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A GENERALIZATION OF DEGREE TWO SIMPLE FINITE-DIMENSIONAL NONCOMMUTATIVE JORDAN ALGEBRAS

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1. Introduction. Let $\mathscr{A}$ be an algebra over a field $\mathscr{F}$. For x, y, z in $\mathscr{A}$, write $(x, y, z) = (xy)z - x(yz)$ and $x \cdot y = xy + yx$. The attached algebra $\mathscr{A}^+$ is the same vector space as $\mathscr{A}$, but the product of x and y is $x \cdot y$. We aim to prove the following result.

THEOREM 1. *Let $\mathscr{A}$ be a finite-dimensional, power-associative, simple algebra of degree two over a field of prime characteristic greater than five. For all x, y, z in $\mathscr{A}$, suppose*

(1) $\quad (x^2, y, z) = x \cdot (x, y, z).$

Then $\mathscr{A}$ is noncommutative Jordan.

The proof of Theorem 1 falls into three main sections. In § 3 we establish some multiplication properties for elements of the subspace $\mathscr{A}_{1/2}$ in the Peirce decomposition $\mathscr{A} = \mathscr{A}_1 + \mathscr{A}_{1/2} + \mathscr{A}_0$. In § 4 we construct an ideal of $\mathscr{A}$, which we then use to show that the nilpotent elements of $\mathscr{A}_i$ form a subalgebra of $\mathscr{A}$ for $i = 0, 1$. Finally, in § 5 we define a trace functional on $\mathscr{A}$ and use it to prove our main result.

Kosier [**5**, p. 39, Theorem 8] has proved that if $\mathscr{A}$ is of degree greater than one, then (1) and the identity $(y, z, x^2) = x \cdot (y, z, x)$ imply $\mathscr{A}$ is either associative or a Cayley algebra. Morgan [**6**, p. 963, Theorem 9] has proved the same result using (1) and $(y, x^2, z) = x \cdot (y, x, z)$. The proofs involve a refinement of the Peirce decomposition of $\mathscr{A}$. Goldman and Kokoris [**4**, p. 481] have shown that the single identity $(y, x^2, z) = x \cdot (y, x, z)$ yields a noncommutative Jordan algebra in the degree two case. Their proof depends on the symmetry of the preceding identity with respect to the second component of the associator. Neither of these approaches is applicable to a degree two algebra satisfying only (1). At present, the cases of degree greater than two and of degree one remain open. The author is investigating the latter.

2. Preliminary results. Let $x, y, z, w \in \mathscr{A}$. Since char $\mathscr{F} \neq 2$, the linearized identity

(2) $\quad (x \cdot w, y, z) = x \cdot (w, y, z) + w \cdot (x, y, z)$

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is equivalent to (1). We shall have occasion to use the Teichmuller identity,

$$0 = (xw, y, z) - (x, wy, z) + (x, w, yz) - x(w, y, z) - (x, w, y)z,$$

which holds in any algebra, and the linearized version of third power associativity,

$$0 = (x, y, z) + (x, z, y) + (y, x, z) + (y, z, x) + (z, x, y) + (z, y, x).$$

Now, since $\mathscr{A}$ has degree 2, there exist orthogonal idempotents u and v in $\mathscr{A}$ such that $u + v = 1$. If $\mathscr{A}_i$ is defined to be $\{a \in \mathscr{A} | u \cdot a = 2ia\}$ for $i = 0, 1/2, 1$, then by [**2**, p. 12, Lemmas 3 and 4], $\mathscr{A}_1$ and $\mathscr{A}_0$ are subalgebras of $\mathscr{A}$. Moreover, $\mathscr{A}$ has Peirce decomposition $\mathscr{A} = \mathscr{A}_1 + \mathscr{A}_{1/2} + \mathscr{A}_0$. Next, Florey [**3**, p. 505, Lemma 2] has proved that $\mathscr{A}$ is stable, i.e., $\mathscr{A}_i\mathscr{A}_{1/2} \subseteq \mathscr{A}_{1/2}$ and $\mathscr{A}_{1/2}\mathscr{A}_i \subseteq \mathscr{A}_{1/2}$ for $i = 0, 1$. Thus, if $\mathscr{N}_i^+$ is the radical of $\mathscr{A}_i^+$, then $\mathscr{A}_1 = \mathscr{F}u + \mathscr{N}_1$ and $\mathscr{A}_0 = \mathscr{F}v + \mathscr{N}_0$. Also, we have the following two results, due to [**1**, p. 517, Lemma 10] and [**4**, p. 473], respectively. First, for $x, y \in \mathscr{A}_{1/2}$ we have

(3) $\quad x \cdot y = \alpha u + \alpha v + n_1 + n_0,$

for some $\alpha \in \mathscr{F}$ and $n_i \in \mathscr{N}_i$. Second, if n is in $\mathscr{N}_1$ or $\mathscr{N}_0$, then

(4) $\quad (x \cdot n) \cdot y \in \mathscr{N}_1 + \mathscr{N}_0.$

Furthermore, let $u = x$ in (1) to get $(u, \mathscr{A}, \mathscr{A}) \subseteq \mathscr{A}_{1/2}$. Also, argue as in [**4**, p. 474] to show that $(u, \mathscr{A}, u) = 0$.

Finally, most of the results in this paper are stated in terms of u, $\mathscr{N}_1$, and $\mathscr{A}_1$. Similar results can be proved for v, $\mathscr{N}_0$, and $\mathscr{A}_0$ by using the following relationship between u and v. Let $x \in \mathscr{A}_{1/2}$. Use the definition of $\mathscr{A}_{1/2}$ and multiply the equality $1 = u + v$ on the left by x to get $xv = ux$. Similarly, $vx = xu$.

3. Multiplication properties of $\mathscr{A}_{1/2}$. Before we can determine the structure of $\mathscr{A}$, we need a series of results regarding multiplication by elements in $\mathscr{A}_{1/2}$.

LEMMA 1. $(\mathscr{A}_1, \mathscr{A}_{1/2}, u) = 0$.

Proof. Let $a \in \mathscr{A}_1$ and let $x \in \mathscr{A}_{1/2}$. Since $\mathscr{A}_1$ is orthogonal to $\mathscr{A}_0$, we have

$$0 = (a \cdot v, x, u) = a \cdot (v, x, u) + v \cdot (a, x, u),$$

from (2). But by stability, $(a, x, u) \in \mathscr{A}_{1/2}$, so that

$$0 = a \cdot (1 - u, x, u) + (a, x, u) = (a, x, u),$$

because $(u, \mathscr{A}, u) = 0$. Hence, $(\mathscr{A}_1, \mathscr{A}_{1/2}, u) = 0$.

LEMMA 2. *If* $a \in \mathscr{A}_1$ *and* $x \in \mathscr{A}_{1/2}$, *then* $(ax)u = u(xa)$.

Proof. Third power associativity and Lemma 1 yield

$$0 = (a, u, x) + (x, a, u) + (x, u, a) + (u, x, a) + (u, a, x).$$

Recall the definitions of $\mathscr{A}_1$ and $\mathscr{A}_{1/2}$ and combine like terms so that

$$(5) \qquad 0 = 2ax - u(xa) - xa - a(ux) + (xa)u - u(ax).$$

Since $ax \in \mathscr{A}_{1/2}$ by stability, use Lemma 1 to conclude

$$u(ax) = ax - (ax)u = ax - a(xu) = a(ux).$$

Hence, by substitution in (5) and because $xa \in \mathscr{A}_{1/2}$ by stability,

$$0 = 2[ax - u(ax) - u(xa)].$$

Since char $\mathscr{F} \neq 2$ and $ax \in \mathscr{A}_{1/2}$, we have shown that $0 = (ax)u - u(xa)$.

LEMMA 3. *Let* $i = 0, 1$. *For all* $x, y \in \mathscr{A}_{1/2}$, *there exist* $\beta, \delta \in \mathscr{F}$, $a_{1/2} \in \mathscr{A}_{1/2}$, *and* $a_i, b_i \in \mathscr{N}_i$ *such that*

$$xy = (\beta u + a_1) + a_{1/2} + (\delta v + a_0),$$
$$yx = (\delta u + b_1) - a_{1/2} + (\beta v + b_0),$$
$$(xu) \cdot y = (yx)_1 + (xy)_0,$$
$$(ux) \cdot y = (xy)_1 + (yx)_0.$$

Proof. Write

$$(6) \qquad xy = (\beta u + a_1) + a_{1/2} + (\delta v + a_0),$$
$$yx = (\gamma u + b_1) + b_{1/2} + (\epsilon v + b_0)$$

for $\beta, \gamma, \delta, \epsilon \in \mathscr{F}$, $a_{1/2}, b_{1/2} \in \mathscr{A}_{1/2}$, and $a_i, b_i \in \mathscr{N}_i$. Then, (3) and (6) yield $\beta + \gamma = \delta + \epsilon$ and $a_{1/2} = -b_{1/2}$. Furthermore, $(u, x, y) \in \mathscr{A}_{1/2}$, and, since $y \in \mathscr{A}_{1/2}$ and $(u, \mathscr{A}, u) = 0$,

$$(y, x, u) = (u \cdot y, x, u) = u \cdot (y, x, u),$$

by (2). Therefore, $(y, x, u) \in \mathscr{A}_{1/2}$. Write $(y, x, u) = r_{1/2}$ and $(u, x, y) = s_{1/2}$. Then, use the first of (6) to compute

$$(7) \qquad (ux)y = s_{1/2} + u(xy) = (\beta u + a_1) + (ua_{1/2} + s_{1/2})$$

and, since $x \in \mathscr{A}_{1/2}$,

$$y(ux) = yx + (y, x, u) - (yx)u$$
$$= (b_{1/2} - b_{1/2}u + r_{1/2}) + (\epsilon v + b_0).$$

But then, the definition of $\mathscr{A}_{1/2}$ and $a_{1/2} = -b_{1/2}$ imply

$$(8) \qquad y(ux) = (r_{1/2} - ua_{1/2}) + (\epsilon v + b_0).$$

Use (3) on $y \cdot (ux)$ and apply (7) and (8) to get $\beta = \epsilon$ and $r_{1/2} + s_{1/2} = 0$. Then, since $\beta + \gamma = \delta + \epsilon$, we have $\gamma = \delta$, so the first two equalities of the lemma are proved.

Next, since $s_{1/2} + r_{1/2} = 0$ and $\beta = \epsilon$,

$$(ux) \cdot y = (\beta u + a_1) + (\beta v + b_0)$$

by (7) and (8). But then, by the equalities just proved, $(ux) \cdot y = (xy)_1 + (yx)_0$. Finally, substitute for $(ux) \cdot y$ and use (3) to conclude

$$(xu) \cdot y = x \cdot y - (ux) \cdot y = (x \cdot y)_1 + (x \cdot y)_0 - (xy)_1 - (yx)_0 = (yx)_1 + (xy)_0.$$

LEMMA 4. *If $x, y \in \mathscr{A}_{1/2}$ and $n \in \mathscr{N}_1$, then $(x \cdot y)n \in \mathscr{N}_1$.*

Proof. Stability implies that $nx \in \mathscr{A}_{1/2}$, so that Lemma 3 yields

$$\begin{aligned} (9) \qquad (nx)y &= (\beta u + a_1) + a_{1/2} + (\delta v + a_0), \\ y(nx) &= (\delta u + b_1) - a_{1/2} + (\beta v + b_0). \end{aligned}$$

Now, Lemmas 2 and 3 give

$$[y(nx)]_1 + [(nx)y]_0 = [(nx)u] \cdot y = [u(xn)] \cdot y = [(xn)y]_1 + [y(xn)]_0.$$

Therefore,

$$\begin{aligned} (10) \qquad [y(nx)]_1 &= [(xn)y]_1, \\ [(nx)y]_0 &= [y(xn)]_0. \end{aligned}$$

Since $xn \in \mathscr{A}_{1/2}$ by stability, use Lemma 3 again to get

$$\begin{aligned} (11) \qquad (xn)y &= (\gamma u + c_1) + c_{1/2} + (\epsilon v + c_0), \\ y(xn) &= (\epsilon u + d_1) - c_{1/2} + (\gamma v + d_0). \end{aligned}$$

Then, (9), (10), and (11) yield $\gamma = \delta$, $b_1 = c_1$, and $a_0 = d_0$. So (11) becomes

$$\begin{aligned} (12) \qquad (xn)y &= (\delta u + b_1) + c_{1/2} + (\epsilon v + c_0), \\ y(xn) &= (\epsilon u + d_1) - c_{1/2} + (\delta v + a_0). \end{aligned}$$

But then, $(x \cdot n) \cdot y \in \mathscr{N}_1 + \mathscr{N}_0$ by (4), so that (9) and (12) imply $\beta + 2\delta + \epsilon = 0$. A similar argument yields

$$\begin{aligned} (13) \qquad (yn)x &= (\rho u + r_1) + r_{1/2} + (\sigma v + r_0), \\ x(yn) &= (\sigma u + s_1) - r_{1/2} + (\rho v + s_0), \\ x(ny) &= (\rho u + r_1) + t_{1/2} + (\tau v + t_0), \\ (ny)x &= (\tau u + w_1) - t_{1/2} + (\rho v + s_0), \\ 2\rho + \sigma + \tau &= 0. \end{aligned}$$

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Now, third power associativity applied to n, x, and y can be written as

(14) $0 = (y \cdot n)x + (x \cdot y)n + (x \cdot n)y - y(x \cdot n) - n(x \cdot y) - x(y \cdot n).$

Also, by (4), $(y \cdot n) \cdot x$ and $(x \cdot n) \cdot y$ are both in $\mathscr{N}_1 + \mathscr{N}_0$. Moreover, $(x \cdot y) \cdot n \in (\mathscr{A}_1 + \mathscr{A}_0) \cdot \mathscr{N}_1 \subseteq \mathscr{N}_1$ by (3), orthogonality of $\mathscr{A}_1$ and $\mathscr{A}_0$, and because $\mathscr{N}_1^+$ is an ideal of $\mathscr{A}_1^+$. Thus,

(15) $$(x \cdot n)y + y(x \cdot n) + (y \cdot n)x + x(y \cdot n) + n(x \cdot y) + (x \cdot y)n \in \mathscr{N}_1 + \mathscr{N}_0,$$

so that adding (14) and (15), we have

$$(x \cdot y)n + (y \cdot n)x + (x \cdot n)y \in \mathscr{N}_1 + \mathscr{N}_0,$$

since char $\mathscr{F} \neq 2$. Substituting from (9), (12), and (13), we get

$$(x \cdot y)n + (\beta + \delta + \rho + \tau)u + (a_1 + b_1 + r_1 + w_1) + (a_{1/2} + c_{1/2} + r_{1/2} - t_{1/2}) + (\delta + \epsilon + \rho + \sigma)v + (a_0 + c_0 + r_0 + s_0) \in \mathscr{N}_1 + \mathscr{N}_0.$$

Now, $(x \cdot y)n \in (\mathscr{A}_1 + \mathscr{A}_0)\mathscr{N}_1 \subseteq \mathscr{A}_1$ by (3), and since $\mathscr{A}_1$ and $\mathscr{A}_0$ are orthogonal subalgebras. Hence,

$$(x \cdot \mathrm{y})n + (\beta + \delta + \rho + \tau)u + (\delta + \epsilon + \rho + \sigma)v \in \mathscr{N}_1 + \mathscr{N}_0,$$

so that

(16) $\delta + \epsilon + \rho + \sigma = 0$

and

$$(x \cdot y)n + (\beta + \delta + \rho + \tau)u \in \mathscr{N}_1.$$

But we showed earlier that

(17) $(\beta + 2\delta + \epsilon) + (2\rho + \sigma + \tau) = 0.$

Subtraction of (16) from (17) leaves

(18) $\beta + \delta + \rho + \tau = 0,$

so that $(x \cdot y)n \in \mathscr{N}_1$.

We conclude this section with the following theorem, whose proof utilizes the two preceding lemmas.

THEOREM 2. *If* $x, y \in A_{1/2}$ *and* $n \in \mathscr{N}_1$, *then* $(xu)(yn)$, $(yn)(xu)$, $(xu)(ny)$, $(ny)(xu) \in \mathscr{N}_1 + \mathscr{A}_{1/2} + \mathscr{N}_0$.

Proof. Use (2) and the definition of $\mathscr{A}_1$ to get

$$2(n, x, y) = (n \cdot u, x, y) = n \cdot (u, x, y) + u \cdot (n, x, y),$$

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so that

$$(n, x, y) \in \mathscr{A}_1 \cdot (\mathscr{A}_1 + \mathscr{A}_{1/2} + \mathscr{A}_0) \subseteq \mathscr{A}_1 + \mathscr{A}_{1/2},$$

because char $\mathscr{F} \neq 2$, $\mathscr{A}$ is stable, and $\mathscr{A}_1$ is a subalgebra orthogonal to $\mathscr{A}_0$. Also,

$$n(xy) \in \mathscr{A}_1(\mathscr{A}_1 + \mathscr{A}_{1/2} + \mathscr{A}_0) \subseteq \mathscr{A}_1 + \mathscr{A}_{1/2}.$$

Hence,

$$(nx)y = (n, x, y) + n(xy) \in \mathscr{A}_1 + \mathscr{A}_{1/2},$$

so that, by (9), $\delta = 0$ and $a_0 = 0$. But since we have already shown that $2\delta + \beta + \epsilon = 0$, we now have $\beta = -\epsilon$. A similar argument yields $\rho = 0$, $s_0 = 0$, and $-\sigma = \tau$ in (13). Moreover, compare (16) and (18) to get $\sigma + \epsilon = \beta + \tau$, or $\sigma + \epsilon = -(\sigma + \epsilon)$, because $\beta = -\epsilon$ and $\tau = -\sigma$. Hence, since char $\mathscr{F} \neq 2$, $\tau = -\sigma = \epsilon$.

Next, for $p, q \in \mathscr{A}$, we define $p \equiv q$ if and only if $p - q \in \mathscr{N}_1 + \mathscr{A}_{1/2} + \mathscr{N}_0$. It is easy to prove that $\equiv$ is an equivalence relation on $\mathscr{A}$. Now, stability and Lemma 3 imply that

$$(ux) \cdot (yn) = [x(yn)]_1 + [(yn)x]_0 \equiv -\epsilon u - \epsilon v,$$

by (13) and $\sigma = -\epsilon$. Hence, by comparison with (13), there exists some α in $\mathscr{F}$ such that

$$(19)\quad \begin{aligned} (yn)(ux) &\equiv (-\epsilon - \alpha)u + \alpha v, \\ (ux)(yn) &\equiv \alpha u + (-\epsilon - \alpha)v, \\ (ux)(ny) &\equiv (-\epsilon - \alpha)u - \alpha v, \\ (ny)(ux) &\equiv -\alpha u + (-\epsilon - \alpha)v, \\ &\ -2\epsilon - 2\alpha = 0. \end{aligned}$$

Since char $\mathscr{F} \neq 2$, the last of these equations yields $\epsilon = -\alpha$. Therefore, (13) and (19), with $\rho = 0$ and $-\alpha = \epsilon = \tau = -\sigma$, allow us to conclude

$$\begin{aligned} (yn)(xu) &= (yn)(x - ux) \equiv (\epsilon - \epsilon)v = 0, \\ (xu)(yn) &= (x - ux)(yn) \equiv (\epsilon - \epsilon)u = 0, \\ (xu)(ny) &= (x - ux)(ny) \equiv (\epsilon - \epsilon)v = 0, \\ (ny)(xu) &= (ny)(x - ux) \equiv (\epsilon - \epsilon)u = 0, \end{aligned}$$

because $x \in \mathscr{A}_{1/2}$.

4. An ideal and subalgebras of $\mathscr{A}$. We aim to show that $\mathscr{N}_1$ and $\mathscr{N}_0$ are subalgebras of $\mathscr{A}$. For the moment, suppose this is not true.

LEMMA 5. *If $x \in \mathscr{A}_{1/2}$ and $\mathscr{N}_1$ is not a subalgebra of $\mathscr{A}$, then*

$$ux \in \mathscr{A}_{1/2}\mathscr{N}_1 + \mathscr{N}_1\mathscr{A}_{1/2}.$$

Proof. Although $\mathscr{N}_1$ is not a subalgebra of $\mathscr{A}_1$, $\mathscr{N}_1$ is a subspace of $\mathscr{A}_1$. Thus, we can find $a, b \in \mathscr{N}_1$ such that $ab = \alpha u + n$ for some $n \in \mathscr{N}_1$ and some nonzero $\alpha \in \mathscr{F}$. Then, stability implies that $(a, b, x) \in \mathscr{A}_{1/2}$, so that properties of the Peirce decomposition and (2) yield

$$2(a, b, x) = (a \cdot u, b, x) = a \cdot (u, b, x) + (a, b, x).$$

Therefore, subtracting (a, b, x) from both sides, we get

$$(ab)x - a(bx) = a \cdot [bx - u(bx)] = a \cdot [(bx)u],$$

since $bx \in \mathscr{A}_{1/2}$ by stability. Substitute $\alpha u + n$ for ab and solve the resulting equation for ux to get

$$ux = \alpha^{-1}\{-nx + a(bx) + a[(bx)u] + [(bx)u]a\},$$

since α is a nonzero element of $\mathscr{F}$. By stability, we conclude that $ux \in \mathscr{A}_{1/2}\mathscr{N}_1 + \mathscr{N}_1\mathscr{A}_{1/2}$.

Next, we begin to construct an ideal of $\mathscr{A}$. For $i = 0, 1$, define $\mathscr{C}_i$ as the subspace generated by $\{(xy)_i | x, y \in \mathscr{A}_{1/2}\}$.

LEMMA 6. *$\mathscr{C}_1 + \mathscr{A}_{1/2} + \mathscr{C}_0$ is a right ideal of $\mathscr{A}$.*

Proof. Denote $\mathscr{C}_1 + \mathscr{A}_{1/2} + \mathscr{C}_0$ by $\mathscr{C}$. We only need show $\mathscr{C}\mathscr{A} \subseteq \mathscr{C}$. By stability,

$$\mathscr{C}\mathscr{A}_{1/2} \subseteq \mathscr{A}_{1/2} + \mathscr{A}_{1/2}\mathscr{A}_{1/2} \subseteq \mathscr{C}_1 + \mathscr{A}_{1/2} + \mathscr{C}_0 = \mathscr{C}.$$

Next, consider $\mathscr{C}\mathscr{A}_1 = \mathscr{C}(\mathscr{F}u + \mathscr{N}_1)$. Since $\mathscr{C}$ is a subspace, $\mathscr{A}$ is stable, and $\mathscr{A}_1$ is orthogonal to $\mathscr{A}_0$, it suffices to consider $\mathscr{C}_1\mathscr{N}_1$. Now, let $x, y \in \mathscr{A}_{1/2}$ and $n \in \mathscr{N}_1$. Equation (2) and the definition of $\mathscr{A}_{1/2}$ yield

$$(x, y, n) = (x \cdot u, y, n) = x \cdot (u, y, n) + u \cdot (x, y, n).$$

Substitute $t_1 + t_{1/2} + t_0 = (x, y, n)$ and use properties of the Peirce decomposition to get

$$t_1 + t_{1/2} + t_0 = x \cdot (u, y, n) + 2t_1 + t_{1/2}.$$

Solving for $t_0 - t_1$ and using stability, we have $t_0 - t_1 \in \mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2}$. But the latter is contained in the subspace generated by $\mathscr{A}_{1/2}\mathscr{A}_{1/2}$. Since this subspace is itself contained in $\mathscr{C}$, we conclude that $t_1 \in \mathscr{C}_1$ and $t_0 \in \mathscr{C}_0$. Thus,

$$(xy)n - x(yn) = (x, y, n) \in \mathscr{C}.$$

By stability, $x(yn) \in \mathscr{A}_{1/2}\mathscr{A}_{1/2} \subseteq \mathscr{C}$, so that $(xy)n \in \mathscr{C}$, and thus $(xy)_1 n \in \mathscr{C}_1$ for all x, y in $\mathscr{A}_{1/2}$ and all n in $\mathscr{N}_1$. Thus, $\mathscr{C}_1\mathscr{N}_1 \subseteq \mathscr{C}$, so that $\mathscr{C}\mathscr{A}_1 \subseteq \mathscr{C}$. In a similar manner, $\mathscr{C}\mathscr{A}_0 \subseteq \mathscr{C}$, and the lemma is proved.

We want to show that $\mathscr{C}$ is a (left) ideal of $\mathscr{A}$. But first, we need a few more lemmas.

LEMMA 7. *If* $n \in \mathscr{N}_i$ *for* $i = 0, 1$, *and if* $x, y \in \mathscr{A}_{1/2}$, *then* $n(x \cdot y) \in \mathscr{C}$.

Proof. We prove the result for $i = 1$. The linearized form of third power associativity can be written as

$$n(x \cdot y) = (nx)y + (ny)x + (xn)y - x(ny) + (xy)n - x(yn) + (yn)x - y(nx) + (yx)n - y(xn).$$

By stability and Lemma 6 the terms of the right hand side are elements of

$$\mathscr{A}_{1/2}\mathscr{A}_{1/2} + (\mathscr{A}_{1/2}\mathscr{A}_{1/2})\mathscr{N}_1 \subseteq \mathscr{C} + \mathscr{C}\mathscr{A} \subseteq \mathscr{C},$$

so that $n(x \cdot y) \in \mathscr{C}$.

Next, consider the set $\mathscr{K}$ defined as

$$\mathscr{A}_{1/2} + \text{subspace generated by } \{x \cdot y | x, y \in \mathscr{A}_{1/2}\} + \text{subspace generated by } \{u \cdot (z \cdot w) | z, w \in \mathscr{A}_{1/2}\}.$$

We shall use $\mathscr{K}$ to show that $\mathscr{C}$ is a left ideal of $\mathscr{A}$. However, we need some preliminary results concerning $\mathscr{K}$.

LEMMA 8. $(\mathscr{A}_{1/2}, \mathscr{A}, \mathscr{A}) \subseteq \mathscr{K}$.

Proof. Let $x, z \in \mathscr{A}$ and let $y \in \mathscr{A}_{1/2}$. Then (2) and the definition of $\mathscr{A}_{1/2}$ yield

(20) $\quad (y, x, z) = (y \cdot u, x, z) = y \cdot (u, x, z) + u \cdot (y, x, z).$

Since $(u, x, z) \in (u, \mathscr{A}, \mathscr{A}) \subseteq \mathscr{A}_{1/2}$, (3) implies that $y \cdot (u, x, z)$ is in $\mathscr{A}_1 + \mathscr{A}_0$. Denote $y \cdot (u, x, z)$ by $a_1 + a_0$. Then, use char $\mathscr{F} \neq 2$ and properties of the Peirce decomposition to conclude

$$a_1 = \tfrac{1}{2}\, u \cdot (a_1 + a_0) = \tfrac{1}{2}\, u \cdot [y \cdot (u, x, z)].$$

The latter is in the subspace generated by $u \cdot (\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2})$. Hence, by definition of $\mathscr{K}$, we have shown that $a_1 \in \mathscr{K}$. Also,

$$a_0 = y \cdot (u, x, z) - a_1 \in \mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2} + \mathscr{K} \subseteq \mathscr{K}.$$

Finally, if we let $(y, x, z) = r_1 + r_{1/2} + r_0$, (20) becomes

$$r_1 + r_{1/2} + r_0 = (a_1 + 2r_1) + r_{1/2} + a_0,$$

so that, equating components in $\mathscr{A}_i$, we get $r_1 = -a_1 \in \mathscr{K}$ and $r_0 = a_0 \in \mathscr{K}$. Therefore, $(y, x, z) = r_1 + r_{1/2} + r_0 \in \mathscr{K} + \mathscr{A}_{1/2} \subseteq \mathscr{K}$, and the lemma is proved.

LEMMA 9. $\mathscr{A}_{1/2}\mathscr{A}_{1/2} \subseteq \mathscr{K}$.

Proof. We begin by showing that $u \cdot \mathscr{K} \subseteq \mathscr{K}$. Now,

$$(21) \quad u \cdot \mathscr{K} \subseteq u \cdot \mathscr{A}_{1/2} + \text{subspace generated by } u \cdot (\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2}) + \text{subspace generated by } u \cdot [u \cdot (\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2})].$$

By (3), $u \cdot (\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2}) \subseteq u \cdot (\mathscr{A}_1 + \mathscr{A}_0) \subseteq \mathscr{A}_1$. Thus, since u is the unity of $\mathscr{A}_1$, we have $u \cdot [u \cdot (\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2})] \subseteq u \cdot (\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2})$. Therefore, from (21), $u \cdot \mathscr{K} \subseteq \mathscr{K}$.

Next, let $x, y \in \mathscr{A}_{1/2}$. Use (3), properties of the Peirce decomposition, and char $\mathscr{F} \neq 2$ to get

$$(y \cdot x)u = (y \cdot x)_1 u + (y \cdot x)_0 u = (y \cdot x)_1 = \tfrac{1}{2} u \cdot (y \cdot x) \in \mathscr{K},$$

by definition of $\mathscr{K}$. The same definition shows that $(xu) \cdot y \in \mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2} \subseteq \mathscr{K}$. These results, together with Lemma 8, show that

$$(y, x, u) - (x, y, u) - (x, u, y) + (y \cdot x)u + (xu) \cdot y \in \mathscr{K}.$$

After simplification, we have $-2(xy)u + xy \in \mathscr{K}$. Substituting $xy = s_1 + s_{1/2} + s_0$, we have

$$-s_1 - 2s_{1/2}u + s_{1/2} + s_0 \in \mathscr{K}.$$

Since $\mathscr{A}_{1/2} \subseteq \mathscr{K}$, we conclude that $s_0 - s_1 \in \mathscr{K}$.

Finally, using the preceding paragraphs and char $\mathscr{F} \neq 2$, we get

$$s_1 = -\tfrac{1}{2}\, u \cdot (s_0 - s_1) \in u \cdot \mathscr{K} \subseteq \mathscr{K}.$$

But then, $s_0 = (s_0 - s_1) + s_1 \in \mathscr{K}$, so that $xy = s_1 + s_{1/2} + s_0 \in \mathscr{K} + \mathscr{A}_{1/2} \subseteq \mathscr{K}$.

LEMMA 10. $\mathscr{A}\mathscr{K} \subseteq \mathscr{C}$.

Proof. First, by stability,

$$\mathscr{A}\mathscr{A}_{1/2} \subseteq \mathscr{A}_{1/2} + \mathscr{A}_{1/2}\mathscr{A}_{1/2} \subseteq \mathscr{C}.$$

Next, consider $\mathscr{A}(\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2})$. By (3) and stability, we know that

$$(22) \quad \mathscr{A}_{1/2}(\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2}) \subseteq \mathscr{A}_{1/2} \subseteq \mathscr{C}.$$

Furthermore, using $u + v = 1$ and Lemma 7, we have

$$(23) \quad (\mathscr{A}_1 + \mathscr{A}_0)(\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2}) = (\mathscr{F}u + \mathscr{N}_1 + \mathscr{F}v + \mathscr{N}_0) \times (\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2}) \subseteq \mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2} + (\mathscr{N}_1 + \mathscr{N}_0)(\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2}) \subseteq \mathscr{C}.$$

Hence, (22) and (23) show that $\mathscr{A}(\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2}) \subseteq \mathscr{C}$.

Finally, consider $\mathscr{A}[u \cdot (\mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2})]$. Let $a \in \mathscr{A}$ and let $x, y \in \mathscr{A}_{1/2}$. Then,

$$(24) \quad a[u \cdot (x \cdot y)] = a[u(x \cdot y)] + a[(x \cdot y)u] = (au)(x \cdot y) + [a(x \cdot y)]u - (a, u, x \cdot y) - (a, x \cdot y, u).$$

Using the preceding paragraph, we get

(25) $(au)(x \cdot y) + [a(x \cdot y)]u \in \mathscr{C} + \mathscr{C}u \subseteq \mathscr{C},$

since $\mathscr{C}$ is a right ideal. Moreover, third power associativity leads to

$$(26)\quad (x \cdot y, u, a) + (x \cdot y, a, u) + (u, a, x \cdot y) + (u, x \cdot y, a) = -(a, u, x \cdot y) - (a, x \cdot y, u).$$

Now, from (2) and stability,

$$(x \cdot y, u, a) = x \cdot (y, u, a) + y \cdot (x, u, a) \in \mathscr{A}_{1/2} \cdot \mathscr{A} \subseteq \mathscr{A}_{1/2} + \mathscr{A}_{1/2} \cdot \mathscr{A}_{1/2} \subseteq \mathscr{C}.$$

Similarly, $(x \cdot y, a, u) \in \mathscr{C}$. Then, since $(u, \mathscr{A}, \mathscr{A}) \subseteq \mathscr{A}_{1/2} \subseteq \mathscr{C}$, (24), (25), and (26) yield $a[u \cdot (x \cdot y)] \in \mathscr{C}$ for all $a \in \mathscr{A}$ and all $x, y \in \mathscr{A}_{1/2}$, and the lemma is proved.

We now have the ideal of $\mathscr{A}$ mentioned previously.

THEOREM 3. *$\mathscr{C}$ is an ideal of $\mathscr{A}$.*

Proof. By Lemma 6, we only need show that $\mathscr{A}\mathscr{C} \subseteq \mathscr{C}$. But, by stability and the definition of $\mathscr{C}$,

$$\mathscr{A}\mathscr{A}_{1/2} \subseteq \mathscr{A}_{1/2} + \mathscr{A}_{1/2}\mathscr{A}_{1/2} \subseteq \mathscr{C}.$$

Let $x, y, z, w \in \mathscr{A}_{1/2}$, and let $a \in \mathscr{A}$. Then, by Lemmas 9 and 10, we find that

$$a(xy)_1 + a(zw)_0 \in \mathscr{A}(\mathscr{A}_{1/2}\mathscr{A}_{1/2})_1 + \mathscr{A}(\mathscr{A}_{1/2}\mathscr{A}_{1/2})_0 \subseteq \mathscr{A}\mathscr{K}_1 + \mathscr{A}\mathscr{K}_0 \subseteq \mathscr{A}\mathscr{K} \subseteq \mathscr{C}.$$

Hence, $\mathscr{A}(\mathscr{C}_1 + \mathscr{C}_0) \subseteq \mathscr{C}$, and the proof is complete.

The main result of this section is that $\mathscr{N}_1$ and $\mathscr{N}_0$ are subalgebras of $\mathscr{A}$. Before we can prove this, we need two more preliminary results. The following lemma has been proved by Morgan [**6**, p. 957, Lemma 2].

LEMMA 11. *If $x, y \in \mathscr{A}_{1/2}$, then $(ux)(uy), (xu)(yu) \in \mathscr{A}_{1/2}$.*

LEMMA 12. *If $\mathscr{N}_1$ is not a subalgebra of $\mathscr{A}$, then $\mathscr{A}_{1/2}\mathscr{A}_{1/2} \subseteq \mathscr{N}_1 + \mathscr{A}_{1/2} + \mathscr{N}_0$.*

Proof. Let $x, y \in \mathscr{A}_{1/2}$. Lemma 5 allows us to write

$$ux = a_{1/2}n_1 + m_1b_{1/2} \quad \text{and} \quad uy = c_{1/2}q_1 + k_1d_{1/2}$$

for some $k_1, m_1, n_1, q_1 \in \mathscr{N}_1$. The definition of $\mathscr{A}_{1/2}$ yields

$$xy = (xu + ux)(yu + uy) = (xu)(yu) + (ux)(uy) + (xu)(c_{1/2}q_1 + k_1d_{1/2}) + (a_{1/2}n_1 + m_1b_{1/2})(yu),$$

after substitution for uy and ux. Now, Lemma 11 and Theorem 2 show

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that the right hand side is in $\mathscr{N}_1 + \mathscr{A}_{1/2} + \mathscr{N}_0$, so that the lemma is proved.

We now have the subalgebras of $\mathscr{A}$ mentioned previously.

THEOREM 4. *$\mathscr{N}_1$ and $\mathscr{N}_0$ are subalgebras of $\mathscr{A}$.*

Proof. We prove the theorem for $\mathscr{N}_1$ only. Suppose the result is false. By definition of $\mathscr{C}$,

$$\mathscr{C} \subseteq \text{subspace generated by } (\mathscr{A}_{1/2}\mathscr{A}_{1/2})_1 + \mathscr{A}_{1/2} + \text{subspace generated by } (\mathscr{A}_{1/2}\mathscr{A}_{1/2})_0.$$

By Lemma 12, the latter is a subset of $\mathscr{N}_1 + \mathscr{A}_{1/2} + \mathscr{N}_0 \neq \mathscr{A}$. Therefore, simplicity of $\mathscr{A}$ implies that the ideal $\mathscr{C} = (0)$. But then $\mathscr{A}_{1/2} = (0)$, so that $\mathscr{A} = \mathscr{A}_1 + \mathscr{A}_0$. Thus, $\mathscr{A}_1$ and $\mathscr{A}_0$ are ideals of $\mathscr{A}$, so that simplicity implies that either $\mathscr{A}_1$ or $\mathscr{A}_0$ is (0), a contradiction of $\deg \mathscr{A} = 2$. So $\mathscr{N}_1$ must be a subalgebra of $\mathscr{A}$.

5. A trace functional. Let $a \in \mathscr{A}$. Then a may be uniquely represented as $(\alpha u + n_1) + a_{1/2} + (\beta v + n_0)$ for some α, β in $\mathscr{F}$, $a_{1/2}$ in $\mathscr{A}_{1/2}$, and n_i in $\mathscr{N}_i$ with $i = 0, 1$. Define the function $t : \mathscr{A} \to \mathscr{F}$ by $t(a) = \alpha + \beta$. It is easy to show that t is a well-defined linear functional such that $t(\mathscr{A}_{1/2}) = t(\mathscr{N}_1) = t(\mathscr{N}_0) = 0$ and $t(u) = t(v) = 1$. The following lemma was established by Goldman and Kokoris [**4**, p. 480, Lemma 5].

LEMMA 13. *If x and y are in $\mathscr{A}$, then $t(xy - yx) = 0$.*

We need one final lemma regarding the trace of associators before we can prove our main result.

LEMMA 14. *If $a, b, c \in \mathscr{A}$, then $t[(a, b, c)] = 0$.*

Proof. There are five types of associators to consider. (The rest are trivially shown to have trace zero.) Let $a_1 = \alpha u + k_1$, $b_1 = \beta u + m_1$, $c_1 = \gamma u + n_1$ for some $\alpha, \beta, \gamma \in \mathscr{F}$ and $k_1, m_1, n_1 \in \mathscr{N}_1$. Then, since $\mathscr{N}_1$ is a subalgebra of $\mathscr{A}$, $(k_1, m_1, n_1) \in \mathscr{N}_1$, and so

$$t[(a_1, b_1, c_1)] = t[(k_1, m_1, n_1)] = 0.$$

Similarly, $t[(a_0, b_0, c_0)] = 0$.

Next, the definition of $\mathscr{A}_{1/2}$ and (2) yield

$$(a_{1/2}, b, c) = (u \cdot a_{1/2}, b, c) = u \cdot (a_{1/2}, b, c) + a_{1/2} \cdot (u, b, c).$$

Let $(a_{1/2}, b, c) = s_1 + s_{1/2} + s_0$, substitute into the preceding equation, and solve for $a_{1/2} \cdot (u, b, c)$ to get

$$s_0 - s_1 = a_{1/2} \cdot (u, b, c) \in \mathscr{A}_{1,2} \cdot \mathscr{A}_{1/2}.$$

Then (3) yields $s_0 - s_1 = \tau u + \tau v + p_1 + p_0$ for some $\tau \in \mathscr{F}$ and some $p_i \in \mathscr{N}_i$ with $i = 0, 1$. Thus, $s_1 = -\tau u - p_1$ and $s_0 = \tau v + p_0$, so that $t[(a_{1/2}, b, c)] = 0$.

Finally,

$$(27) \quad t[(a_1, b_{1/2}, c_{1/2})] = t[(\alpha u + k_1, b_{1/2}, c_{1/2})] = t[(k_1 b_{1/2})c_{1/2}] - t[k_1(b_{1/2}c_{1/2})],$$

since $(u, \mathscr{A}, \mathscr{A}) \subseteq \mathscr{A}_{1/2}$. Now, by stability and the orthogonality of the subalgebras $\mathscr{A}_1$ and $\mathscr{A}_0$, $k_1(b_{1/2}c_{1/2}) \in \mathscr{A}_1 + \mathscr{A}_{1/2}$. Write $(b_{1/2}c_{1/2})_1 = \delta u + q_1$ for some $\delta \in \mathscr{F}$ and some $q_1 \in \mathscr{N}_1$. Thus, since $\mathscr{N}_1$ is a subalgebra, we have

$$(28) \quad t[k_1(b_{1/2}c_{1/2})] = t[k_1(\delta u + q_1)] = 0.$$

Next, recalling that u is the unity of $\mathscr{A}_1$ and using the Teichmuller identity, we get

$$(k_1, b_{1/2}, c_{1/2}) = (uk_1, b_{1/2}, c_{1/2}) = (u, k_1 b_{1/2}, c_{1/2}) - (u, k_1, b_{1/2}c_{1/2}) + (u, k_1, b_{1/2})c_{1/2} + u(k_1, b_{1/2}, c_{1/2}).$$

Subtract $u(k_1, b_{1/2}, c_{1/2})$ from both sides of the preceding equation and use $(u, \mathscr{A}, \mathscr{A}) \subseteq \mathscr{A}_{1/2}$ to conclude

$$(29) \quad t[v(k_1, b_{1/2}, c_{1/2})] = t[(u, k_1, b_{1/2})c_{1/2}].$$

But (2) and char $\mathscr{F} \neq 2$ imply that

$$(k_1, b_{1/2}, c_{1/2}) = \tfrac{1}{2} k_1 \cdot (u, b_{1/2}, c_{1/2}) + \tfrac{1}{2} u \cdot (k_1, b_{1/2}, c_{1/2}) \in \mathscr{A}_1 + \mathscr{A}_{1/2},$$

so that stability and orthogonality of $\mathscr{A}_1$ and $\mathscr{A}_0$ cause (29) to become

$$0 = t\{(k_1 b_{1/2})c_{1/2} - [u(k_1 b_{1/2})]c_{1/2}\} = t\{[(k_1 b_{1/2})u]c_{1/2}\}.$$

But then,

$$0 = t[(k_1 b_{1/2}, u, c_{1/2})] + t[(k_1 b_{1/2})(uc_{1/2})] = t[(k_1 b_{1/2})(uc_{1/2})]$$

by an earlier part of this proof. Hence,

$$(30) \quad t[(k_1 b_{1/2})c_{1/2}] = t[(k_1 b_{1/2})(c_{1/2} \cdot u)] = 0,$$

by the last equation and Theorem 2. Therefore, (27), (28), and (30) yield $t[(a_1, b_{1/2}, c_{1/2})] = 0$. Similarly, $t[(a_0, b_{1/2}, c_{1/2})] = 0$, and the proof is complete.

Now we are ready to prove our main result.

Proof of Theorem 1. Let $\mathscr{S} = \{a \in \mathscr{A} | t(\mathscr{A}a) = 0\}$. Since t is a linear functional, it is easy to show that $\mathscr{S}$ is a subspace of $\mathscr{A}$. We wish to prove that $\mathscr{S}$ is an ideal of $\mathscr{A}$. Let $b \in \mathscr{S}$ and let $c, d \in \mathscr{A}$. Lemmas 13 and 14 and the definition of $\mathcal{S}$ yield

$$t[c(db)] = t[(cd)b] = 0$$

and

$$t[c(bd)] = t[(cb)d] = t[d(cb)] = t[(dc)b] = 0,$$

so that db and bd are in $\mathcal{S}$. Therefore, $\mathcal{S}$ is an ideal of the simple algebra $\mathcal{A}$, so that $\mathcal{S} = \mathcal{A}$ or $\mathcal{S} = (0)$. Then, since $t(u^2) = t(u) = 1$, we have $u \notin \mathcal{S}$. Thus, $\mathcal{S} = (0)$.

Next, let $w, x, y, z \in \mathcal{A}$. Equation (2), together with Lemmas 13 and 14, yields

$$0 = t[x \cdot (w, y, z) + w \cdot (x, y, z)] = 2t[x(w, y, z) + w(x, y, z)].$$

Because char $\mathcal{F} \neq 2$ and t is a linear functional, we get

$$(31) \quad t[x(w, y, z)] = -t[w(x, y, z)].$$

Also, use the Teichmuller identity and Lemma 14 to conclude

$$0 = t[-w(x, y, z) - (w, x, y)z],$$

so that

$$(32) \quad t[w(x, y, z)] = -t[(w, x, y)z].$$

We now use (31) and (32) to show that $\mathcal{A}$ is flexible. From third power associativity, char $\mathcal{F} \neq 2$, and (31) and (32) we conclude

$$t[w(x, y, x)] = -t[w(x, x, y) + w(y, x, x)]$$
$$= t[x(w, x, y) + (w, y, x)x].$$

Then, by Lemmas 13 and (31) and (32), we have

$$t[w(x, y, x)] = t[(w, x, y)x + x(w, y, x)] = -2t[w(x, y, x)].$$

Since char $\mathcal{F} \neq 3$, $t[w(x, y, x)] = 0$, and so $(x, y, x) \in \mathcal{S}$. Hence, $(x, y, x) = 0$; i.e., $\mathcal{A}$ is flexible.

Finally, by (1) and flexibility, $(x^2, y, x) = x \cdot (x, y, x) = 0$. Therefore, $\mathcal{A}$ is a noncommutative Jordan algebra, and the proof is finished.

Remark. If we replace (1) by $(x, y, z^2) = z \cdot (x, y, z)$ in the statement of Theorem 1, the results of this paper can be adapted to show that $\mathcal{A}$ is again a noncommutative Jordan algebra.

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