioning disk of T, and then delete all the remaining sectioning disks of T and combine sections accordingly. Then any section of T lies in six sections of N, and T is a solid torus canonical ϵ-neighbourhood of J. Let D_i' be a sectioning disk of T, and let J_1 and J_2 be simple closed curves in $D_i' - J$ each of which has linking number one with J. Then J_1 and J_2 bound a singular annulus of diameter less than $\eta/2$ missing J. This singular annulus must then intersect at most the section of N containing D_i' plus the two adjacent sections of N. Thus, the singular annulus can only intersect the sections of T adjacent to D_i'.

LEMMA 14. *Let $\epsilon > 0$. Then there is a solid torus canonical ϵ-neighbourhood T of J and a $\delta > 0$ so that if T' is any solid torus canonical δ-neighbourhood of J, and if D_i is a sectioning disk of T and l is a simple closed curve of $D_i \cap \partial T'$, then ∂D_i and l bound an annulus A in $T - \text{Int}\, T'$ such that*

$$\text{Int}\, A \subset (\text{Int}\, T) - T'$$

and A lies in a chain of four sections of T.

Proof. Let T be a solid torus canonical ϵ-neighbourhood of J constructed as in Lemma 13. Since T is a canonical neighbourhood of J, for each sectioning disk D_j of T, $D_j \cap J$ is contained in a subarc of J which intersects no other sectioning disk of T. We choose δ so small that if T' is a solid torus canonical δ-neighbourhood of J and D_j is a sectioning disk of T, then $T' \cap D_j$ is contained in a chain of sections of T' which intersects no other sectioning disk of T.

We fix a solid torus canonical δ-neighbourhood T', a sectioning disk D_i of T, and a simple closed curve l of $D_i \cap \partial T'$ which links J. Let l^* be a simple closed curve of $D_{i+1} \cap \partial T'$ which links J. Then ∂D_{i+1} and l^* bound a singular annulus which intersects T only in the sections of T adjacent to D_{i+1}. Hence this singular annulus misses D_i. We can now piece together an annulus on ∂T from ∂D_i to ∂D_{i+1}, the singular annulus just constructed, and an annulus on $\partial T'$ from l to l^* to obtain a singular annulus contained in the union of a chain of 3-sections of T with no singularities in a neighbourhood of its boundary. We apply Dehn's lemma to this annulus to obtain a nonsingular annulus A_0 with the same properties. We suppose that $\text{Int}(A_0)$ is in general position with respect to ∂T and $\partial T'$, and thus that $\text{Int}(A_0) \cap (\partial T \cup \partial T')$ is a finite collection of simple closed curves. By a disk-trading argument we can suppose that none of these simple closed curves bounds a disk on A_0, ∂T or $\partial T'$. We can then find a subannulus A_0' of A_0 which spans from ∂T to $\partial T'$ with $\text{Int}\, A_0' \subset (\text{Int}\, T) - T'$. Note that either $\partial D_i \subset \partial A_0'$ or $\partial D_i \cap \partial A_0' = \emptyset$. We then piece together a subannulus of ∂T from ∂D_i to $A_0' \cap \partial T$ (if necessary), A_0', and a subannulus of $\partial T'$ from l to $A_0' \cap \partial T'$ to obtain an annulus bounded by ∂D_i and l which lies in $T - \text{Int}\, T'$. We push the interior of this annulus off $\partial T \cup \partial T'$ to form the annulus A.

THEOREM 15. *At each point $p \in J$, there is a disk D so that J pierces D at p. Hence, J is tame.*

Proof. Let $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ be a sequence of positive numbers with $\epsilon_i < 1/i$. Using Lemma 4, we can construct a sequence of solid torus canonical ϵ_i-neighbourhoods $T^1, T^2, T^3, \ldots$ so that T^{i+1} lies in T^i as specified by Lemma 14. For each i, let D^i be a sectioning disk of T^i which lies in a section of T^i which contains p. Using Lemma 14, we can construct an $8\epsilon_i$-annulus A^i from ∂D^i to ∂D^{i+1} in $T^i - \text{Int}\, T^{i+1}$. Then $D = \bigcup A_i \cup \{p\}$ is the required disk. Theorem 1 of [**4**] then shows that J is tame.

Remark. At this point it would not be difficult to complete an elementary proof that J is tame which would not require reference to McMillan's paper [**4**]. We have all the necessary elements to construct a 'regular' neighbourhood of J.

Cannon [**2**] now has a proof of the corresponding theorem for graphs.

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