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LOWER BOUNDS FOR BLOW-UP TIME IN SOME NON-LINEAR PARABOLIC PROBLEMS UNDER NEUMANN BOUNDARY CONDITIONS

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Abstract. This paper deals with some non-linear initial-boundary value problems under homogeneous Neumann boundary conditions, in which the solutions may blow up in finite time. Using a first-order differential inequality technique, lower bounds for blow-up time are determined.

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1. Introduction. The study of the blow-up phenomena in parabolic problems has received a great deal of attention in the last decades (we refer the reader especially to the books of Straughan [**12**] and Quittner–Souplet [**11**], the survey papers of Levine [**4**] and Galaktionov [**2**] and the references therein). Therefore, nowadays a variety of methods are known and used in the study of various questions regarding the blow-up phenomena in parabolic problems. But, most of the methods used to show that solutions blow-up provide only an upper bound for the blow-up time, while in applications, due to the explosive nature of the solutions, it is more important to determine the lower bounds on the blow-up time. We note, however, that during the last four years, beginning with the paper of Payne and Schaefer [**6**], such lower bounds on blow-up time have been obtained in various parabolic problems, by mean of a first-order differential inequality technique (see, for instance, [**5**]–[**9**] and some references therein).

In this paper, we will consider the following type of non-linear parabolic problems in divergence form:

$$\begin{cases} \left(\rho(\mathbf{x}, u, |\nabla u|^2)u_{,i}\right)_{,i} - u_{,t} = -f(u) & \text{in } \Omega \times (0, t^*)\,, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0, t^*)\,, \\ u(\mathbf{x}, 0) = g(\mathbf{x}) \geq 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u_{,t}$ denotes the partial derivative of $u(\mathbf{x}, t)$ with respect to t, the symbol $_{,i}$ denotes the partial differentiation with respect to x_i, $i = 1, 2, 3$, $\partial u/\partial n$ is the outward normal derivative of $u(\mathbf{x}, t)$ on the boundary $\partial\Omega$ and the summation is understood on repeated indices. Moreover, the domain $\Omega \subset \mathbb{R}^3$ is assumed to be bounded, starshaped, convex in two orthogonal directions and with smooth boundary $\partial\Omega$, while ρ is a positive C^1 function that satisfies the ellipticity condition throughout Ω, i.e.

$$\rho(\mathbf{x}, u, s) + 2s\frac{\partial}{\partial s}\rho(\mathbf{x}, u, s) > 0, \qquad s > 0,\ \mathbf{x} \in \Omega. \tag{1.2}$$

We also ask that ρ and f satisfy the conditions

$$0 < f(s) \le a_1 + a_2 s^p, \qquad \rho(\mathbf{x}, u, s) \ge b_1, \qquad s > 0,\ \mathbf{x} \in \Omega, \tag{1.3}$$

where $p > 1$ and $a_1 \in \mathbb{R}_+$, a_2, $b_1 \in \mathbb{R}_+^*$. In addition, g is assumed to satisfy the compatibility condition $\partial g/\partial n = 0$ on $\partial\Omega$. Under these assumptions on the data, it follows from the parabolic maximum principles (see Protter–Weinberger [**10**]) that the solution of the problem (1.1) is non-negative. Moreover, it is well-known that the solution may not exist for all time, and the only way that it can fail to exist is by becoming unbounded at some finite time t^* (see, for instance, the works of Ball [**1**] and Kielhöfer [**3**] in the case $\rho \equiv 1$). This phenomena depends on the form of $f(u)$ and $\rho(\mathbf{x}, u, |\nabla u|^2)$, the initial data $g(\mathbf{x})$ or the geometry of the given domain Ω.

In what follows, we shall assume that a non-negative classical solution of the problem (1.1)–(1.3) exists and become unbounded at time $t = t^*$. Our aim is to determine an explicit lower bound for the the blow-up time t^* in some appropriate measure. We notice that lower bounds for blow-up time in non-linear parabolic problems with particular divergence form, but under Dirichlet boundary conditions and different assumptions on the data, have been recently obtained by Payne–Philippin–Schaefer in [**5**]. A key ingredient in their proof was the Sobolev inequality, which is no longer applicable in our case, since we deal with homogeneous Neumann boundary conditions. However, for a class of semi-linear heat equations under homogeneous Neumann boundary conditions, Payne and Schaefer succeeded [**6**] to overpass this difficulty by the determination of an appropriate Sobolev-type inequality for C^1-functions. In order to handle the more general problem (1.1)–(1.3), our approach is inspired by their technique, the main ingredient of our argument being again the determination of an appropriate Sobolev-type inequality for C^1-functions on Ω.

2. Lower bound on blow-up time. Let us introduce the auxiliary function

$$\Phi(t) := \int_\Omega u^{2n}\, dx, \tag{2.1}$$

for some constant $n > 1$ to be chosen. We compute

$$\begin{aligned}\Phi'(t) &= 2n \int_\Omega u^{2n-1}\left[\left(\rho(\mathbf{x}, u, |\nabla u|^2) u_{,i}\right)_{,i} + f(u)\right] dx \\ &= -2n(2n-1) \int_\Omega u^{2n-2} \rho(\mathbf{x}, u, |\nabla u|^2)\, |\nabla u|^2\, dx + 2n \int_\Omega u^{2n-1} f(u)\, dx \\ &\le -2n(2n-1)\, b_1 \int_\Omega u^{2n-2}\, |\nabla u|^2\, dx + 2n \int_\Omega u^{2n-1} (a_1 + a_2 u^p)\, dx,\end{aligned} \tag{2.2}$$

where we have used successively the differential equation (1.1), the divergence theorem, the boundary condition (1.1) and the assumption (1.3). Next, we notice that

$$|\nabla u^n|^2 = n^2 u^{2(n-1)}\, |\nabla u|^2, \tag{2.3}$$

and we use Holder's inequality to obtain

$$\Phi'(t) \le -\frac{2n(2n-1)}{n^2} b_1 \int_\Omega \left|\nabla u^n\right|^2 dx + 2n a_1\, |\Omega|^{\frac{1}{2n}}\, \Phi(t)^{\frac{2n-1}{2n}} + 2n a_2 \int_\Omega u^{2n+p-1}\, dx. \tag{2.4}$$

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Now, our aim is to transform the right side of (2.4) in terms of $\Phi(t)$ and obtain a first-order differential inequality for Φ. To accomplish this, we begin by using Holder's inequality to write:

$$\int_\Omega u^{2n+p-1}\,dx \le \left(\int_\Omega u^{4n}\,dx\right)^{\frac{1}{3}} \left(\int_\Omega u^{\frac{2n+3p-3}{2}}\,dx\right)^{\frac{2}{3}}. \tag{2.5}$$

To bound the integral of $u^{(2n+3p-3)/2}$, we use again Holder's inequality and obtain

$$\int_\Omega u^{\frac{2n+3p-3}{2}}\,dx \le |\Omega|^{1-\mu}\,|\Phi(t)|^{\mu}, \quad \text{with} \quad \mu := \frac{2n+3p-3}{4n}, \tag{2.6}$$

where $|\Omega|$ denotes the volume of Ω and, in order to ensure that $\mu < 1$ in (2.6), the constant n must be chosen to satisfy $n > 3(p-1)/2$.

Next, to bound the integral of u^{4n} in (2.5), we seek to determine an appropriate Sobolev-type inequality. For this aim, we denote by x_{im} and x_{iM} the minimum and the maximum values, respectively, of the coordinates x_i, $i = 1, 2, 3$, relative to Ω and by ν_i, $i = 1, 2, 3$, the components of the unit outer normal to $\partial\Omega$. We also denote by D_z the intersection of Ω with the plane $x_3 = z$ and, for clarity, we let $w := u^n$. Then, using Schwarz's inequality, we can write

$$\int_\Omega w^4\,dx = \int_{x_{3m}}^{x_{3M}} \left(\int_{D_z} w^4 dA\right) d\xi \le \int_{x_{3m}}^{x_{3M}} \left[\int_{D_z} w^2 dA \int_{D_z} w^6 dA\right]^{\frac{1}{2}} d\xi. \tag{2.7}$$

Now, let $\mathbf{P} = (\overline{x}_1, \overline{x}_2, z)$ be an arbitrary point in D_z and $\mathbf{P}_1 := (\xi_1, \overline{x}_2, z)$ and $\mathbf{P}_2 := (\xi_2, \overline{x}_2, z)$ denotes the points on the boundary ∂D_z where the line $x_2 = \overline{x}_2$ in D_z intersects the boundary ∂D_z. Similarly, let $\mathbf{Q}_1 := (\overline{x}_1, \eta_1, z)$ and $\mathbf{Q}_2 := (\overline{x}_1, \eta_2, z)$ be the points on the boundary ∂D_z, where the line $x_2 = \overline{x}_2$ in D_z intersects ∂D_z. We then have

$$\begin{aligned} w^3(\mathbf{P}) &= w^3(\mathbf{P}_1) + 3\int_{\mathbf{P}_1}^{\mathbf{P}} w^2 w_{,1}\,dx_1, \\ w^3(\mathbf{P}) &= w^3(\mathbf{P}_2) - 3\int_{\mathbf{P}_2}^{\mathbf{P}} w^2 w_{,1}\,dx_1, \end{aligned} \tag{2.8}$$

from which we obtain

$$w^3(\mathbf{P}) \le \frac{1}{2}\left[w^3(\mathbf{P}_1) + w^3(\mathbf{P}_2)\right] + \frac{3}{2}\int_{\mathbf{P}_1}^{\mathbf{P}_2} w^2\left|w_{,1}\right|dx_1. \tag{2.9}$$

In a similar way, one may show that

$$w^3(\mathbf{P}) \le \frac{1}{2}\left[w^3(\mathbf{Q}_1) + w^3(\mathbf{Q}_2)\right] + \frac{3}{2}\int_{\mathbf{Q}_1}^{\mathbf{Q}_2} w^2\left|w_{,2}\right|dx_2. \tag{2.10}$$

Therefore, multiplying (2.9) and (2.10) and integrating over D_z, we get

$$\begin{aligned} \int_{D_z} w^6 dA \le \frac{1}{4}&\left\{\int_{x_{2m}}^{x_{2M}} \left[w^3(\mathbf{P}_1) + w^3(\mathbf{P}_2)\right] dx_2 + 3\int_{D_z} w^2\left|w_{,1}\right| dA\right\}\cdot \\ &\cdot\left\{\int_{x_{1m}}^{x_{1M}} \left[w^3(\mathbf{Q}_1) + w^3(\mathbf{Q}_2)\right] dx_1 + 3\int_{D_z} w^2\left|w_{,2}\right| dA\right\}. \end{aligned} \tag{2.11}$$

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Next, making use of the fact that

$$\begin{aligned} &\int_{x_{2m}}^{x_{2M}} \left[w^3(\mathbf{P}_1) + w^3(\mathbf{P}_2)\right] dx_2 \le \int_{\partial D_z} w^3 |\nu_1| \, ds, \\ &\int_{x_{1m}}^{x_{1M}} \left[w^3(\mathbf{Q}_1) + w^3(\mathbf{Q}_2)\right] dx_1 \le \int_{\partial D_z} w^3 |\nu_2| \, ds, \end{aligned} \tag{2.12}$$

together with the facts that $|\nu_k| < 1$, $\left|w_{,k}\right| < |\nabla w|$, $k = 1, 2$, and Schwarz's inequality, it follows from (2.11) that

$$\int_{D_z} w^6 dA \le \frac{1}{4}\left\{\int_{\partial D_z} w^3 ds + 3\left[\int_{D_z} w^4 dA \int_{D_z} |\nabla w|^2 dA\right]^{\frac{1}{2}}\right\}^2. \tag{2.13}$$

Therefore, making use of Schwarz's inequality and (2.13), we deduce that

$$\int_{D_z} w^4 dA \le \frac{1}{2}\left[\max_z \int_{D_z} w^2 dA\right]^{\frac{1}{2}} \left\{\int_{\partial D_z} w^3 ds + 3\left[\int_{D_z} w^4 dA \int_{D_z} |\nabla w|^2 dA\right]^{\frac{1}{2}}\right\}. \tag{2.14}$$

Integrating now (2.14) over z we get

$$\int_{\Omega} w^4 \, dx \le \frac{1}{2}\left[\max_z \int_{D_z} w^2 dA\right]^{\frac{1}{2}} \left\{\int_{\partial\Omega} w^3 ds + 3\left[\int_{\Omega} w^4 \, dx \int_{\Omega} |\nabla w|^2 \, dx\right]^{\frac{1}{2}}\right\}, \tag{2.15}$$

where we have used Schwarz's inequality to obtain the last term.

We now seek to bound $\int_{\partial\Omega} w^3 ds$ and $\max\limits_z \int_{D_z} w^2 dA$. For this aim, we denote by

$$p_0 := \min_{\partial\Omega} (\mathbf{x} \cdot \mathbf{n}), \quad d^2 := \max_{\Omega} |\mathbf{x}|, \tag{2.16}$$

and make use of the divergence theorem to write

$$p_0 \int_{\partial\Omega} w^3 ds \le \int_{\partial\Omega} x_i n_i w^3 ds = 3\int_{\Omega} w^3 \, dx + 3\int_{\Omega} x_i w^2 w_{,i} \, dx. \tag{2.17}$$

It then follows that

$$\int_{\partial\Omega} w^3 ds \le \frac{3}{p_0}\int_{\Omega} w^3 \, dx + \frac{3d}{p_0}\left[\int_{\Omega} w^4 \, dx \int_{\Omega} |\nabla w|^2 \, dx\right]^{\frac{1}{2}}, \tag{2.18}$$

where we have used Schwarz's inequality to get the last term. Replacing (2.18) in (2.15), we obtain

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$$
\begin{aligned}
\int_\Omega w^4\,dx &\le \frac{3}{2}\left[\max_z \int_{D_z} w^2 dA\right]^{\frac12}\left\{\frac{1}{p_0}\int_\Omega w^3\,dx + \left(1+\frac{d}{p_0}\right)\left[\int_\Omega w^4\,dx\int_\Omega |\nabla w|^2\,dx\right]^{\frac12}\right\}\\
&\le \frac{3}{2}\left[\max_z \int_{D_z} w^2 dA\right]^{\frac12}\left(\int_\Omega w^4\,dx\right)^{\frac12}\\
&\quad\times\left\{\frac{1}{p_0}\left(\int_\Omega w^2\,dx\right)^{\frac12} + \left(1+\frac{d}{p_0}\right)\left(\int_\Omega |\nabla w|^2\,dx\right)^{\frac12}\right\}, \qquad (2.19)
\end{aligned}
$$

where we have used again Schwarz's inequality to get the last expression.

Next, in order to bound $\max_z \int_{D_z} w^2 dA$ in (2.19), we let Ω^+ be the portion of Ω above D_z, with $\partial\Omega^+$ the portion of $\partial\Omega$ above D_z, and Ω^- the portion of Ω below D_z, with $\partial\Omega^-$ the portion of $\partial\Omega$ below D_z. Then, the divergence theorem gives

$$
\int_{D_z} w^2 dA - \int_{\partial\Omega^+} w^2 \nu_3 ds = -2\int_{\Omega^+} w w_{,3}\,dx, \qquad (2.20)
$$

$$
\int_{D_z} w^2 dA + \int_{\partial\Omega^-} w^2 \nu_3 ds = 2\int_{\Omega^-} w w_{,3}\,dx. \qquad (2.21)
$$

Combining (2.20) and (2.21) and making use of Schwarz's inequality, we obtain

$$
\int_{D_z} w^2 dA \le \frac{1}{2}\int_{\partial\Omega} w^2 ds + \left[\int_\Omega w^2\,dx\int_\Omega |\nabla w|^2\,dx\right]^{\frac12}. \qquad (2.22)
$$

On the other hand, from the definition of p_0 (see (2.16)) and the divergence theorem, we have

$$
p_0\int_{\partial\Omega} w^2 ds \le \int_{\partial\Omega} x_i n_i w^2 ds = 3\int_\Omega w^2\,dx + 2\int_\Omega x_i w w_{,i}\,dx, \qquad (2.23)
$$

so that we obtain

$$
\int_{\partial\Omega} w^2 ds \le \frac{3}{p_0}\int_\Omega w^2\,dx + \frac{2d}{p_0}\left[\int_\Omega w^2\,dx\int_\Omega |\nabla w|^2\,dx\right]^{\frac12}. \qquad (2.24)
$$

Therefore, replacing (2.24) in (2.22), we get

$$
\int_{D_z} w^2 dA \le \frac{3}{2p_0}\int_\Omega w^2\,dx + \left(1+\frac{d}{p_0}\right)\left[\int_\Omega w^2\,dx\int_\Omega |\nabla w|^2\,dx\right]^{\frac12}. \qquad (2.25)
$$

Going back to (2.19) we find, after some manipulations, that

$$
\begin{aligned}
\left(\int_\Omega w^4\,dx\right)^{\frac12} &\le \frac{3}{2}\left(\int_\Omega w^2\,dx\right)^{\frac14}\left\{\frac{3}{2p_0}\left(\int_\Omega w^2\,dx\right)^{\frac12} + \left(1+\frac{d}{p_0}\right)\left(\int_\Omega |\nabla w|^2\,dx\right)^{\frac12}\right\}^{\frac32}\\
&= \frac{3}{2}\left\{\frac{3}{2p_0}\left(\int_\Omega w^2\,dx\right)^{\frac23} + \left(1+\frac{d}{p_0}\right)\left(\int_\Omega w^2\,dx\right)^{\frac16}\left(\int_\Omega |\nabla w|^2\,dx\right)^{\frac12}\right\}^{\frac32}.
\end{aligned} \qquad (2.26)
$$

Next, with $w := u^n$, we replace (2.26) in (2.5) to obtain

$$\int_\Omega u^{2n+p-1}\,dx \le \left(\frac{3}{2}\right)^{\frac{2}{3}} |\Omega|^{\frac{2}{3}(1-\mu)}\,\Phi(t)^{\frac{2}{3}\mu} \times \left\{ \frac{3}{2p_0}\Phi(t)^{\frac{2}{3}} + \left(1+\frac{d}{p_0}\right)\Phi(t)^{\frac{1}{6}} \left(\int_\Omega |\nabla u^n|^2\,dx\right)^{\frac{1}{2}} \right\}. \quad (2.27)$$

Moreover, making use of the inequality $ab \le \frac{a^2}{2\alpha} + \frac{b^2\alpha}{2}$, where α is an, as yet, unspecified positive weight to be chosen, we have

$$\Phi(t)^{\frac{4\mu+1}{6}} \left(\int_\Omega |\nabla u^n|^2\,dx\right)^{\frac{1}{2}} \le \frac{1}{2\alpha}\Phi(t)^{\frac{4\mu+1}{3}} + \frac{\alpha}{2}\int_\Omega |\nabla u^n|^2\,dx. \quad (2.28)$$

Therefore, replacing (2.28) in (2.27) and, thereafter, (2.27) in (2.4), we get

$$\Phi'(t) \le -\frac{2n(2n-1)}{n^2} b_1 \int_\Omega |\nabla u^n|^2\,dx + 2na_1 |\Omega|^{\frac{1}{2n}}\,\Phi(t)^{\frac{2n-1}{2n}} + na_2 \times \left(\frac{3^5}{2^2}\right)^{\frac{1}{3}} \frac{1}{p_0} |\Omega|^{\frac{2}{3}(1-\mu)}\,\Phi(t)^{\frac{2}{3}(1+\mu)} + na_2 \left(\frac{3}{2}\right)^{\frac{2}{3}} \frac{1}{\alpha}\left(1+\frac{d}{p_0}\right)|\Omega|^{\frac{2}{3}(1-\mu)}\,\Phi(t)^{\frac{4\mu+1}{3}} + na_2\alpha\left(\frac{3}{2}\right)^{\frac{2}{3}}\left(1+\frac{d}{p_0}\right)|\Omega|^{\frac{2}{3}(1-\mu)}\int_\Omega |\nabla u^n|^2\,dx. \quad (2.29)$$

Choosing now the parameter α in (2.29) such that

$$-\frac{2n(2n-1)}{n^2} b_1 + na_2\alpha\left(\frac{3}{2}\right)^{\frac{2}{3}}\left(1+\frac{d}{p_0}\right)|\Omega|^{\frac{2}{3}(1-\mu)} = 0, \quad (2.30)$$

we obtain the following differential inequality for $\Phi(t)$:

$$\Phi'(t) \le K_1\Phi(t)^{\frac{2n-1}{2n}} + K_2\Phi(t)^{\frac{2}{3}(1+\mu)} + K_3\Phi(t)^{\frac{4\mu+1}{3}}, \quad (2.31)$$

where

$$K_1 := 2na_1|\Omega|^{\frac{1}{2n}},\ K_2 := na_2\left(\frac{3^5}{2^2}\right)^{\frac{1}{3}}\frac{1}{p_0}|\Omega|^{\frac{2}{3}(1-\mu)}, \quad K_3 := na_2\left(\frac{3}{2}\right)^{\frac{2}{3}}\frac{1}{\alpha}\left(1+\frac{d}{p_0}\right)|\Omega|^{\frac{2}{3}(1-\mu)}. \quad (2.32)$$

Next, an integration of the differential equation (2.31) from 0 to t gives

$$\int_{\Phi(0)}^{\Phi(t)} \frac{d\eta}{K_1\eta^{\frac{2n-1}{2n}} + K_2\eta^{\frac{2}{3}(1+\mu)} + K_3\eta^{\frac{4\mu+1}{3}}} \le t. \quad (2.33)$$

Therefore, if $u(\mathbf{x}, t)$ blows up in the measure Φ as $t \longrightarrow t^*$, we obtain the lower bound

$$t^* \ge \int_{\Phi(0)}^{\infty} \frac{d\eta}{K_1\eta^{\frac{2n-1}{2n}} + K_2\eta^{\frac{2}{3}(1+\mu)} + K_3\eta^{\frac{4\mu+1}{3}}}, \quad (2.34)$$

where μ was given in (2.6). Clearly, since $2(\mu+1)/3 > 1$ and $(4\mu+1)/3 > 1$, the integral in (2.34) is bounded.

We summarise this result in the following theorem:

THEOREM. *If $n > 3(p-1)/2$ and $u(\mathbf{x}, t)$ is a non-negative classical solution of the problem (1.1)–(1.3), which becomes unbounded at time $t = t^*$ in the measure $\Phi(t)$ given by (2.1), then t^* is bounded below by (2.34), where K_1, K_2 and K_3 are given in (2.32).*

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