A RELATION BETWEEN THE PERMANENTAL AND DETERMINANTAL ADJOINTS

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Let H_n denote the set of complex n-square positive semidefinite hermitian matrices. We partially order H_n: If $A, B, A-B \in H_n$, write $A > B$. For $A \in H_n$, write $P(A)$ for the permanental adjoint of A, i.e., $P(A)$ is the n-square matrix whose i, j entry is per $A(j \mid i)$, where $A(j \mid i)$ is the submatrix of A obtained by deleting row j and column i. Now, $P(A)$ is a principal submatrix of the $(n-1)$st induced power matrix of A^T. Hence, $P(A) \in H_n$. Also $D(A)$, the classical adjoint, is in H_n.

THEOREM. *If $A \in H_n$ is positive definite then*

(1) $$(\operatorname{per} A)^{-1} P(A) < n(\det A)^{-1} D(A).$$

PROOF. Rewrite (1) as follows:

(2) $$P(A) < n(\operatorname{per} A) A^{-1}.$$

Pre- and post-multiply both sides of (2) by $A^{\frac{1}{2}} > 0$ to obtain the equivalent statement

(3) $$A^{\frac{1}{2}} P(A) A^{\frac{1}{2}} < \mathrm{n}(\operatorname{per} A) I_n.$$

Statement (3) is equivalent to the statement that the maximum eigenvalue of $A^{\frac{1}{2}} P(A) A^{\frac{1}{2}}$ satisfies

$$\lambda_1(A^{\frac{1}{2}} P(A) A^{\frac{1}{2}}) \leqq n \operatorname{per} A.$$

Now, the eigenvalues of $A^{\frac{1}{2}} P(A) A^{\frac{1}{2}}$ are all nonnegative. Hence, it suffices to prove

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that $\operatorname{tr}(A^{\frac{1}{2}}P(A)A^{\frac{1}{2}}) \leqq n \operatorname{per} A$. But, $\operatorname{tr}(A^{\frac{1}{2}}P(A)A^{\frac{1}{2}}) = \operatorname{tr}(AP(A))$. The main diagonal elements of $AP(A)$ are

$$\sum_{k=1}^{n} a_{ik} \operatorname{per} A(i \mid k) = \operatorname{per} A.$$

Hence, $\operatorname{tr}(AP(A)) = n \operatorname{per} A$, and the proof is complete.

Indeed, the proof shows that (3) holds for all $A \in H_n$, not just for A positive definite.

COROLLARY. *Suppose $A \in H_n$. Let σ_i be the ith row sum of A. Let $\sigma(A)$ be the sum of the elements of A. Then*

$$0 \leqq \sum_{i,j=1}^{n} \sigma_i \bar{\sigma}_j \operatorname{per} A(i \mid j) \leqq n\, \sigma(A) \operatorname{per} A.$$

PROOF. Display (3) is congruent to $AP(A)A < n(\operatorname{per} A)A$. Now, if $A \in H_n$ then $\sigma(A) \geqq 0$. It follows that

$$0 \leqq \sigma(AP(A)A) \leqq n(\operatorname{per} A)\sigma(A).$$

But

$$\sigma(AP(A)A) = \sum_{i,j=1}^{n} \sigma_i \bar{\sigma}_j \operatorname{per} A(i \mid j).$$

We point out that if $A \in H_n$ is doubly stochastic then the corollary becomes

$$1/n^2 \sum_{i,j=1}^{n} \operatorname{per} A(i \mid j) \leqq \operatorname{per} A. \tag{4}$$

Display (4) is the first of a class of inequalities conjectured by Djoković [1] to hold for all doubly stochastic A. It was proved in [2] using other methods.

References

[1] D. Ž. Djoković, 'On a conjecture by van der Waerden', *Mat. Vesnik* (4) 19 (1967), 272-276.
[2] Marvin Marcus and Henryk Minc, 'Extensions of classical matrix inequalities', *Linear Algebra Appl.* 1 (1968), 421–444.

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