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# TREES AS COMMUTATIVE BCK-ALGEBRAS

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A new method of constructing commutative BCK-algebras is given. It depends upon the notion of a valuation of a lower semilattice in a given commutative BCK-algebra. Any tree with the descending chain condition has a valuation in the natural numbers, considered as a commutative BCK-algebra; the valuation is the height-function. Thus, any tree of finite height possesses a uniquely determined commutative BCK-structure. The finite trees with at most one atom and height at most $n$ are precisely the finitely generated subdirectly irreducible (simple) algebras in the subvariety of commutative BCK-algebras which satisfy the identity $(E_n)$ : $xy^n = xy^{n+1}$ . Due to congruence-distributivity, it is then possible to describe the associated lattice of subvarieties.

## Introduction

The concept of a lower semilattice with a valuation in a commutative BCK-algebra is introduced, and it is shown that such a semilattice can be converted into a commutative BCK-algebra. Any tree, which satisfies the descending chain condition, provides an example; the valuation is the height-function. Thus, any tree of finite height possesses a uniquely determined commutative-BCK-algebra-structure. It is then possible to

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completely describe the lattice of subvarieties of the variety of commutative BCK-algebras satisfying the identity $xy^n = xy^{n+1}$ .

## 1. Valuations

Because of Yutani [15], a commutative BCK-algebra can be considered as a groupoid with a nullary operation $0$ , which satisfies the identities: $xx = 0$ , $x0 = x$ , $x(xy) = y(yx)$ , $(xy)z = (xz)y$ . We will presume a familiarity with BCK-algebras and especially commutative BCK-algebras; good references are supplied by Iséki and Tanaka [9] and Traczyk [14], but see also [12], [3], [4] and [5].

Let $(A; \wedge, 0)$ be a lower semilattice with smallest element $0$ , and $(C; 0)$ be a commutative BCK-algebra. Then, the semilattice $A$ is said to have a *valuation*, $v$ , in the commutative BCK-algebra $C$ if $v$ is a function mapping $A$ into $C$ such that

(V1) $v(a \wedge b) = v(a) \wedge v(b)$ *for any* $a, b \in A$ *, which possess a common upper bound;*

(V2) *for any* $a \in A$ *, the restriction* $v_a$ *of* $v$ *to the interval* $[0, a]$ *possesses an inverse*

$$v_a^{-1} : [0, v(a)] \to [0, a] ;$$

(V3) *for any* $a, b \in A$ *, with* $a \leq b$ *, and* $x \in [0, v(a)]$ $v_a^{-1}(x) = v_b^{-1}(x)$ .

Any commutative BCK-algebra is a lower semilattice, wherein the infimum is given as the derived operation $x \wedge y = x(xy) = y(yx)$ . Thus, (V1)-(V3) make sense. Also, each interval $[0, x]$ in a commutative BCK-algebra is a distributive lattice; *cf.* [3, Section 3], [14]. Due to (V1) and (V2), $v_a$ and $v_a^{-1}$ are then mutually inverse lattice-isomorphisms. Also $v$ is isotone and $v(0) = 0$ .

**THEOREM 1.1.** *Let* $(A; \wedge, 0)$ *be a lower semilattice which possesses a valuation* $v$ *in a commutative* BCK-*algebra* $(C; 0)$ . *Define a binary operation on* $A$ *by*

$$ab = v_a^{-1}\big(v(a)v(a \wedge b)\big) .$$

*With respect to this operation,* $A$ *is a commutative* BCK-*algebra and the original semilattice infimum is given by* $a \wedge b = a(ab) = b(ba)$ *. Moreover, for each* $a \in A$ *,* $v_a$ *and* $v_a^{-1}$ *are mutually inverse* BCK-*isomorphisms between the* BCK-*subalgebras* $([0, a]; 0)$ *and* $\big([0, v(a)]; 0\big)$ .

Proof. We must show that Yutani's identities hold. However, before doing this, we should note that (V1) and (V2) imply that $v(ab) = v(a)v(a \wedge b)$ and $ab \le a$ for any $a, b \in A$ .

As $v_a^{-1}(0) = 0$ ,

$$aa = v_a^{-1}\big(v(a)v(a \wedge a)\big) = v_a^{-1}\big(v(a)v(a)\big) = v_a^{-1}(0) = 0 .$$

As $v(0) = 0$ ,

$$a0 = v_a^{-1}\big(v(a)v(a \wedge 0)\big) = v_a^{-1}\big(v(a)0\big) = v_a^{-1}\big(v(a)\big) = a .$$

Due to (V3), $v_a^{-1}\big(v(a \wedge b)\big) = v_{a\wedge b}^{-1}\big(v(a \wedge b)\big) = a \wedge b$ . Hence,

$$\begin{aligned} a(ab) &= v_a^{-1}\big(v(a)v\big(a \wedge (ab)\big)\big) = v_a^{-1}\big(v(a)v(ab)\big) = v_a^{-1}\big(v(a)\big(v(a)v(a \wedge b)\big)\big) \\ &= v_a^{-1}\big(v(a) \wedge v(a \wedge b)\big) = v_a^{-1}\big(v(a \wedge b)\big) = v_{a\wedge b}^{-1}\big(v(a \wedge b)\big) = a \wedge b . \end{aligned}$$

As $a \wedge b = b \wedge a$ , $a(ab) = b(ba)$ .

Because $ab \le a$ , (V1) implies that

$$v\big((ab) \wedge c\big) = v\big((ab) \wedge (a \wedge c)\big) = v(ab) \wedge v(a \wedge c) .$$

Hence,

$$\begin{aligned} (ab)c &= v_{ab}\big(v(ab)v\big((ab) \wedge c\big)\big) = v_{ab}\big(v(ab)\big(v(ab) \wedge v(a \wedge c)\big)\big) \\ &= v_{ab}\big(v(ab)v(a \wedge c)\big) = v_{ab}\big(\big(v(a)v(a \wedge b)\big)v(a \wedge c)\big) \\ &= v_a\big(\big(v(a)v(a \wedge b)\big)v(a \wedge c)\big) = v_a\big(\big(v(a)v(a \wedge c)\big)v(a \wedge b)\big) . \end{aligned}$$

Because of the symmetric roles of $b$ and $c$ , we conclude that $(ab)c = (ac)b$ . Thus $A$ is a commutative BCK-algebra.

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Finally suppose $b, c \in [0, a]$ . Due to (V1) and (V3),

$$bc = v_b^{-1}\big(v(b)v(b \wedge c)\big) = v_b^{-1}\big(v(b)\big(v(b) \wedge v(c)\big)\big)$$

$$= v_b^{-1}\big(v(b)v(c)\big) = v_a^{-1}\big(v_a(b)v_a(c)\big) .$$

That is, $v_a(bc) = v_a(b)v_a(c)$ , and so $v_a : [0, a] \to [0, v(a)]$ is a BCK-isomorphism.

When $(A; 0)$ is a commutative BCK-algebra and $(A; \wedge, 0)$ is its lower semilattice reduct, the identity function on $A$ provides a valuation of $(A; \wedge, 0)$ in the BCK-algebra $(A; 0)$ . We now give less trivial examples.

EXAMPLE 1.2. Consider the unit interval $[0, 1]$ of the real numbers as a commutative BCK-algebra, wherein $xy = \max(x-y, 0) = x - \min(x, y)$ . Let $(A; \wedge, 0)$ be the tree with two distinct maximal chains $\{a(x) : x \in [0, 1]\}$ , $\{b(x) : x \in [0, 1]\}$ , each of which is order-isomorphic to $[0, 1]$ , and such that $a(y) = b(y)$ , when $y \in [0, \frac{1}{2}]$ , while $a(z) \wedge b(w) = a(\frac{1}{2}) = b(\frac{1}{2})$ for all $z, w \in (\frac{1}{2}, 1]$ . Then $v : A \to [0, 1]$ , defined by $v\big(a(x)\big) = v\big(b(x)\big) = x$ for all $x \in [0, 1]$ , is a valuation.

EXAMPLE 1.3. Let $C$ be a commutative BCK-algebra and for each $i$ in an index set $I$ with at least two elements, let $C_i$ be a copy of the underlying semilattice of $C$ . Form the semilattice $(A; \wedge, 0)$ where $A = \bigcup\{C_i : i \in I\}$ and $C_i \cap C_j = \{0\}$ if $i \neq j$ . Each $C_i$ is order-isomorphic to $C$ under $v_i$ , say, and $a$ and $b$ are incomparable when $a \in C_i$ , $b \in C_j$ and $i \neq j$ . Then $v : A \to C$ , given by $v(a) = v_i(a)$ if $a \in C_i$ , is a valuation. When $C$ is taken as the 2-element BCK-chain, the resulting BCK-algebra is the one given in Example 3 of Iseki and Tanaka [8]. When $C$ is the BCK-algebra which is the set of natural numbers $N = \{0, 1, 2, \ldots\}$ with BCK-product $ab = \max(a-b, 0)$ , the resulting BCK-algebra is the one constructed in Example 4 of Iséki and Tanaka [8].

By a *tree*, we mean a lower semilattice $(A; \wedge, 0)$ with a smallest element $0$ , in which any two elements have a common upper bound only if

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they are comparable or equivalently, each initial interval $[0, a]$ is a chain. When a tree $(A; \wedge, 0)$ satisfies the descending chain condition, each element $a \in A$ has finite height $h(a)$ ; $h(a)$ is the length of the chain $[0, a]$ . A tree has *finite height* equal to $n$ , if $n$ is the maximum of the lengths of its subchains.

Let $(N; 0)$ be the commutative BCK-algebra, wherein $N = \{0, 1, 2, \ldots\}$ is the set of natural numbers and the BCK-product on $N$ is given by $xy = \max(x-y, 0) = x - \min(x, y)$ , for each $x, y \in N$ . We are now ready to give the most important instance of Theorem 1.1; we formulate it as a theorem.

**THEOREM 1.4.** *Let* $(A; \wedge, 0)$ *be a tree with the descending chain condition and let* $v : A \to N$ *be given by* $v(a) = h(a)$ *for each* $a \in A$ . *Then* $v$ *is a valuation of the tree* $(A; \wedge, 0)$ *in the commutative* BCK-*algebra* $(N; 0)$ . *Thus the tree* $A$ *becomes a commutative* BCK-*algebra, wherein the* BCK-*product* $ab$ *of* $a, b \in A$ *is the unique element of height* $h(a) - h(a \wedge b)$ *in the interval* $[0, a]$ . *What is more, this is the only product which is definable on* $A$ *so that the resulting structure is a commutative* BCK-*algebra, whose lower semilattice reduct coincides with the original semilattice* $(A; \wedge, 0)$ .

Proof. We only have to establish the uniqueness of the BCK-structure. Suppose $(A; {}^*, 0)$ is a commutative BCK-algebra such that the original infimum is given by $a \wedge b = a^*(a^*b) = b^*(b^*a)$ , for any $a, b \in A$ . Then the finite chain $[0, a]$ is a subalgebra of $(A; {}^*, 0)$ and $a, a \wedge b \in [0, a]$ . But Traczyk [*14*, Theorem 3.5] has shown that there is a unique way to turn a finite chain into a commutative BCK-algebra so that the original order and the induced BCK-order coincide. Hence $a^*(a \wedge b) = a(a \wedge b)$ . But in $(A; {}^*, 0)$ , $a^*(a \wedge b) = a^*b$ and, in $(A; 0)$ , $a(a \wedge b) = ab$ . Hence $a^*b$ is the unique element of height $h(a) - h(a \wedge b)$ in $[0, a]$ , as asserted.

Some examples of trees of finite height supporting a commutative BCK-structure have already been studied; see, for example, Iséki and Tanaka [*8*, Example 5] and Sető [*13*].

We now exploit Theorem 1.4 to study the lattice of subvarieties of a certain variety of commutative BCK-algebras.

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## 2. Lattice of subvarieties

For $n \geq 0$, the polynomials $xy^n$ are defined inductively by $xy^0 = x$, $xy^{k+1} = (xy^k)y$.

**LEMMA 2.1.** *Let* $(A; 0)$ *be a commutative* BCK-*algebra whose underlying semilattice is a tree with the descending chain condition. Let* $a, b \in A$ *and* $n$ *be a natural number. Then*

$$h(ab^n) = \max(h(a)-nh(a \wedge b), 0) .$$

*Moreover, if* $a \wedge b > 0$ *then* $ab^{h(a)} = 0$ *and* $h(a) \geq 1$.

Proof. The second assertion is an immediate consequence of the first assertion. We use induction to establish the first one.

It is evidently true for $n = 0$. Suppose $m \geq 0$ and

$$h(ab^m) = \max(h(a)-mh(a \wedge b), 0) .$$

Then

$$h(ab^{m+1}) = h((ab^m)b) = h(ab^m) - h((ab^m) \wedge b) = h(ab^m) - h((ab^m) \wedge (a \wedge b))$$
$$= h(ab^m) - \min(h(ab^m), h(a \wedge b)) = \max(h(ab^m)-h(a \wedge b), 0)$$
$$= \max(\max(h(a)-mh(a \wedge b), 0)-h(a \wedge b), 0)$$
$$= \max(\max(h(a)-(m+1)h(a \wedge b), -h(a \wedge b)), 0) = \max(h(a)-(m+1)h(a \wedge b), 0) .$$

The proof is now complete.

We now come to the important role played by trees.

**THEOREM 2.2.** *Let* $(A; \wedge, 0)$ *be a lower semilattice with smallest element* $0$, *which satisfies the descending chain condition. Then the following conditions are equivalent:*

*(i)* $A$ *is a reduct of a subdirectly irreducible commutative* BCK-*algebra;*

*(ii)* $A$ *is a reduct of a simple commutative* BCK-*algebra;*

*(iii)* $A$ *is a tree in which* $0$ *is meet-irreducible.*

Proof. Because of Theorem 1.4 and Lemma 2.1, *(iii)* implies *(i)*, in view of the correspondence between ideals and congruences in any variety of BCK-algebras. For this correspondence, see the remarks of [4] which immediately precede Theorem 2.4, therein; the observation on simplicity is an immediate consequence, *cf.* the proof of [*4*, Corollary 3.2] and also Iséki [*7*, Proposition 4].

Of course, *(ii)* follows from *(i)*. The implication *(i)* ⇒ *(iii)* is the content of Lemmas 5.1 and 5.2 of Romanowska and Traczyk [*12*]. The fact that *(i)* implies that 0 is meet-irreducible is their Lemma 5.1; for a different explanation involving the notion of prime ideal, see [*5*, Theorem 4.3]. Why does *(i)* then imply that $A$ is a tree? Well, for each $a \in A$, $[0, a]$ is a lattice with the map $b \to ab$ $(b \in [0, a])$ as an involution, due to the commutativity of $A$, and so $a$ is then join-irreducible in $[0, a]$. Thus $[0, a]$ is a chain, and the underlying semilattice is a tree. This argument is due to Romanowska and Traczyk [*12*, Lemma 5.2].

**COROLLARY 2.3.** *A commutative* BCK-*algebra of finite height is subdirectly irreducible (simple) if and only if it is a tree with a unique atom, endowed with the* BCK-*structure of Theorem* 1.4.

In [*4*], the author showed that the class of BCK-algebras, satisfying the identity $(E_n)$ : $xy^n = xy^{n+1}$, is a congruence-distributive variety. He denoted this variety by $\underline{\underline{E}}_n$, and the variety of commutative BCK-algebras by $\underline{\underline{T}}$. The variety $\underline{\underline{T}}$ is also congruence-distributive, see [*3*, Section 3] for a list of proofs; in [*4*, Theorem 3.3], the author extended the proof of [*4*, Theorem 2.1] to show that any quasicommutative variety of BCK-algebras is, in fact, congruence-3-distributive. In order to conform with the notation of [*4*] and [*5*], the variety of commutative BCK-algebras, satisfying the identity $(E_n)$ is denoted by $\underline{\underline{T}} \cap \underline{\underline{E}}_n$. The fundamental result on the subdirectly irreducible algebras in this variety has been proved by Komori [*10*, Theorem 3.13] and is discussed immediately before Lemma 3.4 in [*4*]. Using Theorems 1.4, 2.1, and Corollary 2.3, together with Theorem 2.4, we can state:

**THEOREM 2.4.** *The subdirectly irreducible (simple) algebras in the variety* $\underline{\underline{T}} \cap \underline{\underline{E}}_n$ *are precisely those trees of height less than or equal to* $n$*, which possess a unique atom and whose* BCK-*structure is determined by Theorem* 1.4.

Proof. Komori's [*10*, Theorem 3.13] says that a commutative BCK-chain, which satisfies $(E_n)$ , must have at most $n$ elements.

Because of the isomorphism in Theorem 1.1, it is not hard to see that the set of maximal elements and the unique atom form a generating set of a subdirectly irreducible algebra having finite height. Sometimes, the unique atom can be omitted, but no maximal element can ever be eliminated.

Hence, we obtain:

THEOREM 2.5. *Each finitely generated subdirectly irreducible algebra in the variety* $\underline{\underline{T}} \cap \underline{\underline{E}}_n$ *is both simple and finite. Consequently, the variety* $\underline{\underline{T}} \cap \underline{\underline{E}}_n$ *is locally finite, that is, each of its finitely generated subalgebras is finite.*

Proof. There are only finitely many finite trees of height $n$ .

The number of non-isomorphic finite trees having a given number of elements was determined by Cayley in 1857, according to Knuth [*11*, p. 405]. By adding a new smallest element to a finite tree, we produce a finite subdirectly irreducible algebra. Hence, Cayley's work applies to the variety $\underline{\underline{T}} \cap \underline{\underline{E}}_n$ ; details are given by Knuth [*11*, p. 386, pp. 395-396, Exercises 1-4].

As we mentioned after Corollary 2.3, the variety $\underline{\underline{T}} \cap \underline{\underline{E}}_n$ is congruence-distributive. This fact and Theorem 2.5 allow us to apply Theorem 3.3 of Davey [*6*]: the lattice of subvarieties of a locally finite congruence-distributive variety is isomorphic to the lattice of all hereditary subsets of the partially ordered set of isomorphism-classes of the finite subdirectly irreducible algebras; for two representative such algebras $A$ and $B$ , $A \le B$ if and only if $A$ is a homomorphic image of a subalgebra of $B$ . Combining this with Theorem 2.4, we obtain:

THEOREM 2.6. *Let* $P_n$ *be the partially ordered set of isomorphism-classes of finite trees with a unique atom and height at most* $n$ ; *for two representative such trees* $A$ *and* $B$ , $A \le B$ *if and only if* $A$ *is isomorphic to a subtree of* $B$ *under a semilattice-homomorphism which preserves smallest elements. Then the lattice of subvarieties of the variety* $\underline{\underline{T}} \cap \underline{\underline{E}}_n$ *is isomorphic to the lattice of hereditary subsets of* $P_n$ .

*Moreover, each algebra in* $\underline{\underline{T}} \cap \underline{\underline{E}}_n$ *is isomorphic to a subalgebra of a direct power of the tree of height* $n$ *having at most one atom and countable many elements covering each of its elements of height* $1, \ldots, n-1$ *, if* $n \geq 2$ *, endowed with the* BCK*-structure of Theorem* 1.4.

The variety $\underline{\underline{T}} \cap \underline{\underline{E}}_1$ is the variety of implicative BCK-algebras. Theorem 2.6 gives the well known result that this variety is equationally complete and generated by the 2-element algebra. For a history see [2]; another proof was given recently by Comer [1].

From Theorem 2.6 it also follows that the lattice of subvarieties of $\underline{\underline{T}} \cap \underline{\underline{E}}_2$ is a chain of type $\omega + 1$ . This was established by the author in [5, Theorem 5.4], using a different approach. In [5, Theorem 5.3], an equational base was given for each subvariety of $\underline{\underline{T}} \cap \underline{\underline{E}}_2$ : the variety generated by the tree of height 2 with one atom and $n \geq 1$ maximal elements has an equaltional base which consists of a base for $\underline{\underline{T}} \cap \underline{\underline{E}}_2$ together with the identity

$$(S_n) : \bigwedge_{1 \leq i \leq n} (x_i x_{i+1}) \wedge x_{n+1} x_1 = 0 .$$

It would be interesting to find an equaltional base for the variety generated by a finite (simple) tree.

## References

[1] Stephen D. Comer, "The representation of implicative *BCK*-algebras", *Math. Japonica* 25 (1980), 111-115.

[2] William H. Cornish, "A multiplier approach to implicative BCK-algebras", *Math. Sem. Notes Kobe Univ.* 8 (1980), 157-169.

[3] William H. Cornish, "3-permutability and quasicommutative BCK-algebras", *Math. Japonica* 25 (1980), 477-496.

[4] William H. Cornish, "Varieties generated by finite BCK-algebras", *Bull. Austral. Math. Soc.* 22 (1980), 411-430.

[5] William H. Cornish, "On Iséki's BCK-algebras", *Proceedings of the Western Australian Algebra Conference*, Perth, 1980 (to appear).

[6] Brian A. Davey, "On the lattice of subvarieties", *Houston J. Math.* 5 (1979), 183-192.

[7] Kiyoshi Iséki, "On finite BCK-algebras", *Math. Japonica* 25 (1980), 225-229.

[8] Kiyoshi Iséki and Shotaro Tanaka, "Ideal theory of BCK-algebras", *Math. Japonica* 21 (1976), 351-366.

[9] Kiyoshi Iséki and Shotaro Tanaka, "An introduction to the theory of BCK-algebras", *Math. Japonica* 23 (1978), 1-26.

[10] Yuichi Komori, "Super-Łukasiewicz implicational logics", *Nagoya Math. J.* 72 (1978), 127-133.

[11] Donald E. Knuth, *The art of computer programming. Volume 1: Fundamental algorithms*, 2nd edition (Addison-Wesley, Reading, Massachusetts; Menlo Park, California; London; 1973).

[12] Anna Romanowska and Tadeusz Traczyk, "On commutative BCK-algebras", preprint.

[13] Yasuo Setō, "Some examples of BCK-algebras", *Math. Sem. Notes Kobe Univ.* 5 (1977), 397-400.

[14] Tadeusz Traczyk, "On the variety of bounded commutative BCK-algebras algebras", *Math. Japonica* 24 (1979), 283-292.

[15] Hiroshi Yutani, "On a system of axioms of a commutative BCK-algebra", *Math. Sem. Notes Kobe Univ.* 5 (1977), 255-256.

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