Can. J. Math., Vol. XXXI, No. 6, 1979, pp. 1307–1321

BOUNDING THE VALENCY OF POLYGONAL GRAPHS WITH ODD GIRTH

MANLEY PERKEL

1. Introduction. In this paper we investigate the action of finite groups G on finite polygonal graphs. The notion of a polygonal graph was introduced in [**17**]: A *polygonal graph* is a pair $(\mathscr{H}, \mathscr{E})$ consisting of a graph $\mathscr{H}$ which is regular, connected and has girth m for some $m \geqq 3$, and a set $\mathscr{E}$ of m-gons of $\mathscr{H}$ such that every 2-claw of $\mathscr{H}$ is contained in an unique element of $\mathscr{E}$. (See Section 2 for the definitions of the terms used here.) If $\mathscr{E}$ is the set of all m-gons of $\mathscr{H}$, so that there is in $\mathscr{H}$ an unique m-gon on every one of its 2-claws, then we write $\mathscr{H}$ for $(\mathscr{H}, \mathscr{E})$ and call $\mathscr{H}$ a *strict polygonal graph*. If we wish to emphasize the integer m, then we call $(\mathscr{H}, \mathscr{E})$ an *m-gon-graph* (respectively, a *strict m-gon-graph*).

Examples of polygonal graphs not arising from regular solids are known mainly with girth $m \leqq 6$ and with valency $k \leqq 5$. Fewer examples with $m > 6$ or $k > 5$ are known, the most notable arising from J_1, Janko's first simple group ($m = 5$ and $k = 11$), which in fact can be characterized by this action on a polygonal graph [**15**]. These examples will be discussed in Section 3. In Section 2 we define the terms used in this paper and prove some basic lemmas about strict polygonal graphs and their automorphism groups.

In Sections 4 and 5 we shall assume that $(\mathscr{H}, \mathscr{E})$ is a polygonal graph of valency $k \geqq 3$ on a set Ω, with girth m, m odd, $m \geqq 5$, and that $G \leqq \mathrm{Aut}(\mathscr{H})$ is a group of automorphisms of $\mathscr{H}$ transitive on Ω. We also suppose that for any 2-claw $(x{:}y, z)$, $x, y, z \in \Omega$, every involution in G_{xyz} fixes (pointwise) the m-gon in $\mathscr{E}$ on $(x{:}y, z)$, but no other m-gon on $(x{:}y, z)$. This latter hypothesis is automatically satisfied if $\mathscr{H}$ is a strict m-gon-graph, and in the case that G_{xyz} has no involutions we interpret this hypothesis to mean that G_{xyz} fixes the m-gon in $\mathscr{E}$ on $(x{:}y, z)$, and no other m-gon on $(x{:}y, z)$.

We shall then prove the following two theorems.

THEOREM 1. *Let $x \in \Omega$. Suppose that for some prime p and integer $n > 0$, $PSL(2, p^n) \leqq G_x^{\Delta(x)} \leqq P\Gamma L(2, p^n)$ on $p^n + 1$ points. Then either $k = 3$ and $G_x \simeq \Sigma_3$, $k = 4$ and $G_x \simeq A_4$ or Σ_4, $k = 5$ and $G_x \simeq A_5$ or Σ_5, $k = 6$ and $G_x \simeq PSL(2, 5)$, or $k = 10$ and $G_x \simeq PSL(2, 9)$ or $PSL(2, 9)\langle\alpha\rangle$, where α is the non-trivial automorphism of the field of 9 elements.*

Received November 10, 1977. The results of this paper formed part of the author's doctoral thesis written at the University of Michigan under the direction of Professor D. G. Higman. The author also wishes to thank Professor W. M. Kantor of the University of Oregon.

THEOREM 2. *If k is odd, G_x is 3-transitive on $\Delta(x)$ for $x \in \Omega$, and $\mathscr{H}$ contains no strict m-gon-graph of valency 3 as a subgraph, then $k = 5$ and $G_x \simeq A_5$.*

All the examples of polygonal graphs with m odd from Section 3 (except the Petersen graph) satisfy the hypotheses of Theorem 1. As for Theorem 2, the only example arising from Section 3 which satisfies its hypotheses is the pentagraph $\mathscr{H}_{31}$ with Aut $(\mathscr{H}_{31}) \simeq PSL(2, 31)$.

Remark. If we remove the restriction that m be odd, then there are further examples of polygonal graphs satisfying the remaining hypotheses of Theorems 1 and 2. However, I know of no example of a polygonal graph with k and m odd ($k > 3$) which does contain a strict m-gon-graph of valency 3 as a subgraph (whether or not $G_x^{\Delta(x)}$ is 3-transitive), whereas there are such examples if we allow either k or m (or both) to be even.

Finally it should be remarked that with $m = 5$, Theorem 2 provides a characterization of $PSL(2, 31)$ in its action on a 5-gon-graph. This is done in [**16**].

2. Notation and preliminary results. All groups and graphs to be considered will be finite, and the graphs will be undirected with no loops or multiple edges.

If $\mathscr{H}$ is a graph on a set Ω and if $x, y \in \Omega$, we write $x \backsim y$ to mean x is adjacent to y, i.e. (x, y) is an edge of $\mathscr{H}$. A *path of length* $n > 0$ in $\mathscr{H}$ is a sequence $(x_0, x_1, \ldots, x_n)$ of $n + 1$ vertices $x_i \in \Omega$ such that $x_i \backsim x_{i+1}$ for all $i = 0, \ldots, n - 1$, and $x_i \neq x_{i+2}$ for $i = 0, \ldots, n - 2$. The above path is a *circuit of length* n if $x_0 = x_n$ and $x_{n-1} \neq x_1$, in which case we write $(x_1, \ldots, x_n)$. It is called a *simple path* (respectively, a *simple circuit*) if $x_i \neq x_j$ for any $i \neq j$, $0 \leqq i, j \leqq n$ (except of course $x_0 = x_n$ in the case of a circuit).

Remark. We shall not distinguish the circuit $(x_1, \ldots, x_n)$ from the circuits $(x_i, \ldots, x_n, x_1, \ldots, x_{i-1})$ and $(x_i, \ldots, x_1, x_n, \ldots, x_{i+1})$, for $1 \leqq i \leqq n$; for our purposes these circuits are considered to be the same.

The definitions of *connected graphs*, *distance* from x to y, $x, y \in \Omega$, *diameter* of a graph $\mathscr{H}$, *subgraph* of $\mathscr{H}$, *induced subgraph* of $\mathscr{H}$, and *connected component* of $\mathscr{H}$ are as in [**11**]. The *girth* of $\mathscr{H}$ is the minimum of the lengths of all circuits of $\mathscr{H}$.

For all $i \geqq 0$ and $x \in \Omega$, define $\Delta_i(x)$ to be the set of all $y \in \Omega$ at distance i from x (where $\Delta_0(x) = \{x\}$). Of course, if the diameter of $\mathscr{H}$ is n, then $\Delta_i(x) = \emptyset$ for $i > n$. Often we will just write $\Delta(x)$ for $\Delta_1(x)$, the points adjacent to x.

An *m-claw*, denoted by $(x_0{:}x_1, \ldots, x_m)$, is a subgraph $\bar{\mathscr{H}}$ of $\mathscr{H}$ on the $m + 1$ distinct points $\{x_0, \ldots, x_m\} \subseteq \Omega$, where $\{x_1, \ldots, x_m\} \subseteq \Delta(x_0)$ and there are no further adjacencies between the x_i's in $\bar{\mathscr{H}}$.

The *valency* of a vertex x of $\mathscr{H}$ is $|\Delta(x)|$. $\mathscr{H}$ is said to be *regular* (of valency k) if $|\Delta(x)| = k$ for all $x \in \Omega$. If $k = 3$, $\mathscr{H}$ is called a *cubic* graph, while if

$k = 2$, $\mathscr{H}$ is connected and $|\Omega| = m \geqq 3$, $\mathscr{H}$ is called an m-gon. An m-gon, with $m = 3, 4, 5, \ldots$ will be called, respectively, a *triangle*, *rectangle*, *pentagon*, ... The automorphism group of $\mathscr{H}$ will be denoted by Aut $(\mathscr{H})$.

A *polygonal graph* is a pair $(\mathscr{H}, \mathscr{E})$ consisting of a graph $\mathscr{H}$ which is regular, connected and has girth m for some $m \geqq 3$, and a set $\mathscr{E}$ of m-gons of $\mathscr{H}$ such that every 2-claw of $\mathscr{H}$ is contained in an unique element of $\mathscr{E}$. If $\mathscr{E}$ is the set of all m-gons of $\mathscr{H}$, so that there is in $\mathscr{H}$ an unique m-gon on every one of its 2-claws, then we write $\mathscr{H}$ for $(\mathscr{H}, \mathscr{E})$ and call $\mathscr{H}$ a *strict polygonal graph.* If we wish to emphasize the integer m, then we call $(\mathscr{H}, \mathscr{E})$ an *m-gon-graph* (respectively a *strict m-gon-graph*). For example, if the valency is k, then with $k = 2$, an m-gon-graph is just an m-gon, while if k is arbitrary and $m = 3$ then $\mathscr{H}$ is just the complete graph on $k + 1$ vertices (or the 1-skeleton of the k-dimensional tetrahedron). For this reason we shall assume that $k > 2$ and $m > 3$. Strict m-gon-graphs with $m = 4, 5, 6, \ldots$ will also be called, respectively, *rectagraphs*, *pentagraphs*, *hexagraphs*, ...

If G is a group acting on a set Ω, then we shall denote by x^g the image of $x \in \Omega$ by an element $g \in G$. $\Omega(g) = \{x \in \Omega : x^g = x\}$ and for $H \subseteq G$, $\Omega(H) = \bigcap_{g \in H} \Omega(g)$. If $\Delta \subseteq \Omega$, $G_\Delta = \{g \in G: x^g \in \Delta$ for all $x \in \Delta\}$ is the setwise stabilizer of Δ, and $G_{[\Delta]} = \{g \in G: x^g = x$ for all $x \in \Delta\}$ is the pointwise stabilizer of Δ. If $\Delta = \{x, y, z, \ldots\}$ we also write $G_{xyz\ldots}$ for $G_{[\Delta]}$. $G_\Delta{}^\Delta$ denotes the group of permutations induced by G_Δ on Δ, so that $G_\Delta{}^\Delta \simeq G_\Delta/G_{[\Delta]}$.

If G is transitive on Ω, the *rank* of G on Ω is the number of orbits of G_x on Ω; these orbits are called the *suborbits*. Clearly $\{x\}$ is a suborbit for G_x, called a trivial suborbit. If $\Delta \subseteq \Omega - \{x\}$ is a non-trivial suborbit for G_x, then we can construct a graph $\mathscr{H} = \mathscr{H}(\Delta)$ as follows. The vertices of $\mathscr{H}$ are the elements of Ω, and $y \backsim z$ if and only if there is a $g \in G$ with $y^g = x$ and $z^g \in \Delta$. $\mathscr{H}(\Delta)$ is undirected if and only if there is a $g \in G$ with $x^g \in \Delta$ and $g^2 \in G_x$. Clearly since Δ is a suborbit, $\Delta = \Delta_1(x)$. Also $G^\Omega \leqq$ Aut $(\mathscr{H})$ and G_x is transitive on $\Delta(x)$. We call $\mathscr{H} = \mathscr{H}(\Delta)$ the *graph constructed with respect to the suborbit* Δ.

If $G \leqq$ Aut $(\mathscr{H})$ is a group of automorphisms of a graph $\mathscr{H}$ with vertex set Ω, if $x \in \Omega$ and $H \leqq G_x$, then we denote by $\Omega_x(H)$ the set $\Omega(H) \cap \Delta(x)$, i.e. $\Omega_x(H)$ is the set of vertices fixed by g at distance i from x.

In this paper Σ_n, A_n, D_n and Z_n will denote respectively, the symmetric group of degree n, the alternating group of degree n, and the dihedral and cyclic groups of order n. Q_8 is the quaternion group of order 8. By a (finite) regular nearfield we shall mean a nearfield constructed from a (finite) field as in Theorem 20.7.2 of M. Hall, Jr. [**10**].

LEMMA 2.1. *Let $\mathscr{H}$ be a connected, undirected graph (no loops or multiple edges) and suppose that every 2-claw of $\mathscr{H}$ is contained in a unique m-gon of $\mathscr{H}$. If the girth of $\mathscr{H}$ is m then $\mathscr{H}$ is a strict m-gon-graph.*

Proof. All we need to show is that $\mathscr{H}$ is regular. So let x be a vertex of $\mathscr{H}$ of valency k, where k is the maximal valency of all vertices of $\mathscr{H}$. Let

$\Delta(x) = \{y_1, \ldots, y_k\}$. Suppose that the valency of y_1 is $l < k$ and let $\Delta(y_1) = \{x = x_1, x_2, \ldots, x_l\}$.

On each of the 2-claws $(x:y_1, y_i)$, $2 \leqq i \leqq k$, there is in $\mathscr{H}$ a unique m-gon Π_i, say. Since there are $l < k$ vertices of $\mathscr{H}$ adjacent to y_1, some vertex x_j, $2 \leqq j \leqq l$, must occur in at least two of the m-gons Π_i. But then there are two m-gons in $\mathscr{H}$ on the 2-claw $(y_1:x, x_j)$, a contradiction.

Thus y_1 has valency k, and since $\mathscr{H}$ is connected, so does every vertex of $\mathscr{H}$. Thus $\mathscr{H}$ is regular.

LEMMA 2.2. *Suppose that $\mathscr{H}$ is a strict m-gon-graph on Ω and that $\mathscr{H}_1$ and $\mathscr{H}_2$ are two induced subgraphs of $\mathscr{H}$ which are also strict m-gon-graphs, on Ω_1 and Ω_2 respectively, where Ω_1, Ω_2 are subsets of Ω with $\Omega_1 \cap \Omega_2 \neq \emptyset$. Then connected components of the subgraph of $\mathscr{H}$ induced by $\Omega_1 \cap \Omega_2$ are points, edges, or strict m-gon-graphs.*

Proof. Let C be a connected component of $\Omega_1 \cap \Omega_2$. If C is not a point or an edge, then C contains 2-claws. On each 2-claw of C there is an unique m-gon Π in $\mathscr{H}$. However the 2-claw is in both $\mathscr{H}_1$ and $\mathscr{H}_2$ by definition of C, so that Π is in $\mathscr{H}_1$ and in $\mathscr{H}_2$, whence Π is in C. Thus each 2-claw of C is on a (necessarily unique) m-gon of C.

Now apply Lemma 2.1.

LEMMA 2.3. *Let $\mathscr{H}$ be a graph of girth $m \geqq 3$, and $G \leqq \mathrm{Aut}\,(\mathscr{H})$. Suppose that $(x: y, z)$ is a 2-claw in $\mathscr{H}$ and $g \in G_{xyz}$. If g fixes the vertices of an m-gon of $\mathscr{H}$ on $(x: y, z)$ setwise, then g fixes these vertices pointwise.*

Proof. This is obvious, since the girth of $\mathscr{H}$ is m.

LEMMA 2.4. *Let $\mathscr{H}$ be a strict m-gon-graph and $G \leqq \mathrm{Aut}\,(\mathscr{H})$ with $\Omega(G) \neq \emptyset$. Then connected components of the subgraph of $\mathscr{H}$ induced by $\Omega(G)$ are points, edges, or strict m-gon-graphs.*

Proof. Let C be a connected component of $\Omega(G)$ which is not a point or an edge. Then for $g \in G$, C is contained in a connected component $C(g)$ of $\Omega(g)$, and clearly $C = \bigcap_{g \in G} C(g)$.

Now by Lemma 2.3, the unique m-gon in $\mathscr{H}$ on any 2-claw of $C(g)$ is in $C(g)$. Thus by Lemma 2.1, $C(g)$ is a strict m-gon-graph whence by Lemma 2.2, $C = \bigcap_{g \in G} C(g)$ is also a strict m-gon-graph.

LEMMA 2.5. *Let $\mathscr{H}$ be a strict m-gon-graph and $G \leqq \mathrm{Aut}\,(\mathscr{H})$. For x a vertex of $\mathscr{H}$, G_x is faithful on $\Delta(x)$.*

Proof. Let $g \in G_x$ and suppose g fixes $\Delta(x)$ pointwise. We show by induction that g fixes $\Delta_n(x)$ pointwise for all $n \geqq 2$, and thus, since $\mathscr{H}$ is connected, g fixes $\mathscr{H}$.

So suppose g fixes $\Delta_i(x)$ for $i < n$. Let $y \in \Delta_n(x)$ and choose $u \in \Delta_{n-1}(x)$, $v \in \Delta_{n-2}(x)$ with $v \backsim u \backsim y$. Let $\Pi = (y, u, v, w, \ldots)$ be the unique m-gon on the 2-claw $(u: v, y)$. Then Π is also the unique m-gon on the 2-claw $(v: u, w)$

and since $w \in \Delta_i(x)$ for some $i < n$, g fixes this latter 2-claw, and hence Π, pointwise by Lemma 2.3. Thus g fixes y. Since y was arbitrary, g fixes $\Delta_n(x)$.

So g fixes every vertex of $\mathscr{H}$, and thus $g = 1$.

Remark. Lemma 2.5 is false for general (non-strict) polygonal graphs: a counterexample is given by the Petersen graph and its full automorphism group (see Section 3).

The following t-transitive version of a theorem of Jordan (see [**18**], Theorem 3.7) is given without proof.

LEMMA 2.6. *Let the group G act t-transitively on the set Ω. Let S be a Sylow subgroup of the stabilizer of some t points of Ω. Then $N_G(S)$ is t-transitive on $\Omega(S)$.*

We conclude this section by mentioning that m-gon-graphs give rise to incidence structures belonging to the diagram $\cdot\overset{(m)}{\text{——}}\cdot\overset{c}{\text{——}}\cdot$ of F. Buekenhout [**2**].

3. Examples of polygonal graphs. In all the following examples, $k > 2$ will denote the valency of the polygonal graph $(\mathscr{H}, \mathscr{E})$, and $m > 3$ its girth.

The most obvious examples of polygonal graphs are those which arise from regular solids. In particular the points and edges of the regular cube in k-dimensional real Euclidean space gives rise to a rectagraph $\mathscr{H}(k)$ of valency k on 2^k vertices, which contains, as subgraphs, the rectagraphs $\mathscr{H}(k')$ for any $2 \leqq k' \leqq k$. Aut $(\mathscr{H}(k))$ is isomorphic with the wreath product $Z_2 \wr \Sigma_k$ of order $2^k.k!$ afforded by the obvious action of Σ_k on $Z_2{}^k$.

Another example of a rectagraph of valency $k \geqq 5$ can be obtained from the above rectagraph $\mathscr{H}(k)$ by identifying antipodal points. The resulting quotient graph on 2^{k-1} vertices is a rectagraph with automorphism group isomorphic with Aut $(\mathscr{H}(k))/Z$ (Aut $(\mathscr{H}(k))$ of order $2^{k-1}.k!$

There are two pentagraphs arising from regular solids. One of valency 3 on 20 vertices consists of the points and edges of the dodecahedron. For convenience this will be called the *dodecahedral* graph. Its automorphism group is isomorphic with $A_5 \times Z_2$, and has point stabilizers isomorphic with Σ_3.

The other pentagraph consists of the points and edges of the 4-dimensional polytope known as the 120-cell (see [**6**] and [**14**]). This is a regular solid in 4-dimensional real space which has 120 dodecahedra as its 3-dimensional "faces" or "cells". The corresponding pentagraph of valency $k = 4$ has 600 vertices, and on each 3-claw contains a (unique) dodecahedral subgraph. Its automorphism group is isomorphic with $H/Z(H)$, where $H = SL(2, 5) \wr Z_2$, and has point stabilizers isomorphic with Σ_4.

Other examples of m-gon-graphs are known. With $k = 3$ and $m = 5$ we have the Petersen graph (see for example [**11**]) which is a (non-strict) 5-gon-graph on 10 points and has automorphism group isomorphic with Σ_5. For the distinguished set $\mathscr{E}$ of pentagons take any pentagon and its images under the subgroup of the automorphism group isomorphic with A_5. Examples with $k = 3$ and $m = 6, 7, 8$ and 9 exist, most of which come from regular maps (see [**5**], Chapter 8). For a more detailed discussion of these, and their groups, see [**17**].

An example of a rectagraph of valency 4 on 14 vertices is given by the incidence graph of the unique 2-(7, 4, 2) design (see for example [**4**], Theorem 4.5). This graph has automorphism group isomorphic with $PGL(2, 7)$ and the stabilizer of a vertex is isomorphic with Σ_4.

The action of $PGL(2, 11)$ on the right cosets of a subgroup isomorphic with A_5 and defining a graph with respect to the suborbit of length 5 gives an example of a rectagraph $\mathscr{H}_{11}'$ of valency 5. Similar constructions with the actions of $PSL(2, 31)$ and $PSL(2, 41)$ on right cosets of subgroups isomorphic with A_5 yield examples of a pentagraph $\mathscr{H}_{31}$ and heptagraph $\mathscr{H}_{41}$, respectively, of valency 5. It can be shown that $\mathscr{H}_{11}'$ does not contain a subgraph isomorphic with the rectagraph $\mathscr{H}(3)$, and $\mathscr{H}_{31}$ does not contain a subgraph isomorphic with the dodecahedral graph mentioned before; however it is not known whether or not $\mathscr{H}_{41}$ contains a heptagraph of valency 3 as a subgraph. The significance of this can be seen from Theorem 2. Also, Aut $(\mathscr{H}_{11}') \simeq PGL(2, 11)$, Aut $(\mathscr{H}_{31}) \simeq PSL(2, 31)$ and $|\text{Aut}\ (\mathscr{H}_{41})\colon PSL(2, 41)| \leqq 2$. In these three examples the sets $\mathscr{E}$ of simple circuits of minimal length are the fixed points of elements of order three in the actions of the respective groups on the respective cosets.

Remark. Let $q = p^n$ be a prime power, $3 \nmid q$ and $q^2 \equiv 1 \pmod{80}$. Let $G_q = PSL(2, q)$ and consider G_q acting on the set Ω of right cosets of a subgroup isomorphic with A_5 (such exists since $q \equiv \pm 1 \pmod 5$). Let $\mathscr{H}_q$ be the graph defined with respect to the suborbit of length 5 (which exists since $q \equiv \pm 1 \pmod 8$). Let l be the length of a (simple) circuit of fixed points in Ω of an element of order three in G_q. Then the following can be shown (see [**17**]):

(a) If $q \equiv (-1)^\epsilon \pmod 3$, then $l|(q - (-1)^\epsilon)/6$, $\epsilon = 0$ or 1.

(b) Define sequences $\{a_m\}$ and $\{b_m\}$ $(m \geqq 1)$ by $a_{m+2} = a_{m+1} - 4a_m$, $a_1 = 1$, $a_2 = 3$, and $b_{m+2} = b_{m+1} - 4b_m$, $b_1 = 1$, $b_2 = 1$. Then if l is odd, $p|a_{\frac{1}{2}(l+1)}$ or $p|a_{\frac{1}{2}(3l+1)}$, while if l is even, $p|b_{l/2}$ or $p|b_{3l/2}$.

Then if it can be shown that the girth of $\mathscr{H}_q$ is l, connected components of $\mathscr{H}_q$ will be l-gon-graphs of valency 5. This has not as yet been done except for $p = 11$, 31 and 41 ($l = 4$, 5 and 7, respectively).

A 5-gon-graph, which is not a pentagraph, of valency 6 can be obtained from the action of the group $G = PSL(2, 19)$ on the right cosets of an A_5 subgroup, and defining $\mathscr{H}$ with respect to a suborbit of length 6. Here the set $\mathscr{E}$ is the set of pentagons fixed pointwise by involutions of G. Another 5-gon-graph (of valency 11) can be obtained from the action of the group J_1, Janko's first simple group, on the cosets of a subgroup isomorphic with $PSL(2, 11)$, and defining the graph $\mathscr{H}$ with respect to the suborbit of length 11. The set $\mathscr{E}$ is the set of pentagons fixed pointwise by subgroups isomorphic with Σ_3. This example is discussed in more detail in [**15**] where J_1 is characterized in terms of this action.

4. Proof of theorem 1. For the remainder of this paper, we shall be assuming that $(\mathscr{H}, \mathscr{E})$ is an m-gon-graph of valency $k \geqq 3$ on a set Ω, with m odd, $m \geqq 5$, and that $G \leqq \operatorname{Aut}(\mathscr{H})$ is a group of automorphisms of $\mathscr{H}$ transitive on Ω. We also suppose that for any 2-claw $(x:y, z)$, $x, y, z \in \Omega$, every involution in G_{xyz} fixes (pointwise) the m-gon in $\mathscr{E}$ on $(x:y, z)$, but no other m-gon on $(x:y, z)$. Note that this latter hypothesis is automatically satisfied if $\mathscr{H}$ is a strict m-gon-graph (even if m is even), and in the case that G_{xyz} has no involutions we interpret this hypothesis to mean that G_{xyz} fixes the m-gon in $\mathscr{E}$ on $(x: y, z)$, and no other m-gon on $(x:y, z)$.

LEMMA 4.1. *If $\Pi \in \mathscr{E}$ is the m-gon containing $(x:y, z)$ then G_{xyz} fixes Π pointwise (and no other m-gon on $(x:y, z)$).*

Proof. Let $H = \langle t: t \text{ an involution in } G_{xyz}\rangle$. H char G_{xyz}, so G_{xyz} acts on the fixed points of H. But H fixes Π and no other m-gon on $(x:y, z)$, so G_{xyz} also fixes Π.

LEMMA 4.2. *G_x is faithful on $\Delta(x)$.*

Proof. Suppose $g \in G_x$ fixes $\Delta(x)$ pointwise. Assume g fixes $\Delta_i(x)$ for all $i < n$. We show g fixes $\Delta_n(x)$, whence by induction g fixes $\mathscr{H}$ (since $\mathscr{H}$ is connected), which implies that $g = 1$.

Take $y \in \Delta_n(x)$, $u \in \Delta_{n-1}(x)$, $v \in \Delta_{n-2}(x)$ with $v \backsim u \backsim y$. Let $\Pi = (y, u, v, w, \ldots)$ be the element of $\mathscr{E}$ on $(u:v, y)$. Then Π is also the m-gon in $\mathscr{E}$ containing $(v:u, w)$, so by Lemma 4.1, Π is fixed by G_{uvw}. But by the inductive hypothesis $g \in G_{uvw}$ because $w \in \Delta_i(x)$ for some $i \leqq n - 1$. So g fixes Π, whence g fixes y. Since y was arbitrary in $\Delta_n(x)$, this completes the proof.

LEMMA 4.3. *Let X be a 2-transitive Frobenius group, V the Frobenius kernel, σ an involutory automorphism of X such that $M^\sigma = M$ for some complement $M \leqq X$.*

(i) *If σ is inner, then σ centralizes M.*

(ii) *If σ is outer, then there is a nearfield $(N, +, \circ)$ with $V \simeq (N, +)$, $M \simeq (N - \{0\}, \circ)$ and $\sigma \in \operatorname{Aut}(N)$.*

Proof. (For properties of Frobenius groups used here see [**8**], Theorems 2.7.6 and 10.3.1).

(i) Suppose σ is inner. Then there is $x \in X$ so that conjugation by x induces σ on X. Since $X = MV$, $x = mv$, say, with $m \in M$ and $v \in V$. Since $M^\sigma = M$, we have for all $n \in M$, $n^x \in M$, so $n^x = n^{mv} = n^m(v^{n^m})^{-1}v \in M$. Thus $v^{n^m} = v$ for all $n \in M$. Thus $n^m \in C_M(v)$ for some $n \neq 1$ and so $v = 1$.

Now $\sigma^2 = 1$, so $x^2 \in Z(X) = 1$ and thus $m^2 = 1$. Hence $x = m$ is the unique involution in M, so $x \in Z(M)$ and σ centralizes M.

(ii) Suppose σ is outer. We claim that σ fixes an element $u \neq 1$ of V. If $|V|$ is even, since $V^\sigma = V$ and σ fixes $1 \in V$, σ must fix some $u \neq 1$ in V.

So suppose $|V|$ is odd and σ fixes no $u \neq 1$ in V. V is abelian (see, for example, [**10**], Section 20.7). Then σ fixes vv^σ for all $v \in V$, so $vv^\sigma = 1$, and hence $v^\sigma = v^{-1}$, for all $v \in V$. Thus for any $m \in M$, $v^{\sigma m} = (v^m)^{-1}$. Also $v^m \in V$, so $(v^m)^\sigma = (v^m)^{-1}$.

Thus $(v^m)^\sigma = v^{\sigma m^\sigma} = v^{\sigma m}$. Hence $m^\sigma m^{-1} \in C_M(v^\sigma)$, since $m^\sigma \in M$, so that for $v \neq 1$, $C_M(v^\sigma) = 1$ gives $m^\sigma = m$. This is true for all $m \in M$, so σ fixes M elementwise. Now clearly if t is the unique involution in M (which exists since $|M| = |V| - 1$ is even) then $m^t = m$ for all $m \in M$ and $v^t = v^{-1}$ for all $v \in V$, so that conjugation by t induces σ on X. This contradicts σ being outer.

So there is a $u \in V$ with $u^\sigma = u \neq 1$.

Now define multiplication $\circ$ on V as follows: $1 \circ v = v \circ 1 = 1$ for all $v \in V$. If $v_1, v_2 \in V - \{1\}$, then there are unique elements $m_1, m_2 \in M$ with $v_1 = u^{m_1}$ and $v_2 = u^{m_2}$. Define $v_1 \circ v_2 = u^{m_1 m_2}$. Clearly this makes V into a nearfield N with operations $+$ and $\circ$, where $+$ is the multiplication of V in X, 1 is the $+$ identity (so denote it by O_N), and u is the $\circ$ identity (so denote it by 1_N). Since $(u^m)^\sigma = u^{\sigma m^\sigma} = u^{m^\sigma}$, it is now an easy matter to show that σ is an automorphism of N.

LEMMA 4.4. *Let Y be a rank 3 Frobenius group contained in a 2-transitive Frobenius group X with kernel V. Let σ be an involutory automorphism of Y such that $\hat{M}^\sigma = \hat{M}$ for some complement $\hat{M} \leqq Y$. Let M be a complement in X with $\hat{M} < M$.*

(i) *If σ is induced by an inner automorphism of X, then σ centralizes $\hat{M}$.*

(ii) *If σ is not induced by an inner automorphism of X, then we again get the conclusion of Lemma* 4.3(ii).

Proof. (i) The proof of Lemma 4.3(i) goes through with minor changes.

(ii) Necessarily, $|V|$ is odd. The proof of Lemma 4.3(ii) again goes through with some minor changes to show that there is an element $u \in V$ with $u^\sigma = u \neq 1$, so we get the nearfield N as before. It remains to show that $\sigma \in \text{Aut}(N)$.

Let $S = \{u^m : m \in \hat{M}\}$, so that $|S| = \frac{1}{2}(|V| - 1)$. Suppose $1 \neq v \in V - S$. Now $1 \notin S \cup S^{-1}v$ and also $v \notin S \cup S^{-1}v$, where $S^{-1} = \{s^{-1} : s \in S\}$. Thus $|S \cup S^{-1}v| \leqq |V| - 2$ and hence $S \cap S^{-1}v \neq \emptyset$. Thus there are $s, t \in S$ such that $t = s^{-1}v$, i.e., $v = st$.

Hence for all $v \in V$, either $v = 1$, $v \in S$, or $v = st$ for $s,t \in S$. Now since $\hat{M}^\sigma = \hat{M}$, $(u^m)^\sigma = u^{m^\sigma}$ implies that S is fixed by σ, i.e. $S^\sigma = S$. Further, for $m_1, m_2 \in \hat{M}$,

$$(u^{m_1} \circ u^{m_2})^\sigma = (u^{m_1 m_2})^\sigma = u^{m_1^\sigma m_2^\sigma} = u^{m_1^\sigma} \circ u^{m_2^\sigma} = (u^{m_1})^\sigma \circ (u^{m_2})^\sigma$$

implies that $(s \circ t)^\sigma = s^\sigma \circ t^\sigma$ for all $s, t \in S$.

It is now an easy calculation to see that $(v \circ w)^\sigma = v^\sigma \circ w^\sigma$ for all $v, w \in V$ and thus $\sigma \in \text{Aut}(N)$.

LEMMA 4.5. *Let H be the multiplicative group of the field $N = GF(p^n)$ of order p^n, p a prime and $n \geqq 1$, and let $A \leqq \text{Aut}(N)$ be a subgroup of the automorphism group of N. Then H is characteristic in $G = AH$.*

Proof. We show that H is the unique cyclic subgroup of G of order $p^n - 1$. So suppose $K \neq H$ is cyclic of order $p^n - 1$, and let $K = \langle k \rangle$, with $k = \sigma h$ for some $\sigma \in A$, $h \in H$. Suppose $|\sigma| = a$, with $1 < a|n$. (The result is clear for $n = 1$.) $k^2 = \sigma h \sigma h = \sigma^2 h^\sigma h$. Similarly,

$$k^a = \sigma^a h^{\sigma^{a-1}} h^{\sigma^{a-2}} \dots h^\sigma h = h^{\sigma^{a-1}} \dots h^\sigma h \in H.$$

Thus $(k^a)^\sigma = k^a$ since H is abelian, so $k^a \in C_H(\sigma)$. But $|C_H(\sigma)| = p^{n/a} - 1$, so $k^{a(p^{n/a}-1)} = 1$, and so $a(p^{n/a} - 1) \equiv 0(p^n - 1)$. Now

$$p^n - 1 = (p^{n/a} - 1)[(p^{n/a})^{a-1} + (p^{n/a})^{a-2} + \dots + p^{n/a} + 1] > a(p^{n/a} - 1).$$

This contradiction proves the lemma.

LEMMA 4.6. *Let n be an even integer and p an odd prime. Let N be the regular nearfield of order p^n with center isomorphic to the field of $p^{n/2}$ elements. Let H be the multiplicative group of N and $A \leqq \text{Aut}(N)$ with $|A|$ odd. Then H is characteristic in $G = AH$.*

Proof. First suppose $p^n \neq 9$. Suppose that N is constructed from the field $GF(p^n)$ of order p^n. Then it can be deduced from [**13**] that $\text{Aut}(N) \simeq \text{Aut}(GF(p^n))$. Let U be the set of squares in $GF(p^n) - \{0\}$, so that $U \leqq H$ and $|H:U| = 2$. We claim that it suffices to prove that $U^\alpha \leqq H$ for any $\alpha \in \text{Aut}(G)$, for if $\alpha \in \text{Aut}(G)$ and $U^\alpha \leqq H$, suppose $H^\alpha \neq H$. Then $H, H^\alpha \lhd G$ and $U^\alpha = H \cap H^\alpha$. Further

$$|H^\alpha H:H| = |H^\alpha:H \cap H^\alpha| = 2,$$

which contradicts the fact that $|G:H| = |A|$ is odd.

Now U is a cyclic subgroup of order $(p^n - 1)/2$, so that we will be done if we show that every cyclic subgroup of G of order $|U|$ lies in H. So suppose V is a cyclic subgroup of G of order $(p^n - 1)/2$ and $V \nleqq H$. Then $V = \langle k \rangle$, say, where $k = \sigma h$, $\sigma \in A$ and $h \in H$. Suppose $|\sigma| = a$, odd with $a \geqq 3$. $k^a = h^{\sigma^{a-1}} h^{\sigma^{a-2}} \dots h^\sigma h$. Now $h \in U$ if and only if $h^{\sigma^i} \in U$ for any i. Thus

$$h^{\sigma^i} h^{\sigma^{i-1}} \in U \text{ for all } i.$$

Thus

$$k^{2a} = h^{\sigma^{a-1}} h^{\sigma^{a-2}} \dots h^\sigma h h^{\sigma^{a-1}} \dots h^\sigma h \in U.$$

Further,

$$(k^{2a})^{\sigma^2} = h^\sigma h \dots h^{\sigma^3} h^{\sigma^2} h^\sigma \dots h^{\sigma^3} h^{\sigma^2} = k^{2a}$$

since U is abelian, so $k^{2a} \in C_U(\sigma^2)$. Thus $k^{2a} \in C_U(\sigma)$, since $\langle \sigma \rangle = \langle \sigma^2 \rangle$ as a is odd. But

$$|C_H(\sigma)| = p^{n/a} - 1 \text{ and } |C_U(\sigma)| = \tfrac{1}{2}(p^{n/a} - 1),$$

since

$$(p^n - 1)/(p^{n/a} - 1) = [(p^{n/a})^{a-1} + \ldots + p^{n/a} + 1] \equiv 1 (\text{mod } 2),$$

so that $C_H(\sigma) \nleq U$.

Hence $k^{a(p^{n/a}-1)} = 1$ and so $a(p^{n/a} - 1) \equiv 0(\tfrac{1}{2}(p^n - 1))$. But

$$p^n - 1 = (p^{n/a} - 1)[(p^{n/a})^{a-1} + \ldots + p^{n/a} + 1],$$

so that since $p \geqq 3$ and $n/a \geqq 2$ we have

$$p^n - 1 \geqq (p^{n/a} - 1)[9(a - 1) + 1] = (p^{n/a} - 1)(9a - 8) > 2a(p^{n/a} - 1),$$

since $a > 2$, a contradiction. Thus $a = 1$ and $\sigma = 1$. But then $V \leqq H$ and we are done in this case.

Now suppose $p^n = 9$. Then the regular nearfield of order 9 has an automorphism of order 3 ([**7**], 5.2.2), but in this case $|H| = 8$ and so H is the characteristic Sylow 2-subgroup of $G = AH$.

Hence the lemma is proved.

Remark. It can be shown that if N is the regular nearfield of order q^2, q a prime power, with center the field of q elements, then except for the case $q = 3$, the cyclic subgroup U of order $(q^2 - 1)/2$ of the multiplicative group N^* of N is in fact the unique cyclic subgroup of N^* of order $(q^2 - 1)/2$. This is not true for $q = 3$.

COROLLARY 4.1. *With the hypotheses of Lemma* 4.5, *H is the unique subgroup of G isomorphic with H.*

Proof. This is what was proven in the proof of Lemma 4.5.

COROLLARY 4.2. *With the hypotheses of Lemma* 4.6, *H is the unique subgroup of G isomorphic with H.*

Proof. This is clear, since if $H_1 \simeq H$, then either $H_1 = H$ or from what was proven in Lemma 4.6, $|HH_1{:}H| = 2$, a contradiction.

COROLLARY 4.3. *Let N, H and A be as in the hypotheses of Lemma* 4.5 *or Lemma* 4.6. *Let V be the additive group of N. Suppose $A'H'$ is a subgroup of AHV such that $A' \simeq A$, $H' \simeq H$ and $A'H' \simeq AH$. Then H' is conjugate in AHV to H.*

Proof. Write $A = B \times C$ where $\pi(|B|) = \pi(\text{g.c.d. } (|A|, |H|))$, and $A' = B' \times C'$ where $|B'| = |B|$. Now AHV is solvable and both BH and $B'H'$ are Hall $\pi(|BH|)$ subgroups of AHV. All such are conjugate ([**8**], Theorem 6.4.1), and so $(B'H')^g = BH$, for some $g \in AHV$. But then $(H')^g \leqq BH$, so that by Corollaries 4.1 and 4.2, applied with B in place of A, $(H')^g = H$.

Remark. We mention without proof that the direct analogues of Lemma 4.5 and Corollaries 4.1 and 4.3, with H the squares of the multiplicative group of a field with p^n elements, hold true.

Let $x \in \Omega$ and $u, v \in \Delta(x)$. Let $\Pi \in \mathscr{E}$ be the unique m-gon in $\mathscr{E}$ on $(x:u, v)$. Let u', v' be the two points in Π at a distance $(m-1)/2$ from x. We now have three cases to consider.

Case 1. $PGL(2, p^n) \leqq G_x^{\Delta(x)} = G_x$, by Lemma 4.2, and G_x is 3-transitive on $\Delta(x)$. G_{xuv} fixes Π and hence u' and v', so $G_{xuv} \leqq G_{u'v'}$. Now $G_{u'v'}$ has a characteristic subgroup V of order p^n, and since no non-identity element of V fixes more than the one point v' of $\Delta(u')$, $G_{xuv} \cap V = \{1\}$ and G_{xuv} is a complement to V in $G_{u'v'}$. Also $G_{xuv} = AH$ where $A \simeq G_{xuvw}$ for some $w \in \Delta(x)$, and by Corollary 4.3, HV is a Frobenius group of order $p^n(p^n - 1)$ with H isomorphic to the multiplicative group of a field of order p^n, and A is isomorphic to a subgroup of the automorphism group of this field. Hence by Lemma 4.5, H is characteristic in G_{xuv}. There is an involution $\sigma \in G_x$ which interchanges u and v, hence normalizes G_{xuv}, and since Π is the unique m-gon on $(x:u, v)$ fixed by G_{xuv}, σ acts on Π and hence also interchanges u' and v'.

So σ normalizes H and V, and thus HV. Furthermore, unless $p^n = 2$ or 3, we can choose σ so that it inverts (but does not centralize) H (for example, we can choose $\sigma \in PSL(2, p^n)$). So if $p^n \neq 2$ or 3, then by Lemma 4.3(i) σ is not an inner automorphism of HV, whence by Lemma 4.3(ii) σ is an involutory field automorphism on a field N of p^n elements. Thus $p^n = r^2$ and therefore $\tau^\sigma = \tau^r = \tau^{-1}$ for all $\tau \in N - \{0\}$. Therefore $\tau^{r+1} = 1$ for all $\tau \in N - \{0\}$, so $(r^2 - 1)|r + 1$, from which we get $r = 2$. Thus $p^n = 4$. Hence in this case we get either $p = 2, n = 1$, or $p = 3, n = 1$, or $p = 2, n = 2$.

Thus either

$$k = 3 \text{ and } G_x = PGL(2, 2) \simeq \Sigma_3, \text{ or}$$
$$k = 4 \text{ and } G_x = PGL(2, 3) \simeq \Sigma_4, \text{ or}$$
$$k = 5 \text{ and } G_x = PGL(2, 4) \simeq A_5,$$
$$\text{or } G_x = P\Gamma L(2, 4) \simeq \Sigma_5.$$

Case 2. G_x is 3-transitive on $\Delta(x)$, but $G_x \ngeq PGL(2, p^n)$, so that n is even and p is odd. First suppose that $p^n \neq 9$. As in Case 1, we have that $G_{xuv} \leqq G_{u'v'}$, and again $G_{u'v'}$ has a characteristic subgroup V of order p^n which is complemented by G_{xuv}. Now $G_{xuv} = AH$ where $A \simeq G_{xuvw}$, and by Corollary 4.3, HV is a Frobenius group of order $p^n(p^n - 1)$ with H isomorphic to the multiplicative group of a regular nearfield of order p^n (whose center is isomorphic to the field of $p^{n/2}$ elements), and A is isomorphic to a subgroup of the automorphism group of this nearfield. (Note that $|A|$ is odd, or else G_x would contain $PGL(2, p^n)$.) Hence by Lemma 4.6, H is characteristic in G_{xuv}.

Again, there is an involution $\sigma \in G_x$ which interchanges u and v, and also u' and v'. So σ normalizes H and V, and thus HV. Furthermore σ does not centralize H. Thus by Lemma 4.3(ii), σ is an involutory nearfield automorphism on a nearfield N of p^n elements, with center $Z(N)$ isomorphic to the field F of $p^{n/2}$ elements. However we can again choose σ so that it inverts the center of H, and thus inverts the center of $N - \{0\}$ which is $F - \{0\}$, and also σ centralizes F (as F is the fixed field of σ). Thus $\tau^\sigma = \tau = \tau^{-1}$ for all $\tau \in F - \{0\}$, and so $\tau^2 = 1$ for all $\tau \in F - \{0\}$. Hence $p^{n/2} = 3$, and so $p^n = 9$, a contradiction.

Now suppose $p^n = 9$. Then we may regard G_x as the following subgroup of $P\Gamma L(2, 9)$ acting on 10 points:

$$G_x = \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \alpha\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} : ad - bc \text{ a square in } GF(9), a'd' - b'c' \right.$$

$$\left. \text{a non-square in } GF(9), \text{ and } 1 \neq \alpha \in \text{Aut }(GF(9)) \right\rangle \leqq P\Gamma L(2, 9).$$

Let $GF(9) = \{a + ib: a, b \in GF(3), i^2 = -1\}$. The squares in $GF(9)$ are $\{\pm 1, \pm i\} = S$, say, and $i^\alpha = i^3 = -i$. Define a binary operation on $GF(9)$ by

$$w \circ u = \begin{cases} wu & \text{if } u \in S, \\ w^3 u & \text{if } u \notin S. \end{cases}$$

Then $N = (GF(9), +, \circ)$ is the regular nearfield of 9 elements, and Aut $(N) \simeq \Sigma_3$ ([7], 5.2.2).

Without loss of generality,

$$G_{xuv} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \alpha\begin{pmatrix} 1 & 0 \\ 0 & a' \end{pmatrix} : a \in S, a' \notin S \right\}.$$

Let τ be the following map on N:

$$\tau: \begin{cases} 0 \mapsto 0 \\ \pm 1 \mapsto \pm 1 \\ \pm i \mapsto \pm(i - 1) \mapsto \pm(i + 1). \end{cases}$$

Then $\tau \in$ Aut (N), $|\tau| = 3$, and Aut $(N) = \langle \alpha, \tau \rangle$. For the element $\sigma \in G_x$ which interchanges u and v, and also u' and v', choose $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$\left[\alpha\begin{pmatrix} 1 & 0 \\ 0 & i + 1 \end{pmatrix}\right]^\sigma = \alpha\begin{pmatrix} i + 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \alpha\begin{pmatrix} 1 & 0 \\ 0 & i - 1 \end{pmatrix}.$$

So σ does not centralize G_{xuv}.

Further, $(i + 1)^\alpha = i^3 + 1 = -i + 1$; $(i + 1)^{\alpha\tau} = -i - 1$; and

$(i + 1)^{\alpha\tau^2} = -i$.

Thus no involution of Aut (N) agrees with σ on $\alpha\begin{pmatrix}1 & 0\\ 0 & i+1\end{pmatrix}$, contradicting Lemma 4.3. Thus Case 2 does not occur.

Case 3. $G_x = PSL(2, p^n)\ \langle\alpha\rangle$, $\langle\alpha\rangle \leqq \text{Aut}\, GF(p^n)$, and p odd. We proceed exactly as in Case 1, observing that H is isomorphic to the squares in the multiplicative group of a field of p^n elements and $A \simeq \langle\alpha\rangle$, using the remark after Corollary 4.3, and using Lemma 4.4 in place of Lemma 4.3.

Then if $p^n \neq 3$ or 5, σ inverts but does not centralize H and so

$$(\tau^2)^\sigma = (\tau^2)^{p^{n/2}} = \tau^{-2}$$

for all $\tau \in N - \{0\}$, N a field of p^n elements.

This gives $(p^n - 1)|2p^{n/2} + 2$, whence $p^n = 9$. Thus either

$k = 4$ and $G_x = PSL(2, 3) \simeq A_4$, or

$k = 6$ and $G_x = PSL(2, 5)$, or

$k = 10$ and $G_x = PSL(2, 9)$,

or $G_x = PSL(2, 9)\ \langle\alpha\rangle$, $1 \neq \alpha \in \text{Aut}\ (GF(9))$.

5. Proof of theorem 2. Suppose Theorem 2 is false and from the set of pairs $(\mathscr{H}, G)$ of polygonal graphs $\mathscr{H}$ and groups $G \leqq \text{Aut}\ (\mathscr{H})$ satisfying the hypotheses, choose a counterexample with $|\Omega|$ a minimum, and $|G|$ a minimum.

By Lemma 4.2, $k \neq 3$ and if $k = 5$ then $G_x \simeq \Sigma_5$. But then by Lemma 5.1, which follows below, if $u, v, w \in \Delta(x)$, G_{xuvw} has order 2 and on $\Omega(G_{xuvw})$ there will be a subgraph of valency 3 which is a strict m-gon-graph, contradicting the hypotheses of the theorem. Thus $k > 5$.

So choose $x \in \Omega$ and $u, v, w \in \Delta(x)$. Let $K = G_{xuvw}$. If $K = 1$, then G_x is sharply 3-transitive on $\Delta(x)$, so by [**9**], Theorem 1 applies and we get a contradiction. So we may assume that $K \neq 1$.

LEMMA 5.1. *Let L be K or a 2-subgroup of K. Then connected components of the induced subgraph of $\mathscr{H}$ whose points are $\Omega(L)$ are regular, and if such a connected component has valency $\geqq 2$ it is a strict m-gon-graph.*

Proof. Take a connected component Γ of $\Omega(L)$. From the points of Γ pick one, y say, whose valency n in Γ is maximal, and let $\{y_1, \ldots, y_n\} = \Omega_y(L)$. We claim that the valency of each y_i is n (in Γ).

Suppose on the contrary that y_1, say, has valency $l < n$ in Γ, and let $\Omega_{y_1}(L) = \{y, z_2, \ldots, z_l\}$. Since there are no triangles in $\mathscr{H}$, $(y{:}y_1, y_i)$ is a 2-claw for $2 \leqq i \leqq n$, so let Π_i be the unique element of $\mathscr{E}$ on $(y{:}y_1, y_i)$. Then L fixes each Π_i pointwise, so each Π_i is in fact in Γ. Thus the points not equal to y in Π_i $(2 \leqq i \leqq n)$ which are adjacent to y_1 must lie in $\Omega_{y_1}(L)$. Since $l < n$, some z_j $(2 \leqq j \leqq l)$ occurs in at least two of the Π_i, both of which would then contain the 2-claw $(y_1{:}y, z_j)$, which contradicts the hypotheses on the set $\mathscr{E}$. This proves the lemma.

Let $S \in \mathrm{Syl}_2(K)$ (possibly $S = 1$), and let $N = N_G(S)$. Let Γ be that connected component of the induced subgraph on $\Omega(S)$ containing x, u, v and w, so that by Lemma 5.1, Γ is a strict m-gon-graph of valency $l \geqq 4$. Since k is odd, so is l.

By Lemma 2.6, $N_x = N_{G_x}(S)$ is 3-transitive on $\Omega_x(S)$.

Now let Π_1, Π_2 be the elements of $\mathscr{E}$ on $(x: u, v)$ and $(x: u, w)$ respectively. Let v_1 and w_1 be the points in Π_1 and Π_2 other than x which are adjacent to u. Since G is transitive on ordered 3-claws, there is a $g \in G$ with $(x:u, v, w)^g = (u: x, v_1, w_1)$ and $u^g = x$, $v^g = v_1$, $w^g = w_1$. Then

$$K^g = (G_{xuvw})^g = G_{uxv_1w_1} = K,$$

by Lemma 4.1, so $g \in N(K)$. Thus by Sylow's theorem, there is $h \in K$ with $S^{gh} = S$, and thus $gh \in N_G(S)$. Hence $\Omega(S)^{gh} = \Omega(S)$ and it is clear that $(x: u, v, w)^{gh} = (u: x, v_1, w_1)$ so we may assume without loss of generality that $g \in N$ and hence $\Gamma = \Gamma^g$. Hence N is transitive on Γ as m is odd.

So by minimality of $\mathscr{H}$ and G, either (a) $l = 5$, or (b) $S = 1$.

Case (a). $l = 5$. Note that by the hypotheses of the theorem, $|\Omega_x(K)| \geqq 4$, so that $|\Omega_x(K)| = 4$ or 5. Let $\Omega_x(S) = \{u, v, w, y, z\}$. If $|\Omega_x(K)| = 4$, say $\Omega_x(K) = \{u, v, w, y\}$, then Γ has a subgraph Λ which is a strict m-gon-graph of valency 4 on $\{x, u, v, w, y\}$ by Lemma 5.1. Now $S \in \mathrm{Syl}_2(G_{xwyz})$ and

$$\{w, y, z\} \subset \Omega_x(G_{xwyz}) \subset \{u, v, w, y, z\}.$$

So some conjugate $\Lambda^k \neq \Lambda$ of Λ, $k \in G_x$, is a subgraph of Γ of valency 4, which is a strict m-gon-graph of valency 4 on $\{x, u, w, y, z\}$ or $\{x, v, w, y, z\}$. Then by Lemma 2.2, $\Lambda \cap \Lambda^k$ contains a strict m-gon-graph of valency 3, a contradiction.

So $|\Omega_x(K)| = 5$. Take $T < S$ of maximal order with respect to fixing > 5 points of $\Delta(x)$. By ([**12**], corollary to, and proof of, Theorem 1), $N_{G_x}(T)^{\Omega_x(T)} \triangleright PSL(2, 16)$, which is 3-transitive. Clearly $|\Omega_x(T)|$ is odd. So if we show that $N_G(T)$ is transitive on that connected component Γ' of the induced subgraph on $\Omega(T)$ which contains x, then the minimality of $\mathscr{H}$ and G would imply that $T = 1$.

Now $T < S \leqq G_{uxv_1w_1} = K^g$ is also of maximal order with respect to fixing > 5 points of $\Delta(u)$, for if not, there is $T_1 < S$ with $|T| < |T_1|$, and T_1 fixes > 5 points of $\Delta(u)$; then, however, $S > T_1^{g^{-1}}$, and $T_1^{g^{-1}}$ fixes > 5 points of $\Delta(x)$, contradicting the maximality of T, since $|T_1^{g^{-1}}| > |T|$. So again, by [**12**], $N_{G_u}(T)$ is 3-transitive on $\Omega_u(T)$. Thus there is an element of $N_G(T)$ taking x to v_1. In a similar way, we see that since m is odd, $N_G(T)$ is transitive on the m-gon Π_1, so that by connectivity of Γ', $N_G(T)$ is transitive on Γ'.

Hence $T = 1$ and thus all involutions of G_x fix 1 or 5 points of $\Delta(x)$.

By ([**3**], Theorem 3), either

(i) $|\Delta(x)| = k = 17$ and $|P\Gamma L(2, 16):G_x| = 1$ or 2, or
(ii) $k = 9$ and $G_x \simeq A_9$, or
(iii) $k = 7$ and $G_x \simeq \Sigma_7$.

Theorem 1 excludes (i). In (ii) and (iii), the stabilizer of 3 points fixes exactly those 3 points, whereas $|\Omega_x(K)| = 5$. Thus Case (a) does not occur.

Case (b). $S = 1$. Then involutions of G_x fix one point of $\Delta(x)$ and so by [**1**], $P\Gamma L(2, 2^j) \geqq G_x \triangleright PSL(2, 2^j)$, for some j. But again, by Theorem 1, this possibility leads to a contradiction.

This proves Theorem 2.

The following result on strict m-gon-graphs now follows immediately from Theorem 2.

COROLLARY 5.1. *Let $\mathscr{H}$ be a strict m-gon-graph, m odd, of valency k, odd, and let $G \leqq$ Aut $(\mathscr{H})$ be transitive on vertices of $\mathscr{H}$ and G_x be 3-transitive on $\Delta(x)$. If $\mathscr{H}$ contains no m-gon-graph as a subgraph of valency* 3, *then $k = 5$ and $G_x \simeq A_5$.*

REFERENCES

1. H. Bender, *Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festlässt*, J. Alg. *17* (1971), 527–554.
2. F. Buekenhout, *Diagrams for geometries and groups*, J. Comb. Th.(A) *27* (1979), 121–151.
3. ——— *Doubly transitive groups in which the maximum number of fixed points of involutions is four*, Arch. Math. *23* (1972), 362–369.
4. P. J. Cameron, *Suborbits in transitive permutation groups*, in Combinatorics, Part 3, Mathematical Centre Tracts *57*, Amsterdam (1974), 98–129.
5. H. S. M. Coxeter, and W. O. J. Moser, *Generators and relations for discrete groups* (Ergeb. Math. 14, 3rd edition, Springer, Berlin etc., 1972).
6. H. S. M. Coxeter, *Regular polytopes* (3rd edition, Dover, New York, 1973).
7. P. Dembowski, *Finite geometries* (Ergeb. Math. *44*, Springer, Berlin etc., 1968).
8. D. Gorenstein, *Finite groups* (Harper and Row, New York, Evanston and London, 1968).
9. D. Gorenstein and D. R. Hughes, *Triply transitive groups in which only the identity fixes four letters*, Ill. J. Math. *5* (1961), 486–491.
10. M. Hall, Jr., *The theory of groups* (Macmillan, New York, 1959).
11. F. Harary, *Graph theory* (Addison-Wesley, Reading, 1969).
12. W. M. Kantor, *Plane geometries associated with certain 2-transitive groups*, J. Alg. *37* (1975), 489–521.
13. H. Lüneburg, *Uber die Anzahl der Dickson'schen Fastkörper gegebener Ordnung*, Atti del Convegno di Geometria Combinatoria e sue Applicazioni, Perugia (1971), 319–322.
14. H. P. Manning, *Geometry of four dimensions* (1914, Dover, New York, 1958).
15. M. Perkel, *A characterization of J_1 in terms of its geometry*, to appear in Geom. Ded.
16. ——— *A characterization of* PSL(2, 31) *and its geometry*, to appear in Can. J. Math.
17. ——— Ph.D. Thesis (University of Michigan, Ann Arbor, Michigan, 1977).
18. H. Wielandt, *Finite permutation groups* (Academic Press, New York and London, 1964).

Wright State University,
Dayton, Ohio