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Asymptotic propagations of a nonlocal dispersal population model with shifting habitats

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This paper is concerned with the asymptotic propagations for a nonlocal dispersal population model with shifting habitats. In particular, we verify that the invading speed of the species is determined by the speed c of the shifting habitat edge and the behaviours near infinity of the species' growth rate which is nondecreasing along the positive spatial direction. In the case *where the species declines near the negative infinity*, we conclude that extinction occurs if $c > c^*(\infty)$, while $c < c^*(\infty)$, spreading happens with a leftward speed $\min\{-c, c^*(\infty)\}$ and a rightward speed $c^*(\infty)$, where $c^*(\infty)$ is the minimum KPP travelling wave speed associated with the species' growth rate at the positive infinity. The same scenario will play out for the case *where the species' growth rate is zero at negative infinity*. In the case *where the species still grows near negative infinity*, we show that the species always survives 'by moving' with the rightward spreading speed being either $c^*(\infty)$ or $c^*(-\infty)$ and the leftward spreading speed being one of $c^*(\infty)$, $c^*(-\infty)$ and $-c$, where $c^*(-\infty)$ is the minimum KPP travelling wave speed corresponding to the growth rate at the negative infinity. Finally, we give some numeric simulations and discussions to present and explain the theoretical results. Our results indicate that there may exists a solution like a two-layer wave with the propagation speeds analytically determined for such type of nonlocal dispersal equations.

Key words: Nonlocal dispersal model, asymptotic propagation, shifting habitat

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1 Introduction

In this paper, we consider the following nonlocal dispersal population model:

$$u_t(x,t) = d[J * u(x,t) - u(x,t)] + u[r(x-ct) - u(x,t)], \tag{1.1}$$

where d is the dispersal rate and $J * u$ represents the standard spatial convolution, and $J * u - u$ denotes the so-called nonlocal dispersal of individuals. Throughout this paper, we always assume

- **(J)** $J \in C(\mathbb{R}, \mathbb{R}^+)$ is symmetric with $\int_{\mathbb{R}} J(y)dy = 1$ and $\int_{\mathbb{R}} J(y)e^{\lambda y}dy < +\infty$ for any $\lambda > 0$.
- **(R)** $r(x)$ is a continuous and nondecreasing function satisfying $-\infty < r(-\infty) < r(\infty) < \infty$ and $r(\infty) > 0$.

Here the shifting environment is described by $r(x-ct)$ with c being the speed of the shifting habitat edge. Obviously, the habitat may be divided into two regions: the favourable region $\{x \in \mathbb{R} : r(x) > r_0\}$ and the unfavourable or less favourable region $\{x \in \mathbb{R} : r(x) \le r_0\}$ with some r_0 satisfying $r_0 = 0$ for $r(-\infty) < 0$ and $r_0 \in (r(-\infty), r(\infty))$ for $r(-\infty) \ge 0$. Particularly, the unfavourable or less favourable region is expanding and the favourable region is shrinking if $c > 0$. The habitat shifting phenomenon can be caused by many factors such as global climate change and the worsening of the environment due to industrialisation. As it is of significant biological meaning to find out whether species can keep pace with the habitat shifting and understand how species live and spread in such a shifting environment, much attentions have been attracted on this topic, see [1, 2, 9, 10, 14, 15, 16, 22] and reference therein.

To explore the effects of climate changes on the spatial dynamics of a species, Li et al. [22] initially considered a monotone habitat whose favourable part shrinks through time (i.e., **(R)** holds true with $r(-\infty) < 0$ and $c > 0$) and studied the following classical reaction–diffusion population model:

$$u_t(x,t) = du_{xx}(x,t) + u[r(x-ct) - u(x,t)], \quad x \in \mathbb{R}, \tag{1.2}$$

in which the main result states that, if the environment shifting speed $c > c_0 := 2\sqrt{dr(\infty)}$, then the species will die out in the long run, while in the case $0 < c < c_0$, the species will survive and spread into new territory in the direction of the moving environment with asymptotic speed c_0. Later, Hu et al. [17] obtained a similar result as that in [22] to the case $r(-\infty) = 0$. To allow a no-sign change scenario on the resource function (i.e., $r(-\infty) > 0$), Hu et al. [15] derived more comprehensive results on the asymptotic propagation behaviours for the solutions corresponding to the initial value problem of (1.2). Furthermore, by using the asymptotic annihilation features of the heat semigroup, Yi et al. [42] investigated the asymptotic propagation of asymptotical monostable type equations with shifting habitats. Very recently, Lam and Yu [19] characterised the asymptotic spreading of KPP fronts in heterogeneous shifting habitats with any number of shifting speeds by developing the method based on the theory of viscosity solutions of Hamilton–Jacobi equations. Taking the free and large-range migration of the species into consideration, Li et al. [23] and Wang and Zhao [34] studied the spatial propagation of the nonlocal dispersal equation (1.1) for the case $r(-\infty) < 0 < r(\infty)$ and $c > 0$. Regarding the case where favourable habitat is bounded and surrounded by unfavourable zone, we refer the readers to the work of [2, 5, 7, 30] for the one-dimensional and high-dimensional spaces as well as infinite cylindrical-type domains. In addition, some other factors such as Allee effect, seasonal succession and intra-species competition were considered in the framework of shifting environments, see for example [1, 6, 11, 29, 45]. For recent studies on the asymptotic propagations and forced waves for the 'shifting environment' problem, one can refer to [4, 9, 14, 20, 46] on scalar equations and [3, 27, 32, 33, 37, 43, 44] on competition/cooperative systems. Motivated by [15, 42] and as a complement to [23], we continue to study the nonlocal model (1.1) under more general assumptions **(J)** and **(R)** in contrast with [23].

The nonlocal dispersal equations (1.1) without shifting feature have been extensively investigated, one can see [8, 24, 25, 31, 35] and reference therein for travelling waves and spreading speeds. In the present paper, we focus on the asymptotic behaviours of the initial value problem of (1.1) as time goes to infinity. That is to say, we verify the conditions determining whether the species can persist and further obtain the rightward and leftward spreading speeds if the persistence happens, when the speed of the shifting habitat edge falls into different intervals. More specifically, we show that when $r(-\infty) \le 0$, the species goes extinct if $c > c^*(\infty) := \min_{\lambda>0} \frac{1}{\lambda}\left[d\left(\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right) + r(\infty)\right]$ (Theorem 2.4), and if $c < c^*(\infty)$, then the leftward spreading speed is $\min\{-c, c^*(\infty)\}$ and the rightward spreading speed is $c^*(\infty)$ (Theorem 2.5). When $r(-\infty) > 0$, we investigate the persistence and the spreading speeds by dividing the range of the habitat shifting speed c into the cases as follows: $c > c^*(\infty)$ (Theorem 2.6); $-c^*(-\infty) < c < c^*(\infty)$ (Theorem 2.7); $-c^*(\infty) < c < -c^*(-\infty)$ (Theorem 2.8); $c < -c^*(\infty)$ (Theorem 2.9), where $c^*(-\infty) := \min_{\lambda>0} \frac{1}{\lambda}\left[d\left(\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right) + r(-\infty)\right]$.

In summary, our results show that the leftward and rightward spreading speeds are determined by not only the speed c of the shifting habitat edge but also the behaviours near infinity of the species' growth rate which is nondecreasing in the positive spatial direction. We should point out that this work is a complement to [23] since we discuss the spreading properties for not only when $r(-\infty) \le 0$ without asking for the sign of shifting speed c but also when $r(-\infty) > 0$ with all $c \in \mathbb{R}$. For the case $r(-\infty) > 0$, the spreading speed is more complex to be determined due to the appearance of another threshold $c^*(-\infty)$. Particularly, for the persistence of species, we find that it won't make any differences if the unfavourable environment is not so hostile, that is $r(-\infty) = 0$ instead of $r(-\infty) < 0$. Meanwhile, by virtue of a proper truncation function introduced by [37], we remove the compacted supporting condition on the kernel $J(x)$ in our previous work [23]. The methods adopted here mainly depend on constructing several kinds of appropriate sub- and super-solutions as well as the comparison principle. However, the appearance of nonlocal diffusion and the special shifting heterogeneity make the problem (1.1) more troublesome.

The rest of this paper is organised as follows. In Section 2, we give some preliminaries and state the main results of this paper. Then we prove the spreading properties, that is, Theorems 2.5–2.9 in Section 3. Finally, in Section 4, some numeric simulations and discussions are presented to illustrate and explain the analytical results.

2 Preliminaries and main results

2.1 Preliminaries

Let $f(x, u) = u(r(x) - u)$. Then for any $0 \le u \le r(\infty)$ and $x \in \mathbb{R}$, in view of the fact $-\infty < r(-\infty) \le r(x) \le r(\infty) < \infty$, $f(x, u)$ is Lipschitz continuous in $u \in [0, r(\infty)]$, since

$$|f(x, u_1) - f(x, u_2)| = |u_1 - u_2||r(x) - (u_1 + u_2)| \le \left(\max\{|r(-\infty)|, r(\infty)\} + 2r(\infty)\right)|u_1 - u_2|.$$

Meanwhile, we can take some $\rho > 0$ such that $\rho u + f(x, u)$ is nondecreasing in $u \in [0, r(\infty)]$. Furthermore, the equation (1.1) can be rewritten as:

$$u_t(x, t) + \rho u(x, t) = d[J * u(x, t) - u(x, t)] + u[\rho + r(x - ct) - u(x, t)]. \tag{2.1}$$

Let

$$\mathcal{BC} = \{\phi \in C(\mathbb{R}) : \phi \text{ is bounded and uniformly continuous on } \mathbb{R}\}$$

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and $\mathcal{BC}_{r(\infty)} = \{\phi \in \mathcal{BC} : 0 \le \phi \le r(\infty)\}$. The solution of (2.1) or (1.1) with initial value $u(\cdot, 0) = u_0(\cdot) \in \mathcal{BC}_{r(\infty)}$ is the fixed point of the nonlinear integral equation in $C(\mathbb{R}^+, \mathcal{BC}_{r(\infty)})$

$$u(x,t) = [\mathcal{T}u(\cdot,t)](x) := [e^{-\rho t}P(t)u_0](x) + \int_0^t \left[e^{-\rho(t-s)}P(t-s)u(\cdot,s)(\rho + r(\cdot - cs) - u(\cdot,s))\right](x)ds,$$

in which $P(t)$ is the strongly continuous semigroup on $\mathcal{BC}$ generated by:

$$\begin{cases} v_t(x,t) = d[J * v(x,t) - v(x,t)], \ x \in \mathbb{R}, \ t > 0, \\ v(x,0) = \psi(x), \quad x \in \mathbb{R} \end{cases}$$

and $[P(t)\psi](x)$ is the unique solution with the form of

$$[P(t)\psi](x) = e^{-dt} \sum_{n=0}^{\infty} \frac{(dt)^n}{n!} a_n(\psi)(x),$$

where $a_0(\psi)(x) = \psi(x)$ and $a_{n+1}(\psi)(x) = \int_{\mathbb{R}} J(x-y)a_n(\psi)(y)dy$ for every nonnegative integer $n \in \mathbb{Z}$, see Weng and Zhao [36].

In addition, as [15], we introduce the following auxiliary functions:

$$\upsilon(x;\mu) = \begin{cases} e^{-\mu x}\sin(\gamma x), & 0 \le x \le \frac{\pi}{\gamma}, \\ 0, & x \in \mathbb{R}\backslash[0, \frac{\pi}{\gamma}], \end{cases} \qquad \upsilon_-(x;\mu) = \begin{cases} -e^{\mu x}\sin(\gamma x), & -\frac{\pi}{\gamma} \le x \le 0, \\ 0, & x \in \mathbb{R}\backslash[-\frac{\pi}{\gamma}, 0], \end{cases}$$

and denote the maximum point of $\upsilon(x;\mu)$ by $\sigma(\mu)$, that is, $\upsilon(\sigma(\mu);\mu) = \max_{x\in\mathbb{R}} \upsilon(x;\mu)$ with $\mu > 0$ and $\gamma > 0$. Further, $\upsilon_-(-\sigma(\mu);\mu) = \max_{x\in\mathbb{R}} \upsilon_-(x;\mu)$. Moreover, let

$$\varphi(\lambda,\gamma) = d\int_{\mathbb{R}} J(y)C(y)e^{\lambda y}\frac{\sin(\gamma y)}{\gamma}dy = d\int_0^{\infty} J(y)C(y)\left(e^{\lambda y} - e^{-\lambda y}\right)\frac{\sin(\gamma y)}{\gamma}dy,$$

where $C(x)$ is a continuous and symmetric cut-off function introduced by [37] as follows:

$$C(x) = \begin{cases} 1, & |x| \le \frac{\pi}{4\gamma}, \\ e^{\frac{\mu\pi}{4\gamma}} e^{-\mu|x|}\sin(2\gamma|x|), & \frac{\pi}{4\gamma} < |x| < \frac{\pi}{2\gamma}, \\ 0, & |x| \ge \frac{\pi}{2\gamma}. \end{cases}$$

As we can see that $\varphi(0,\gamma) = 0$ and $\varphi(\lambda,\gamma) \to d\int_0^{\infty} yJ(y)\left(e^{\lambda y} - e^{-\lambda y}\right)dy$ as $\gamma \to 0$. For $r(x) > 0$, we set

$$\phi(x;\lambda) = \frac{1}{\lambda}\left[d\left(\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right) + r(x)\right]$$

and let

$$c^*(x) = \min_{\lambda>0} \phi(x;\lambda) = \phi(x;\lambda^*(x)) \quad \text{for some} \quad \lambda^*(x) > 0.$$

Clearly, $c^*(\infty) = \min_{\lambda>0} \phi(\infty;\lambda) = \phi(\infty;\lambda^*(\infty))$ for some $\lambda^*(\infty) > 0$ and for some $\lambda^*(-\infty) > 0$, $c^*(-\infty) = \min_{\lambda>0} \phi(-\infty;\lambda) = \phi(-\infty;\lambda^*(-\infty))$ (if $r(-\infty) > 0$) . It follows from **(J)** and **(R)** that

$c^*(\infty) > c^*(-\infty)$ and $\lambda^*(\infty) > \lambda^*(-\infty)$. Moreover, we define

$$\phi_\gamma(x;\lambda) = \frac{1}{\lambda}\left[d\left(\int_{\mathbb{R}} J(y)C(y)e^{\lambda y}\cos(\gamma y)dy - 1\right) + r(x)\right]$$

and let $c^*_\gamma(x) = \min_{\lambda>0} \phi_\gamma(x;\lambda)$. It follows that $c^*_\gamma(x) \to c^*(x)$ as $\gamma \to 0$. In view of the definition of $c^*(\infty)$, we have

$$\left.\frac{\partial\phi(\infty;\lambda)}{\partial\lambda}\right|_{\lambda=\lambda^*(\infty)} = \frac{1}{(\lambda^*(\infty))^2}\left(\lambda^*(\infty)d\int_{\mathbb{R}} yJ(y)e^{\lambda^*(\infty)y}dy - \left[d\left(\int_{\mathbb{R}} J(y)e^{\lambda^*(\infty)y}dy - 1\right) + r(\infty)\right]\right) = 0.$$

Therefore,

$$d\int_{\mathbb{R}} yJ(y)e^{\lambda^*(\infty)y}dy = \frac{1}{\lambda^*(\infty)}\left[d\left(\int_{\mathbb{R}} J(y)e^{\lambda^*(\infty)y}dy - 1\right) + r(\infty)\right] = \phi(\infty;\lambda^*(\infty)) = c^*(\infty).$$

Consequently,

$$\varphi(\lambda^*(\infty),\gamma) \to c^*(\infty) \text{ as } \gamma \to 0.$$

This, together with $\varphi(0,\gamma) = 0$, implies that for any $\tilde{c} \in (0, c^*(\infty))$ and $\gamma > 0$ being sufficiently small, there exists some $\lambda_{\tilde{c}} \in (0, \lambda^*(\infty))$ such that $\varphi(\lambda_{\tilde{c}},\gamma) = \tilde{c}$. We will use this fact to choose some suitable parameters to construct appropriate sub-solutions in Lemmas 3.1–3.6.

Next, we give the definition of sub-/super-solutions and the comparison principle from [23].

Definition 2.1 ([23], Definition 2.2) $u \in C([0,T),\mathcal{BC}^+)$ with $0 < T \le \infty$ is a sub- or super-solution of (2.1), if $u(x,t) \le [\mathcal{T}u](x,t)$ or $u(x,t) \ge [\mathcal{T}u](x,t)$ for all $t \in [0,T)$ and $x \in \mathbb{R}$.

Theorem 2.2 ([23], Theorem 2.3) *Let $u_0 \in \mathcal{BC}_{r(\infty)}$. Then (2.1) admits a unique solution $u \in C(\mathbb{R}^+,\mathcal{BC}_{r(\infty)})$. Moreover, the comparison principle holds for (2.1), that is, if $u_1(x,t)$ and $u_2(x,t)$ are two solutions of (2.1) associated with initial value u_{10}, $u_{20} \in \mathcal{BC}_{r(\infty)}$, respectively, with $u_{10}(x) \le u_{20}(x)$ for all $x \in \mathbb{R}$, then $u_1(x,t) \le u_2(x,t)$ for all $t \ge 0$ and $x \in \mathbb{R}$. If we further assume that $u_{10} \not\equiv u_{20}$, then $u_1(x,t) < u_2(x,t)$ for all $t > 0$ and $x \in \mathbb{R}$.*

Corollary 2.3 ([23], Corollary 2.4) *For any sub- and super-solutions $u, v \in C(\mathbb{R}^+,\mathcal{BC}_{r(\infty)})$ of (2.1) for all $t > 0$ and $x \in \mathbb{R}$. If $u(x,0) \le v(x,0)$ for all $x \in \mathbb{R}$, then $u(x,t) \le v(x,t)$ for all $t > 0$ and $x \in \mathbb{R}$.*

2.2 The main theorems

Our main results on the persistence and spreading properties for (1.1) under (**J**) and (**R**) are summarised in the following theorems.

Theorem 2.4 *When $r(-\infty) \le 0$ and $c > c^*(\infty)$, then the solution $u(x,t,u_0)$ to the Cauchy problem of (1.1) with compactly supported initial value $u_0 \in \mathcal{BC}_{r(\infty)}$ satisfies that*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}} u(x,t,u_0) = 0.$$

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Theorem 2.5 *When $r(-\infty) \le 0$ and $c < c^*(\infty)$, then for any small $\epsilon > 0$, the solution $u(x, t, u_0)$ to the Cauchy problem of* (1.1) *with initial value $u_0 \in \mathcal{BC}_{r(\infty)}$ satisfies that*

(i) *if $u_0(x) \equiv 0$ for all sufficiently large x, then*

$$\lim_{t\to\infty} \sup_{x \ge (c^*(\infty)+\epsilon)t} u(x, t, u_0) = 0;$$

(ii) *if $u_0(x) \equiv 0$ for all sufficiently negative x and $u_0(x) < r(\infty)$ for $x \in \mathbb{R}$, then*

$$\lim_{t\to\infty} \sup_{x \le -\min\{-c+\epsilon, c^*(\infty)+\epsilon\}t} u(x, t, u_0) = 0;$$

(iii) *if $u_0(x) > 0$ on a closed interval, then for each $\epsilon \in \left(0, \frac{c^*(\infty)-c}{2}\right)$, we have*

$$\lim_{t\to\infty,\ -\min\{-c-\epsilon, c^*(\infty)-\epsilon\}t \le x \le (c^*(\infty)-\epsilon)t} |u(x, t, u_0) - r(\infty)| = 0.$$

Theorem 2.6 *When $r(-\infty) > 0$ and $c > c^*(\infty)$, then for any small $\epsilon > 0$, the solution $u(x, t, u_0)$ to the Cauchy problem of* (1.1) *with initial value $u_0 \in \mathcal{BC}_{r(\infty)}$ satisfies that*

(i) *if $u_0(x) \equiv 0$ for sufficiently large x and further $c > c^*(\infty) + \frac{\lambda^*(-\infty)(c^*(\infty)-c^*(-\infty))}{\lambda^*(\infty)-\lambda^*(-\infty)}$, then*

$$\lim_{t\to\infty} \sup_{x \ge (c^*(-\infty)+\epsilon)t} u(x, t, u_0) = 0;$$

(ii) *if $u_0(x) \equiv 0$ for all sufficiently negative x, then*

$$\lim_{t\to\infty} \sup_{x \le -(c^*(-\infty)+\epsilon)t} u(x, t, u_0) = 0;$$

(iii) *if $u_0(x) > 0$ on a closed interval, then*

$$\lim_{t\to\infty,\ -(c^*(-\infty)-\epsilon)t \le x \le (c^*(-\infty)-\epsilon)t} |u(x, t, u_0) - r(-\infty)| = 0.$$

We remark that, restricted by the current approach, the condition:

$$c > c^*(\infty) + \frac{\lambda^*(-\infty)(c^*(\infty) - c^*(-\infty))}{\lambda^*(\infty) - \lambda^*(-\infty)}$$

is needed to prove Theorem 2.6(i). This may be a technical condition. Indeed, our simulation in Figure 3(b) verifies that statement (i) holds true for any $c > c^*(\infty)$.

Theorem 2.7 *When $r(-\infty) > 0$ and $-c^*(-\infty) < c < c^*(\infty)$, then for any small $\epsilon > 0$, the solution $u(x, t, u_0)$ to the Cauchy problem of* (1.1) *with initial value $u_0 \in \mathcal{BC}_{r(\infty)}$ satisfies that*

(i) *if $u_0(x) \equiv 0$ for all sufficiently large x, then*

$$\lim_{t\to\infty} \sup_{x \ge (c^*(\infty)+\epsilon)t} u(x, t, u_0) = 0;$$

(ii) *if $u_0(x) \equiv 0$ for all sufficiently negative x, then*

$$\lim_{t\to\infty} \sup_{x \le -(c^*(-\infty)+\epsilon)t} u(x, t, u_0) = 0;$$

(iii) if $u_0(x) > 0$ *on a closed interval, we have*

$$\lim_{t\to\infty,\ (c+\epsilon)t\le x\le(c^*(\infty)-\epsilon)t} |u(x,t,u_0) - r(\infty)| = 0,$$

and

$$\lim_{t\to\infty,\ -(c^*(-\infty)-\epsilon)t\le x\le \min\{c^*(-\infty)-\epsilon,\ c-\epsilon\}t} |u(x,t,u_0) - r(-\infty)| = 0.$$

Theorem 2.8 *When* $r(-\infty) > 0$ *and* $-c^*(\infty) < c < -c^*(-\infty)$*, then for any small* $\epsilon > 0$*, the solution* $u(x,t,u_0)$ *to the Cauchy problem of* (1.1) *with initial value* $u_0 \in \mathcal{BC}_{r(\infty)}$ *satisfies that*

(i) if $u_0(x) \equiv 0$ *for all sufficiently large x, then*

$$\lim_{t\to\infty} \sup_{x\ge(c^*(\infty)+\epsilon)t} u(x,t,u_0) = 0;$$

(ii) if $u_0(x) \equiv 0$ *for all sufficiently negative x, then*

$$\lim_{t\to\infty} \sup_{x\le(c-\epsilon)t} u(x,t,u_0) = 0;$$

(iii) if $u_0(x) > 0$ *on a closed interval, then*

$$\lim_{t\to\infty,\ (c+\epsilon)t\le x\le(c^*(\infty)-\epsilon)t} |u(x,t,u_0) - r(\infty)| = 0.$$

Theorem 2.9 *When* $r(-\infty) > 0$ *and* $c < -c^*(\infty) < 0$*, then for any small* $\epsilon > 0$*, the solution* $u(x,t,u_0)$ *to the Cauchy problem of* (1.1) *with initial value* $u_0 \in \mathcal{BC}_{r(\infty)}$ *satisfies that*

(i) if $u_0(x) \equiv 0$ *for all sufficiently large x, then*

$$\lim_{t\to\infty} \sup_{x\ge(c^*(\infty)+\epsilon)t} u(x,t,u_0) = 0;$$

(ii) if $u_0(x) \equiv 0$ *for all sufficiently negative x, then*

$$\lim_{t\to\infty} \sup_{x\le-(c^*(\infty)+\epsilon)t} u(x,t,u_0) = 0;$$

(iii) if $u_0(x) > 0$ *on a closed interval, then*

$$\lim_{t\to\infty,\ -(c^*(\infty)-\epsilon)t\le x\le(c^*(\infty)-\epsilon)t} |u(x,t,u_0) - r(\infty)| = 0.$$

3 The proof of main theorems

Proof of Theorem 2.4 The case $r(-\infty) < 0$ in Theorem 2.4 follows directly from [23, Theorem 3.1]. Notice that by [16, Lemma 2.1], [23, Theorem 4.5] and [45, Theorem 1.2], we can conclude that when $r(-\infty) \le 0$, for any given $c > -c^*(\infty)$, (1.1) admits a nondecreasing forced wave $\Phi(x-ct)$ satisfying $\Phi(-\infty) = 0$ and $\Phi(\infty) = r(\infty)$. Regarding the case $r(-\infty) = 0$, we can prove Theorem 2.4 using exactly the same method as that for [23, Theorem 3.1] together with the existence result of forced waves. We omit the details here. □

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Proof of Theorem 2.5 The case $c>0$ has been proved by Li et al. [23], next we consider the case $c \le 0$.

The proof of (i). Let $\lambda_\epsilon > 0$ satisfy

$$\lambda_\epsilon \left(c^*(\infty)+\frac{\epsilon}{2}\right) = d\left(\int_{\mathbb{R}} J(y)e^{\lambda_\epsilon y}dy - 1\right) + r(\infty),$$

and denote $v(x,t) = Ae^{-\lambda_\epsilon\left(x-\left(c^*(\infty)+\frac{\epsilon}{2}\right)t\right)}$, where A is some sufficiently large constant. Then, $v(x,t)$ satisfies

$$v_t(x,t) = \lambda_\epsilon \left(c^*(\infty)+\frac{\epsilon}{2}\right)v(x,t) = \left[d\left(\int_{\mathbb{R}} J(y)e^{\lambda_\epsilon y}dy - 1\right) + r(\infty)\right]v(x,t).$$

In addition, since $u_0(x) \equiv 0$ for sufficiently large x, we can choose $A>0$ sufficiently large such that $u_0(x) \le v(x,0)$ for all $x \in \mathbb{R}$. Then the comparison principle yields that $u(x,t) \le v(x,t)$, and hence the conclusion easily follows.

The proof of (ii). If $-c^*(\infty) < c < c^*(\infty)$, then $\min\{-c+\epsilon, c^*(\infty)+\epsilon\} = -c+\epsilon$. Since $u_0(x) \equiv 0$ for all sufficiently negative x and $u_0(x) < r(\infty)$, there exists some $x_0 \in \mathbb{R}$ such that the translation of the forced wave $\Phi(x-ct+x_0)$ is a super-solution to (1.1) with $\Phi(x+x_0) \ge u_0(x)$ for all $x \in \mathbb{R}$. Then by Corollary 2.3, we know that

$$\lim_{t\to\infty} \sup_{x\le(c-\epsilon)t} u(x,t) \le \lim_{t\to\infty} \sup_{x\le(c-\epsilon)t} \Phi(x-ct+x_0) \le \lim_{t\to\infty} \Phi(-\epsilon t+x_0) = 0.$$

On the other hand, if $c \le -c^*(\infty)$, then $\min\{-c+\epsilon, c^*(\infty)+\epsilon\} = c^*(\infty)+\epsilon$. Similarly to (i), it can be shown that $u(x,t) \le v(x,t) = Ae^{\lambda_\epsilon\left(x+\left(c^*(\infty)+\frac{\epsilon}{2}\right)t\right)}$ with λ_ϵ defined in the proof of (i). This finishes the proof.

The proof of (iii) The main ingredient of this part is to construct proper sub-solutions.

Lemma 3.1 *For $-c^*(\infty) \le c < 0$ and any sufficiently small $\epsilon,\ \gamma>0$, let $\lambda_i > 0$ for $i=1,2$, such that $\varphi(\lambda_1,\gamma) = -c-\epsilon$ and $\varphi(\lambda_2,\gamma) = c^*(\infty)-\epsilon$. Then there exists some $a>0$ small enough such that for some sufficiently large $l>0$, $av_-(x-l+\varphi(\lambda_1,\gamma)t;\lambda_1)$ and $av(x-l-\varphi(\lambda_2,\gamma)t;\lambda_2)$ are continuous sub-solutions of (1.1). Furthermore, for a solution $u(x,t)$ of (1.1) with $0 \le u(x,0) \le r(\infty)$, if $u_0(x) \ge av_-(x-l;\lambda_1)$ ($u_0(x) \ge av(x-l;\lambda_2)$), then $u(x,t) \ge av_-(x-l+\varphi(\lambda_1,\gamma)t;\lambda_1)$ ($u(x,t) \ge av(x-l-\varphi(\lambda_2,\gamma)t);\lambda_2)$ for all $t>0$ and $x \in \mathbb{R}$.*

Proof Denote $w(x,t) = av_-(x-l+\varphi(\lambda_1,\gamma)t;\lambda_1)$, we are going to show

$$w_t(x,t) \le d[J*w(x,t)-w(x,t)] + w[r(x-ct)-w(x,t)]. \tag{3.1}$$

For $x < l-\varphi(\lambda_1,\gamma)t-\frac{\pi}{\gamma}$ or $x > l-\varphi(\lambda_1,\gamma)t$, $w(x,t) = av_-(x-l+\varphi(\lambda_1,\gamma)t;\lambda_1) \equiv 0$. It is sufficient to show that $w(x,t) = -ae^{\lambda_1(x-l+\varphi(\lambda_1,\gamma)t)}\sin\gamma(x-l+\varphi(\lambda_1,\gamma)t)$ is a sub-solution of (1.1) for $x \in [l-\varphi(\lambda_1,\gamma)t-\frac{\pi}{\gamma}, l-\varphi(\lambda_1,\gamma)t]$. Notice that

$$\begin{aligned} w_t(x,t) = &-a\lambda_1\varphi(\lambda_1,\gamma)e^{\lambda_1(x-l+\varphi(\lambda_1,\gamma)t)}\sin\gamma(x-l+\varphi(\lambda_1,\gamma)t) \\ &-a\gamma\varphi(\lambda_1,\gamma)e^{\lambda_1(x-l+\varphi(\lambda_1,\gamma)t)}\cos\gamma(x-l+\varphi(\lambda_1,\gamma)t) \end{aligned}$$

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and

$$\begin{aligned}
&\int_{\mathbb{R}} J(y)[w(x-y,t)-w(x,t)]dy \\
&\geq \int_{\mathbb{R}} J(y)C(y)w(x-y,t)dy - w(x,t) \\
&\geq -a\int_{-\frac{\pi}{2\gamma}}^{\frac{\pi}{2\gamma}} J(y)C(y)e^{\lambda_1(x-y-l+\varphi(\lambda_1,\gamma)t)} \sin\gamma(x-y-l+\varphi(\lambda_1,\gamma)t)dy \\
&\quad + ae^{\lambda_1(x-l+\varphi(\lambda_1,\gamma)t)} \sin\gamma(x-l+\varphi(\lambda_1,\gamma)t) \\
&= -a\int_{-\frac{\pi}{2\gamma}}^{\frac{\pi}{2\gamma}} J(y)C(y)e^{\lambda_1(x-y-l+\varphi(\lambda_1,\gamma)t)}\big[\sin\gamma(x-l+\varphi(\lambda_1,\gamma)t)\cos\gamma y \\
&\quad -\cos\gamma(x-l+\varphi(\lambda_1,\gamma)t)\sin\gamma y\big]dy \\
&\quad + ae^{\lambda_1(x-l+\varphi(\lambda_1,\gamma)t)} \sin\gamma(x-l+\varphi(\lambda_1,\gamma)t).
\end{aligned}$$

Then (3.1) holds true provided that

$$\begin{aligned}
&-\lambda_1\varphi(\lambda_1,\gamma)\sin\gamma(x-l+\varphi(\lambda_1,\gamma)t) \\
&\leq d\int_{\mathbb{R}} J(y)C(y)e^{-\lambda_1 y}(-1)\big[\sin\gamma(x-l+\varphi(\lambda_1,\gamma)t)\cos\gamma y - \cos\gamma(x-l+\varphi(\lambda_1,\gamma)t)\sin\gamma y\big]dy \\
&\quad + \gamma\varphi(\lambda_1,\gamma)\cos\gamma(x-l+\varphi(\lambda_1,\gamma)t) + d\sin\gamma(x-l+\varphi(\lambda_1,\gamma)t) \\
&\quad - \sin\gamma(x-l+\varphi(\lambda_1,\gamma)t)(r(x-ct)-w(x,t)) \\
&= -\sin\gamma(x-l+\varphi(\lambda_1,\gamma)t)\left[d\left(\int_{\mathbb{R}} J(y)C(y)\cos\gamma y e^{-\lambda_1 y}dy - 1\right) + r(x-ct) - w(x,t)\right] \\
&\quad + \left[d\int_{\mathbb{R}} J(y)C(y)e^{-\lambda_1 y}\sin\gamma y dy + \gamma\varphi(\lambda_1,\gamma)\right]\cos\gamma(x-l+\varphi(\lambda_1,\gamma)t) \\
&= -\sin\gamma(x-l+\varphi(\lambda_1,\gamma)t)\left[d\left(\int_{\mathbb{R}} J(y)C(y)\cos\gamma y e^{\lambda_1 y}dy - 1\right) + r(x-ct) - w(x,t)\right],
\end{aligned}$$

in which we use the facts that $J(y) = J(-y)$ and $C(y) = C(-y)$. This is equivalent to

$$\lambda_1\varphi(\lambda_1,\gamma) \leq d\left(\int_{\mathbb{R}} J(y)C(y)\cos\gamma y e^{\lambda_1 y}dy - 1\right) + r(x-ct) - w(x,t).$$

Since $x \geq l - \varphi(\lambda_1,\gamma)t - \frac{\pi}{\gamma}$, then $x-(c+\epsilon)t \geq l - \frac{\pi}{\gamma}$. Note that $r(x)$ is a nondecreasing function, we only need to verify that

$$\lambda_1\varphi(\lambda_1,\gamma) \leq d\left(\int_{\mathbb{R}} J(y)C(y)\cos\gamma y e^{\lambda_1 y}dy - 1\right) + r\left(l - \frac{\pi}{\gamma}\right) - a. \tag{3.2}$$

Recall that

$$d\left(\int_{\mathbb{R}} J(y)C(y)e^{\lambda_1 y}\cos\gamma y dy - 1\right) + r\left(l - \frac{\pi}{\gamma}\right) = \lambda_1\phi_\gamma\left(l - \frac{\pi}{\gamma};\lambda_1\right).$$

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Thus, we can obtain (3.2) by choosing $a < \lambda_1 \left(\phi_\gamma \left(l - \frac{\pi}{\gamma}; \lambda_1\right) - \varphi(\lambda_1, \gamma)\right)$. Indeed, since $0 < -c \le c^*(\infty)$, there exists $\delta > 0$ such that $-c - \epsilon < c^*(\infty) - 5\delta$. Let $\gamma > 0$ be sufficiently small and l be sufficiently large such that

$$\phi_\gamma \left(l - \frac{\pi}{\gamma}; \lambda_1\right) \ge c^*_\gamma \left(l - \frac{\pi}{\gamma}\right) \ge c^*_\gamma(\infty) - 2\delta \ge c^*(\infty) - 3\delta.$$

It then follows that (3.2) and further (3.1) hold true by choosing $a < \lambda_1(c^*(\infty) - 3\delta - c^*(\infty) + 5\delta) = 2\delta\lambda_1$, which, in return, shows that for sufficiently small $a > 0$ and sufficiently large $l > 0$, $w(x, t) = a\upsilon_-(x - l + \varphi(\lambda_1, \gamma)t; \lambda_1)$ is a continuous sub-solution of (1.1). Furthermore, if $u_0(x) \ge a\upsilon_-(x - l; \lambda_1)$, then it follows from Corollary 2.3 that $u(x, t) \ge a\upsilon_-(x - l + \varphi(\lambda_1, \gamma)t; \lambda_1)$ for all $t > 0$ and $x \in \mathbb{R}$.

Similar to the discussion above, we can also show $a\upsilon(x - l - \varphi(\lambda_2, \gamma)t; \lambda_2)$ is a continuous sub-solution of (1.1). □

Lemma 3.2 *For $c < -c^*(\infty)$ and any sufficiently small ϵ, $\gamma > 0$, let $\lambda_2 > 0$ such that $\varphi(\lambda_2, \gamma) = c^*(\infty) - \epsilon$. Then there exists some $a > 0$ small enough such that for some sufficiently large $l > 0$, $a\upsilon_-(x - l + \varphi(\lambda_2, \gamma)t; \lambda_2)$ and $a\upsilon(x - l - \varphi(\lambda_2, \gamma)t; \lambda_2)$ are continuous sub-solutions of (1.1). Furthermore, for a solution $u(x, t)$ of (1.1) with $0 \le u(x, 0) \le r(\infty)$, if $u_0(x) \ge a\upsilon_-(x - l; \lambda_2)$ ($u_0(x) \ge a\upsilon(x - l; \lambda_2)$), then $u(x, t) \ge a\upsilon_-(x - l + \varphi(\lambda_2, \gamma)t; \lambda_2)$ ($u(x, t) \ge a\upsilon(x - l - \varphi(\lambda_2, \gamma)t; \lambda_2)$) for all $t > 0$ and $x \in \mathbb{R}$.*

Proof In view of the proof of Lemma 3.1, we only need to show that

$$\lambda_2\varphi(\lambda_2, \gamma) \le d\left(\int_{\mathbb{R}} J(y)C(y)\cos\gamma y e^{\lambda_2 y} dy - 1\right) + r(x - ct) - a.$$

By the definition of $\phi_\gamma(x; \lambda)$ and the facts that $x > l - \varphi(\lambda_2, \gamma)t - \frac{\pi}{\gamma}$ and that $r(x)$ is nondecreasing, the inequality above can be obtained by choosing $a > 0$ small enough such that

$$a < \lambda_2 \left(\phi_\gamma \left(l - \frac{\pi}{\gamma}; \lambda_2\right) - \varphi(\lambda_2, \gamma)\right). \tag{3.3}$$

In fact, since $\gamma > 0$ is sufficiently small and $l > 0$ is sufficiently large such that

$$\phi_\gamma \left(l - \frac{\pi}{\gamma}; \lambda_2\right) \ge c^*_\gamma \left(l - \frac{\pi}{\gamma}\right) > c^*_\gamma(\infty) - \frac{\epsilon}{4} > c^*(\infty) - \frac{\epsilon}{2},$$

we have (3.3) by choosing $a < \frac{\lambda_2\epsilon}{2}$. Similar process as that for Lemma 3.1, we finish the proof of Lemma 3.2. □

Now we are ready to finish the proof of Theorem 2.5. Since when $0 < c < c^*(\infty)$, Theorem 2.5(iii) has been proved in [23, Theorem 3.3 (iii)] and we further notice that the conclusion in [23, Theorem 3.3(iii)] as well as its proof is also valid for $c = 0$, we only discuss the case $c < 0$ in the following. Define the function $\mathrm{w}(x, t; \alpha, \lambda^1, \lambda^2)$ by

$$
\mathrm{w}(x,t;\alpha,\lambda^1,\lambda^2)=\begin{cases}\alpha_1^-\upsilon_-(x+\varphi(\lambda^1,\gamma)t;\lambda^1), & -\varphi(\lambda^1,\gamma)t-\dfrac{\pi}{\gamma}\le x\le -\sigma(\lambda^1)-\varphi(\lambda^1,\gamma)t,\\ \alpha, & -\sigma(\lambda^1)-\varphi(\lambda^1,\gamma)t\le x\le \dfrac{3\pi}{\gamma}+\sigma(\lambda^2)+\varphi(\lambda^2,\gamma)t,\\ \alpha_2\upsilon\left(x-\dfrac{3\pi}{\gamma}-\varphi(\lambda^2,\gamma)t;\lambda^2\right), & \dfrac{3\pi}{\gamma}+\sigma(\lambda^2)+\varphi(\lambda^2,\gamma)t\le x\\ & \qquad\le \dfrac{4\pi}{\gamma}+\varphi(\lambda^2,\gamma)t,\\ 0, & \text{elsewhere},\end{cases} \tag{3.4}
$$

where $\alpha>0$ and

$$
\alpha_1^-=\frac{\alpha}{\upsilon_-(-\sigma(\lambda^1);\lambda^1)},\qquad \alpha_2=\frac{\alpha}{\upsilon(\sigma(\lambda^2);\lambda^2)}.
$$

Next, we shall show the solution $u(x,t)$ of (1.1) satisfies $u(x,t)\ge \mathrm{w}(x-l,t-t_0;\alpha,\lambda^1,\lambda^2)$ for some $t_0>0$ and some sufficiently large $l>0$. We divide the proof into two cases.

Case (a): $0<-c<c^*(\infty)$**.** Let $\lambda^1=\lambda_1,\ \lambda^2=\lambda_2$ and $\varphi(\lambda_1,\gamma)=-c-\epsilon$ and $\varphi(\lambda_2,\gamma)=c^*(\infty)-\epsilon$. Then it follows from Lemma 3.1 that

$$
u(x,t)\ge a\upsilon_-(x-l+\varphi(\lambda_1,\gamma)t;\lambda_1),\ u(x,t)\ge a\upsilon(x-l-\varphi(\lambda_2,\gamma)t;\lambda_2),\quad \forall t>0,\ x\in\mathbb{R}.
$$

Since $u_0(x)>0$ on a closed interval, it follows that $u(x,t)>0$ for all $t>0$ and $x\in\mathbb{R}$. Choose $0<t_0<\frac{\sigma(\lambda_1)}{\varphi(\lambda_1,\gamma)}$, $\alpha>0$ and $0<\gamma<1$ such that $u(x,t_0)\ge\alpha$ for $x\in\left[l-\frac{\pi}{\gamma},l+\frac{4\pi}{\gamma}\right]$. Then for any $0\le s\le\frac{3\pi}{\gamma}$ and $x\in\mathbb{R}$, we claim that

$$
\begin{cases}\mathrm{w}(x-l,0;\alpha,\lambda_1,\lambda_2)\ge\alpha_1^-\upsilon_-(x-l-s;\lambda_1),\\ \mathrm{w}(x-l,0;\alpha,\lambda_1,\lambda_2)\ge\alpha_2\upsilon\left(x-l-\dfrac{3\pi}{\gamma}+s;\lambda_2\right).\end{cases} \tag{3.5}
$$

Since the two inequalities in (3.5) can be proved in a similar way, we only show the first inequality:

$$
\mathrm{w}(x-l,0;\alpha,\lambda_1,\lambda_2)\ge\alpha_1^-\upsilon_-(x-l-s;\lambda_1). \tag{3.6}
$$

Indeed, for $x\in[l-\frac{\pi}{\gamma},l-\sigma(\lambda_1)]$, $\mathrm{w}(x-l,0;\alpha,\lambda_1,\lambda_2)=\alpha_1^-\upsilon_-(x-l;\lambda_1)$ and $x-l-s\in[-\frac{4\pi}{\gamma},-\sigma(\lambda_1)]$. Note that $\alpha_1^-\upsilon_-(y;\lambda_1)$ is nondecreasing in $(-\infty,\sigma(\lambda_1)]$. Then $\alpha_1^-\upsilon_-(x-l;\lambda_1)\ge\alpha_1^-\upsilon_-(x-l-s;\lambda_1)$, which yields that (3.6) holds true; for $x\in[l-\sigma(\lambda_1),l+\frac{3\pi}{\gamma}+\sigma(\lambda_2)]$, $\mathrm{w}(x-l,0;\alpha,\lambda_1,\lambda_2)=\alpha$. Since $\upsilon_-(-\sigma(\lambda_1);\lambda_1)$ is the maximum of $\upsilon_-(\cdot;\lambda_1)$ on $\mathbb{R}$, we have $\frac{\upsilon_-(x-l-s;\lambda_1)}{\upsilon_-(-\sigma(\lambda_1);\lambda_1)}\le 1$, and hence, (3.6) holds true; For $x\in[l+\frac{3\pi}{\gamma}+\sigma(\lambda_2),l+\frac{4\pi}{\gamma}]$, $x-l-s\in[\sigma(\lambda_2),\frac{4\pi}{\gamma}]$ and $\mathrm{w}(x-l,0;\alpha,\lambda_1,\lambda_2)=\alpha_2\upsilon_-(x-l;\lambda_2)\ge 0$. Then $\upsilon_-(x-l-s;\lambda_1)=0$, and hence, (3.6) holds; for $x<l-\frac{\pi}{\gamma}$ or $x>l+\frac{4\pi}{\gamma}$, we have $x-l-s<-\frac{\pi}{\gamma}$ or $x-l-s>\frac{\pi}{\gamma}$, then $\mathrm{w}(x-l,0;\alpha,\lambda_1,\lambda_2)=0$. It follows that $\upsilon_-(x-l-s;\lambda_1)=0$, which yields (3.6). Therefore, we have shown the claim.

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Furthermore, by virtue of (3.5) and Lemma 3.1, we have

$$\begin{cases} u(x,t) \geq \alpha_1^- \upsilon_-(x - l + \varphi(\lambda_1,\gamma)(t-t_0) - s; \lambda_1), \\ u(x,t) \geq \alpha_2 \upsilon\left(x - l - \frac{3\pi}{\gamma} - \varphi(\lambda_2,\gamma)(t-t_0) + s; \lambda_2\right). \end{cases} \tag{3.7}$$

Then for $t \geq t_0$, it follows from (3.7) that

$$u(x,t) \geq \begin{cases} \alpha_1^- \upsilon_-(x - l + \varphi(\lambda_1,\gamma)(t-t_0); \lambda_1), & l - \varphi(\lambda_1,\gamma)t - \frac{\pi}{\gamma} \leq x \\ & \leq l - \sigma(\lambda_1) - \varphi(\lambda_1,\gamma)(t-t_0), \\ \alpha, & l - \sigma(\lambda_1) - \varphi(\lambda_1,\gamma)(t-t_0) \leq x \leq l + \frac{3\pi}{\gamma} - \sigma(\lambda_1) - \varphi(\lambda_1,\gamma)(t-t_0), \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$u(x,t) \geq \begin{cases} \alpha, & l + \sigma(\lambda_2) + \varphi(\lambda_2,\gamma)(t-t_0) \leq x \leq l + \frac{3\pi}{\gamma} + \sigma(\lambda_2) + \varphi(\lambda_2,\gamma)(t-t_0), \\ \alpha_2 \upsilon\left(x - l - \frac{3\pi}{\gamma} - \varphi(\lambda_2,\gamma)(t-t_0); \lambda_2\right), & l + \frac{3\pi}{\gamma} + \sigma(\lambda_2) + \varphi(\lambda_2,\gamma)(t-t_0) \\ & \leq x \leq l + \frac{4\pi}{\gamma} + \sigma(\lambda_2) + \varphi(\lambda_2,\gamma)(t-t_0), \\ 0, & \text{elsewhere}. \end{cases}$$

Now set

$$h = \frac{\frac{3\pi}{\gamma} - \sigma(\lambda_1) - \sigma(\lambda_2)}{\varphi(\lambda_1,\gamma) + \varphi(\lambda_2,\gamma)} > 0,$$

we then have

$$l + \frac{3\pi}{\gamma} - \sigma(\lambda_1) - \varphi(\lambda_1,\gamma)(t-t_0) \geq l + \sigma(\lambda_2) + \varphi(\lambda_2,\gamma)(t-t_0), \ \forall t_0 \leq t \leq t_0 + h.$$

This yields that $u(x,t) \geq \mathrm{w}(x-l, t-t_0; \alpha, \lambda_1, \lambda_2)$ for $t_0 \leq t \leq t_0 + h$ and $x \in \mathbb{R}$. Now assume that, for $t_0 \leq t \leq t_0 + nh$ with any positive integer $n \in \mathbb{Z}$, the above inequality holds true. Then

$$\begin{cases} \mathrm{w}(x-l, nh; \alpha, \lambda_1, \lambda_2) \geq \alpha_1^- \upsilon_-(x - l + nh\varphi(\lambda_1,\gamma) - s; \lambda_1), \\ \mathrm{w}(x-l, nh; \alpha, \lambda_1, \lambda_2) \geq \alpha_2 \upsilon\left(x - l - \frac{3\pi}{\gamma} - nh\varphi(\lambda_2,\gamma) + s; \lambda_2\right), \end{cases}$$

where $0 \leq s \leq \frac{3\pi}{\gamma} + (\varphi(\lambda_1,\gamma) + \varphi(\lambda_2,\gamma))nh$. Therefore, in view of the choice of h, we have

$$\begin{cases} u(x,t) \geq \alpha_1^- \upsilon_-(x - l + nh\varphi(\lambda_1,\gamma) + \varphi(\lambda_1,\gamma)(t-(t_0+nh)) - s; \lambda_1), \\ u(x,t) \geq \alpha_2 \upsilon\left(x - l - \frac{3\pi}{\gamma} - nh\varphi(\lambda_2,\gamma) - \varphi(\lambda_2,\gamma)(t-(t_0+nh)) + s; \lambda_2\right). \end{cases} \tag{3.8}$$

This and the range of values for s indicate that $u(x,t) \geq \alpha$ for

$$t_0 + nh \leq t \leq t_0 + nh + \frac{\frac{3\pi}{\gamma} - \sigma(\lambda_1) - \sigma(\lambda_2)}{\varphi(\lambda_1,\gamma) + \varphi(\lambda_2,\gamma)} = t_0 + (n+1)h.$$

We hence obtain $u(x,t) \geq \mathrm{w}(x-l, t-t_0; \alpha, \lambda_1, \lambda_2)$ for $t_0 + nh \leq t \leq t_0 + (n+1)h$ and $x \in \mathbb{R}$. By induction, $u(x,t) \geq \mathrm{w}(x-l, t-t_0; \alpha, \lambda_1, \lambda_2)$ holds true for all $t \geq t_0$ and $x \in \mathbb{R}$.

Assume $u(x, t_0)$ is the initial value and $u(x,t)$ is the solution of (1.1). Let

$$r^* = \sup \left\{ r' \in [0, \infty) : \text{ there exist } t' > 0,\ \frac{\epsilon}{2} < \epsilon' < \epsilon \text{ such that } u(x,t) \geq r' \right.$$
$$\left. \text{for } t \geq t' \text{ and } -(-c-\epsilon')t \leq x \leq (c^*(\infty) - \epsilon')t \right\}.$$

Clearly, $r^* \geq \alpha$. If $r^* \geq r(\infty)$, then the proof is finished. Otherwise, suppose that $0 < r^* < r(\infty)$ and take $r' \in (0, r^*)$ such that $r'(1+2\gamma) \geq (1+\gamma)r^*$. By the definition of r^*, there exist $t' > 0$ and $\frac{\epsilon}{2} < \epsilon' < \epsilon$ with $u(x,t) \geq r'$ for $t \geq t'$ and $-(-c-\epsilon)t \leq x \leq (c^*(\infty) - \epsilon)t$.

For any $\tilde{\epsilon} \in (\epsilon', \epsilon)$, let $\tilde{t} > t'$ be sufficiently large such that

$$\int_0^{\tilde{t}-t'} e^{-\rho(\tilde{t}-t'-s)} ds \geq \frac{1}{\rho(1+\gamma)}$$

and for $t \geq \tilde{t}$ as well as $-(-c-\tilde{\epsilon})t \leq x \leq (c^*(\infty) - \tilde{\epsilon})t$, there holds $u(x,t) \geq r'$, then for $t \geq \tilde{t}$,

$$\begin{aligned}
u(x,t) \geq & \int_0^{t-t'} \left[e^{-\rho(t-t'-s)} P(t-t'-s) u(\cdot, s+t'))(\rho + r(\cdot - c(s+t')) - u(\cdot, s+t')) \right](x) ds \\
\geq & \int_0^{t-t'} e^{-\rho(t-t'-s)} P(t-t'-s) r'(\rho + r(\infty) - r') ds \\
= & r'(\rho + r(\infty) - r') \int_0^{t-t'} e^{-\rho(t-t'-s)} ds \\
\geq & r'(\rho + r(\infty) - r') \frac{1}{(1+\gamma)\rho} \\
\geq & \frac{r^*}{1+2\gamma} \left(1 + \frac{r(\infty) - r^*}{\rho} \right),
\end{aligned}$$

where $r' \in (0, r^*)$ satisfying $r'(1+2\gamma) \geq (1+\gamma)r^*$. This gives that for $\gamma < \frac{r(\infty)-r^*}{3\rho}$, there holds

$$u(x,t) \geq \frac{1+3\gamma}{1+2\gamma} r^* > r^*.$$

Then we reach a contradiction against the choice of r^*.

Case (b): $-c \geq c^*(\infty)$**.** The proof of such case is similar to that for Case (a) with $\lambda^1 = \lambda^2 := \lambda_2$ and we omit the details here. The proof is complete. □

In order to show Theorems 2.6–2.9, we first construct some appropriate sub-solutions in the following lemmas. Starting at the current position, we always suppose that $r(-\infty) > 0$.

Lemma 3.3 *For any sufficiently small* $\epsilon,\ \gamma > 0$, *choosing* $\lambda_3 > 0$ *such that* $\varphi(\lambda_3, \gamma) = c^*(-\infty) - \epsilon$, *there exists some* $a > 0$ *small enough such that for any real number* l, *the functions* $a\upsilon_-(x - l + \varphi(\lambda_3, \gamma)t; \lambda_3)$ *and* $a\upsilon(x - l - \varphi(\lambda_3, \gamma)t; \lambda_3)$ *are continuous sub-solutions of* (1.1). *Furthermore,*

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for a solution $u(x,t)$ *of* (1.1) *with* $0 \le u(x,0) \le r(\infty)$, *if* $u_0(x) \ge a\upsilon_-(x-l;\lambda_3)$ $(u_0(x) \ge a\upsilon(x-l;\lambda_3))$, *then*

$$u(x,t) \ge a\upsilon_-(x-l+\varphi(\lambda_3,\gamma)t;\lambda_3)\ (u(x,t) \ge a\upsilon(x-l-\varphi(\lambda_3,\gamma)t;\lambda_3))$$

for all $t > 0$ *and* $x \in \mathbb{R}$.

Proof Denote $\omega_-(x,t) = a\upsilon_-(x-l+\varphi(\lambda_3,\gamma)t;\lambda_3)$. For $x > l-\varphi(\lambda_3,\gamma)t$ or $x < l-\varphi(\lambda_3,\gamma)t-\frac{\pi}{\gamma}$, $\omega_-(x,t) = a\upsilon_-(x-l+\varphi(\lambda_3,\gamma)t;\lambda_3) \equiv 0$, then the conclusion is natural. We only need to show

$$\frac{\partial\omega_-(x,t)}{\partial t} \le d[J*\omega_-(x,t)-\omega_-(x,t)]+\omega_-(x,t)[r(x-ct)-\omega_-(x,t)]$$

for $x \in [l-\varphi(\lambda_3\gamma)t-\frac{\pi}{\gamma}, l-\varphi(\lambda_3\gamma)t]$ and $t > 0$. Similar to the proof of Lemma 3.1, it is sufficient to prove

$$\lambda_3\varphi(\lambda_3,\gamma) \le d\left(\int_{\mathbb{R}} J(y)C(y)e^{\lambda_3 y}\cos\gamma y dy - 1\right) + r(x-ct) - a. \tag{3.9}$$

Based on the fact that $r(x-ct) \ge r(-\infty)$ and the definition of $\phi_\gamma(x;\lambda)$, we find that (3.9) holds true if $a > 0$ satisfies that

$$a < \lambda_3(\phi_\gamma(-\infty;\lambda_3)-\varphi(\lambda_3,\gamma)). \tag{3.10}$$

In fact, since $\gamma > 0$ is sufficiently small such that

$$\phi_\gamma(-\infty;\lambda_3) \ge c^*(-\infty)-\frac{\epsilon}{2}$$

and $\varphi(\lambda_3,\gamma) = c^*(-\infty)-\epsilon$, we can obtain (3.10), and hence (3.9), by choosing $a > 0$ such that

$$a < \lambda_3\left(c^*(-\infty)-\frac{\epsilon}{2}-c^*(-\infty)+\epsilon\right) = \frac{\lambda_3\epsilon}{2}.$$

This implies that for any $a > 0$ small enough and $l \in \mathbb{R}$, $a\upsilon_-(x-l+\varphi(\lambda_3,\gamma)t;\lambda_3)$ is a continuous sub-solution of (1.1). Similarly, we show $a\upsilon(x-l-\varphi(\lambda_3,\gamma)t;\lambda_3)$ is a continuous sub-solution of (1.1) for any sufficiently small $a > 0$. Furthermore, by Corollary 2.3, if $u_0(x) \ge a\upsilon_-(x-l;\lambda_3)$ $(u_0(x) \ge a\upsilon(x-l;\lambda_3))$, then for all $t > 0$ and $x \in \mathbb{R}$,

$$u(x,t) \ge a\upsilon_-(x-l+\varphi(\lambda_3,\gamma)t;\lambda_3)\ (u(x,t) \ge a\upsilon(x-l-\varphi(\lambda_3,\gamma)t;\lambda_3)).$$

The proof is complete. □

Lemma 3.4 *Let* $0 \le c < c^*(\infty)$. *For any sufficiently small* $\epsilon,\ \gamma > 0$, *choose* $\lambda_4 > 0$ *and* $\lambda_2 > 0$ *such that* $\varphi(\lambda_4,\gamma) = c+\epsilon$ *and* $\varphi(\lambda_2,\gamma) = c^*(\infty)-\epsilon$. *Then there exists some* $a > 0$ *small enough such that for some sufficiently large* $l > 0$, *the function* $a\upsilon(x-l-\varphi(\lambda_i,\gamma)t;\lambda_i)$ $(i = 4,2)$ *is a continuous sub-solution of* (1.1). *Furthermore, for a solution* $u(x,t)$ *of* (1.1) *with* $0 \le u(x,0) \le r(\infty)$, *if* $u(x,0) \ge a\upsilon(x-l;\lambda_i)$ $(i = 4,2)$, *then* $u(x,t) \ge a\upsilon(x-l-\varphi(\lambda_i,\gamma)t;\lambda_i)$ $(i = 4,2)$ *for all* $t > 0$ *and* $x \in \mathbb{R}$.

Proof We can easily conclude from the proof of Lemma 3.1 that it is sufficient to show

$$\lambda_i\varphi(\lambda_i,\gamma)\le d\left(\int_{\mathbb{R}}J(y)C(y)\cos(\gamma y)e^{\lambda_i y}dy-1\right)+r(x-ct)-a,\quad i=2,4.$$

Here we only show the case $i=4$, since the case $i=2$ follows directly from the proofs of Lemmas 3.1 and 3.2. By same arguments as that for Lemmas 3.1–3.3, it follows that we just need to explain

$$a<\lambda_4(\phi_\gamma(l;\lambda_4)-\varphi(\lambda_4,\gamma)). \tag{3.11}$$

In fact, by letting $l>0$ large enough and γ small enough such that $c^*(l)\ge c^*(\infty)-\frac{\epsilon}{4}$ and $c^*_\gamma(l)\ge c^*(l)-\frac{\epsilon}{4}$, we have $\phi_\gamma(l;\lambda_4)\ge c^*(\infty)-\frac{\epsilon}{2}$. This, together with the fact $\varphi(\lambda_4,\gamma)=c+\epsilon<c^*(\infty)-\epsilon$, implies that (3.11) holds true as long as we select $a>0$ such that

$$a<\lambda_4\left(c^*(\infty)-\frac{\epsilon}{2}-c^*(\infty)+\epsilon\right)=\frac{\lambda_4\epsilon}{2}.$$

The remaining proof is similar to that of Lemmas 3.1–3.3, we omit it here. The proof is finished. □

Next, we give another two lemmas, the proofs of which directly follow from Lemmas 3.1–3.4.

Lemma 3.5 *Let $-c^*(-\infty)<c<0$. For any sufficiently small ϵ, $\gamma>0$, choose λ_i for $i=1,2,3,5$ such that $\varphi(\lambda_1,\gamma)=-c-\epsilon$, $\varphi(\lambda_2,\gamma)=c^*(\infty)-\epsilon$, $\varphi(\lambda_3,\gamma)=c^*(-\infty)-\epsilon$ and $\varphi(\lambda_5,\gamma)=-c+\epsilon$. Then there exists some $a>0$ small enough such that for sufficiently large $l>0$, the function $av(x-l-\varphi(\lambda_2,\gamma)t;\lambda_2)$ is a continuous sub-solution of (1.1). For any $\bar{l}\in\mathbb{R}$, $av_-(x-\bar{l}+\varphi(\lambda_i,\gamma)t;\lambda_i)$ $(i=1,3,5)$ is a continuous sub-solution of (1.1). Furthermore, for a solution $u(x,t)$ of (1.1) with $0\le u(x,0)\le r(\infty)$, if $u(x,0)\ge av(x-l;\lambda_2)$ $(u(x,0)\ge av_-(x-\bar{l};\lambda_i)$ $(i=1,3,5))$, then $u(x,t)\ge av(x-l-\varphi(\lambda_2,\gamma)t;\lambda_2)$ $(u(x,t)\ge av_-(x-\bar{l}+\varphi(\bar{\lambda}_i,\gamma)t;\lambda_i)$ $(i=1,3,5))$ for all $t>0$ and $x\in\mathbb{R}$.*

Lemma 3.6 *Let $-c^*(\infty)<c<-c^*(-\infty)$. For any sufficiently small ϵ, $\gamma>0$, let $\lambda_i>0$ for $i=1,2$ such that $\varphi(\lambda_1,\gamma)=-c-\epsilon$ and $\varphi(\lambda_2,\gamma)=c^*(\infty)-\epsilon$. Then there exists some $a>0$ small enough such that for sufficiently large $l>0$, $av_-(x-l+\varphi(\lambda_1,\gamma)t;\lambda_1)$ and $av(x-l-\varphi(\lambda_2,\gamma)t;\lambda_2)$ are continuous sub-solutions of (1.1). Furthermore, for a solution $u(x,t)$ of (1.1) with $0\le u(x,0)\le r(\infty)$, if $u(x,0)\ge av_-(x-l;\lambda_1)$ $(u(x,0)\ge av(x-l;\lambda_2))$, then $u(x,t)\ge av_-(x-l+\varphi(\lambda_1,\gamma)t;\lambda_1)$ $(u(x,t)\ge av(x-l-\varphi(\lambda_2,\gamma)t;\lambda_2))$ for $t>0$ and $x\in\mathbb{R}$.*

Based on the above sub-solutions, we further define the following functions. Let

$$\mathrm{w}_r(x,t;\alpha,\lambda^1,\lambda^2)=\begin{cases}\alpha_1 v(x-\varphi(\lambda^1,\gamma)t;\lambda^1), & \varphi(\lambda^1,\gamma)t\le x\le\sigma(\lambda^1)+\varphi(\lambda^1,\gamma)t,\\ \alpha, & \sigma(\lambda^1)+\varphi(\lambda^1,\gamma)t\le x\le\dfrac{3\pi}{\gamma}+\sigma(\lambda^2)+\varphi(\lambda^2,\gamma)t,\\ \alpha_2 v\left(x-\dfrac{3\pi}{\gamma}-\varphi(\lambda^2,\gamma)t;\lambda^2\right), & \dfrac{3\pi}{\gamma}+\sigma(\lambda^2)+\varphi(\lambda^2,\gamma)t\le x\\ & \le\dfrac{4\pi}{\gamma}+\varphi(\lambda^2,\gamma)t,\\ 0, & \text{elsewhere,}\end{cases}$$

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where $\alpha > 0$ and

$$\alpha_1 = \frac{\alpha}{\upsilon(\sigma(\lambda^1);\lambda^1)}, \quad \alpha_2 = \frac{\alpha}{\upsilon(\sigma(\lambda^2);\lambda^2)}.$$

Notice that for

$$x \in \left[\sigma(\lambda^1) + \varphi(\lambda^1,\gamma)t, \frac{3\pi}{\gamma} + \sigma(\lambda^2) + \varphi(\lambda^2,\gamma)t\right],$$

$\mathrm{w}_r(x,t;\alpha,\lambda^1,\lambda^2) = \alpha$ with the end points shifting rightward at speeds $\varphi(\lambda^1,\gamma)$ and $\varphi(\lambda^2,\gamma)$ as $t \to \infty$. Let $\mathrm{w}(x,t;\alpha,\lambda^1,\lambda^2)$ be defined as (3.4), we see that for

$$x \in \left[-\sigma(\lambda^1) - \varphi(\lambda^1,\gamma)t, \frac{3\pi}{\gamma} + \sigma(\lambda^2) + \varphi(\lambda^2,\gamma)t\right],$$

$\mathrm{w}(x,t;\alpha,\lambda^1,\lambda^2) = \alpha$ with the left end point shifting leftward at speed $\varphi(\lambda^1,\gamma)$ and the right end point shifting rightward at speed $\varphi(\lambda^2,\gamma)$ as $t \to \infty$. Let

$$\mathrm{w}_l(x,t;\alpha,\lambda^1,\lambda^2) = \begin{cases} \alpha_1^- \upsilon_-(x + \varphi(\lambda^1,\gamma)t;\lambda^1), & -\varphi(\lambda^1,\gamma)t - \dfrac{\pi}{\gamma} \le x \le -\sigma(\lambda^1) - \varphi(\lambda^1,\gamma)t, \\ \alpha, & -\sigma(\lambda^1) - \varphi(\lambda^1,\gamma)t \le x \le \dfrac{3\pi}{\gamma} - \sigma(\lambda^2) - \varphi(\lambda^2,\gamma)t, \\ \alpha_2^- \upsilon_-\left(x - \dfrac{3\pi}{\gamma} + \varphi(\lambda^2,\gamma)t;\lambda^2\right), & \dfrac{3\pi}{\gamma} - \sigma(\lambda^2) - \varphi(\lambda^2,\gamma)t \le x \\ & \le \dfrac{3\pi}{\gamma} - \varphi(\lambda^2,\gamma)t, \\ 0, \quad \text{elsewhere}, \end{cases}$$

where $\alpha > 0$ and

$$\alpha_1^- = \frac{\alpha}{\upsilon_-(-\sigma(\lambda^1);\lambda^1)}, \quad \alpha_2^- = \frac{\alpha}{\upsilon_-(-\sigma(\lambda^2);\lambda^2)}.$$

Clearly, for

$$x \in \left[-\sigma(\lambda^1) - \varphi(\lambda^1,\gamma)t, \frac{3\pi}{\gamma} - \sigma(\lambda^2) - \varphi(\lambda^2,\gamma)t\right],$$

$\mathrm{w}_l(x,t;\alpha,\lambda^1,\lambda^2) = \alpha$ with the end points shifting leftward at speeds $\varphi(\lambda^1,\gamma)$ and $\varphi(\lambda^2,\gamma)$, respectively, as $t \to \infty$.

The following lemma shows that the proper translations of $\mathrm{w}_r(x,t;\alpha,\lambda^1,\lambda^2)$, $\mathrm{w}(x,t;\alpha,\lambda^1,\lambda^2)$, and $\mathrm{w}_l(x,t;\alpha,\lambda^1,\lambda^2)$ are sub-solutions of (1.1) as time evolves. It is observed that the size of the region for the density function $u(x,t) \ge \alpha$ is increasing linearly with respect to t under appropriate conditions on α, γ, λ^1 and λ^2.

Lemma 3.7 *Let $u(x,t)$ be a solution of (1.1) with $u(\cdot,0) \in \mathcal{BC}_{r(\infty)}$ and $u(x,0) > 0$ on a closed interval. Then for any small positive $\epsilon > 0$, there exist α, γ and $t_0 > 0$ such that the following conclusions hold true:*

(i) If $c \ge c^*(\infty)$, then for any real number l, we have $u(x,t) \ge \mathrm{w}(x-l, t-t_0;\alpha,\lambda_3,\lambda_3)$ for all $t > t_0$ and $x \in \mathbb{R}$, where $\lambda_3 > 0$ satisfies $\varphi(\lambda_3,\gamma) = c^*(-\infty) - \epsilon$.

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(ii) *If $c^*(-\infty) \le c < c^*(\infty)$, then for some sufficiently large $l > 0$, we have $u(x, t) \ge \mathrm{w}_r(x - l, t - t_0; \alpha, \lambda_4, \lambda_2)$ for all $t > t_0$ and $x \in \mathbb{R}$, where $\lambda_4, \lambda_2 > 0$ satisfy $\varphi(\lambda_4, \gamma) = c + \epsilon$ and $\varphi(\lambda_2, \gamma) = c^*(\infty) - \epsilon$, and for any $l' > 0$, we have $u(x, t) \ge \mathrm{w}(x - l', t - t_0; \alpha, \lambda_3, \lambda_3)$ for all $t > t_0$ and $x \in \mathbb{R}$, where $\lambda_3 > 0$ satisfies $\varphi(\lambda_3, \gamma) = c^*(-\infty) - \epsilon$.*

(iii) *If $0 < c < c^*(-\infty)$, then for some large $l > 0$, $u(x, t) \ge \mathrm{w}_r(x - l, t - t_0; \alpha, \lambda_4, \lambda_2)$ for all $t > t_0$ and $x \in \mathbb{R}$, where $\lambda_4, \lambda_2 > 0$ satisfy $\varphi(\lambda_4, \gamma) = c + \epsilon$ and $\varphi(\lambda_2, \gamma) = c^*(\infty) - \epsilon$, and for any $l' > 0$, $u(x, t) \ge \mathrm{w}(x - l', t - t_0; \alpha, \lambda_3, \lambda_6)$ for all $t > t_0$ and $x \in \mathbb{R}$, where $\lambda_3, \lambda_6 > 0$ satisfy $\varphi(\lambda_3, \gamma) = c^*(-\infty) - \epsilon$ and $\varphi(\lambda_6, \gamma) = c - \epsilon$. The first part of this statement is valid for $c = 0$.*

(iv) *If $-c^*(-\infty) < c < 0$, then for some large $l > 0$, we have $u(x, t) \ge \mathrm{w}(x - l, t - t_0; \alpha, \lambda_1, \lambda_2)$ for all $t > t_0$ and $x \in \mathbb{R}$, where $\lambda_1, \lambda_2 > 0$ satisfy $\varphi(\lambda_1, \gamma) = -c - \epsilon$ and $\varphi(\lambda_2, \gamma) = c^*(\infty) - \epsilon$, and for any $l' > 0$, we have $u(x, t) \ge \mathrm{w}_l(x - l', t - t_0; \alpha, \lambda_3, \lambda_5)$ for all $t > t_0$ and $x \in \mathbb{R}$, where $\lambda_3, \lambda_5 > 0$ satisfy $\varphi(\lambda_3, \gamma) = c^*(-\infty) - \epsilon$ and $\varphi(\lambda_5, \gamma) = -c + \epsilon$. The first part of this statement is valid for $c = 0$.*

(v) *If $c^*(-\infty) \le -c < c^*(\infty)$, $u(x, t) \ge \mathrm{w}(x - l, t - t_0; \alpha, \lambda_1, \lambda_2)$ for all $t > t_0$ and $x \in \mathbb{R}$, where $\lambda_1, \lambda_2 > 0$ satisfy $\varphi(\lambda_1, \gamma) = -c - \epsilon$ and $\varphi(\lambda_2, \gamma) = c^*(\infty) - \epsilon$.*

(vi) *If $c \le -c^*(\infty)$, then for large l, we have $u(x, t) \ge \mathrm{w}(x - l, t - t_0; \alpha, \lambda_2, \lambda_2)$ for all $t > t_0$ and $x \in \mathbb{R}$, where $\lambda_2 > 0$ satisfying $\varphi(\lambda_2, \gamma) = c^*(\infty) - \epsilon$.*

Proof It follows from Lemmas 3.1–3.6 that $u(x, t) \ge aV(x; \lambda)$ with $V(x; \lambda) = \upsilon(x - l - \varphi(\lambda, \gamma)t; \lambda)$ or $\upsilon_-(x - l + \varphi(\lambda, \gamma)t; \lambda)$, where λ can be λ_i for $i = 1, 2, 3, 4, 5$. As for $\lambda = \lambda_6 = c - \epsilon$ with $0 < c < c^*(-\infty)$, the same arguments as that for Lemma 3.4 with $\lambda_4 = c + \epsilon$ replaced by $\lambda_6 = c - \epsilon$ can be applied to obtain that $u(x, t) \ge a\upsilon(x - l - \varphi(\lambda_6, \gamma)t; \lambda)$. By a similar argument to that for Theorem 2.5(iii), we can obtain that $u(x, t) \ge w(x - l, t - t_0; \alpha, \lambda^1, \lambda^2)$, $u(x, t) \ge w_r(x - l, t - t_0; \alpha, \lambda^1, \lambda^2)$ and $u(x, t) \ge w_l(x - l, t - t_0; \alpha, \lambda^1, \lambda^2)$, where the parameters λ^1, λ^2 and l are chosen properly as in Lemma 3.7. We omit the details here. □

Now we are in the position to prove Theorems 2.6–2.9.

Proof of Theorem 2.6 We start with the following lemma.

Lemma 3.8 *Let $u(x, t)$ be the solution of (1.1) with $u(\cdot, 0) \in \mathcal{BC}_{r(\infty)}$.*

(i) *If $u(x, 0) \equiv 0$ for all sufficiently large x, then for any small $\epsilon > 0$, there exist some $A > 0$ and $\lambda_\epsilon > 0$ such that*

$$u(x, t) \le Ae^{-\lambda_\epsilon\left(x - \left(c^*(\infty) + \frac{\epsilon}{2}\right)t\right)}.$$

(ii) *There exist $K_1, K_2 > 0$ and sufficiently negative s satisfying that for any small $\epsilon > 0$, one can choose λ'_ϵ such that $\phi(s; \lambda'_\epsilon) = c^*(s) + \frac{\epsilon}{2}$, furthermore, for $c > \hat{c} = \frac{\lambda_\epsilon c^*(\infty) - \lambda'_\epsilon c^*(s)}{\lambda_\epsilon - \lambda'_\epsilon}$,*

$$\breve{u}(x, t) = \min\left\{K_1 e^{-\lambda_\epsilon\left(x - \left(c^*(\infty) + \frac{\epsilon}{2}\right)t\right)},\ K_2 e^{-\lambda'_\epsilon\left(x - \left(c^*(s) + \frac{\epsilon}{2}\right)t\right)}\right\}$$

is a super-solution of (1.1), where λ_ϵ is defined in the proof of Theorem 2.5(i).

(iii) *Assume that $c > -c^*(-\infty)$ and $u(x, 0) \equiv 0$ for sufficiently negative x. Then there exist some $B > 0$ and $\lambda^\delta_\epsilon > 0$ such that for any small $\epsilon > 0$, there holds*

$$u(x, t) \le Be^{\lambda^\delta_\epsilon (x + c^*(-\infty) + \epsilon)t}.$$

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Proof (i). The proof is similar to that of Theorem 2.5(i), so we omit it here.

(ii). Since s is sufficiently negative, it follows that $c^*(s) < c^*(\infty)$. Recall that in Section 2.1, we derive that $c^*(x) = d\int_{\mathbb{R}} yJ(y)e^{\lambda^*(x)y}dy$, and hence, $\lambda^*(s) < \lambda^*(\infty)$. Note that $\phi(s;\lambda'_\epsilon) = c^*(s) + \frac{\epsilon}{2}$ and $\phi(\infty;\lambda_\epsilon) = c^*(\infty) + \frac{\epsilon}{2}$. Thus, $\lambda_\epsilon > \lambda'_\epsilon$ for any $\epsilon > 0$ small enough. Denote

$$Q(x,t) = \frac{K_2 e^{-\lambda'_\epsilon\left(x-\left(c^*(s)+\frac{\epsilon}{2}\right)t\right)}}{K_1 e^{-\lambda_\epsilon\left(x-\left(c^*(\infty)+\frac{\epsilon}{2}\right)t\right)}}.$$

By a direct calculation, we have

$$Q(x,t) = \frac{K_2}{K_1}\exp\left\{(\lambda_\epsilon - \lambda'_\epsilon)\left[x - \left(\frac{\lambda_\epsilon c^*(\infty) - \lambda'_\epsilon c^*(s)}{\lambda_\epsilon - \lambda'_\epsilon} + \frac{\epsilon}{2}\right)t\right]\right\}.$$

Recall that $c > \hat{c}$, we have

$$Q(x,t) \geq \frac{K_2}{K_1}\exp\{(\lambda_\epsilon - \lambda'_\epsilon)(x - ct)\}.$$

Choose $K_1,\ K_2 > 0$ properly such that $Q(x,t) \geq 1$ for $x - ct \geq s$. Then

$$\breve{u}(x,t) = K_1 e^{-\lambda_\epsilon\left(x-\left(c^*(\infty)+\frac{\epsilon}{2}\right)t\right)}$$

for $x - ct \geq s$. By the proof of (i), we know that

$$\breve{u}_t(x,t) \geq d(J * \breve{u}(x,t) - \breve{u}(x,t)) + \breve{u}(x,t)(r(x-ct) - \breve{u}(x,t)) \tag{3.12}$$

for $x - ct \geq s$. Meanwhile, for $x - ct < s$, $\breve{u}(x,t) = K_2 e^{-\lambda'_\epsilon\left(x-\left(c^*(s)+\frac{\epsilon}{2}\right)t\right)}$ satisfies that

$$\breve{u}_t(x,t) - d(J * \breve{u}(x,t) - \breve{u}(x,t)) = r(s)\breve{u}(x,t) \geq \breve{u}(r(x-ct) - \breve{u}).$$

This implies that statement (ii) holds true.

(iii). Rewrite that

$$c^*(-\infty) = \lim_{\delta\to 0}\min_{\lambda>0}\frac{d\int_{\mathbb{R}} J(y)e^{-\lambda y}dy - d + r(-\infty) + \delta}{\lambda}.$$

Therefore, for any $\epsilon > 0$, there exist $\delta > 0$ and $0 < \lambda_\epsilon^\delta < \lambda^*(-\infty)$ such that

$$c^*(-\infty) + \epsilon = \frac{d\int_{\mathbb{R}} J(y)e^{-\lambda_\epsilon^\delta y}dy - d + r(-\infty) + \delta}{\lambda_\epsilon^\delta}. \tag{3.13}$$

Indeed, let λ_ϵ^δ be the smaller positive root of (3.13). Then it is not difficult to find that $\lim_{\delta\to 0}\lambda_\epsilon^\delta = \lambda_\epsilon^-$ with $\lambda_\epsilon^- < \lambda^*(-\infty)$ being the smaller positive root of

$$c^*(-\infty) + \epsilon = \frac{d\int_{\mathbb{R}} J(y)e^{-\lambda_\epsilon^- y}dy - d + r(-\infty)}{\lambda_\epsilon^-}.$$

It follows that we can find $\delta > 0$ being sufficiently small such that $\lambda_\epsilon^\delta < \frac{\lambda_\epsilon^- + \lambda^*(-\infty)}{2} < \lambda^*(-\infty)$.

Since $r(x)$ is continuous and nondecreasing in $x \in \mathbb{R}$, there exists x_1 such that if $x < x_1$, then

$$r(x) \leq r(-\infty) + \delta.$$

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It follows that for any $t > 0$, if $x < x_1 + ct$ then

$$r(x - ct) \le r(-\infty) + \delta.$$

Let $B > 0$ be sufficiently large such that $Be^{\lambda_\epsilon^\delta x_1} \ge r(\infty)$ and $u(x, 0) \le Be^{\lambda_\epsilon^\delta x}$ for all $x \in \mathbb{R}$. Moreover, denote $w^+(x, t) = Be^{\lambda_\epsilon^\delta (x+(c^*(-\infty)+\epsilon)t)}$. It then follows that

$$\begin{aligned}
&w_t^+(x,t) - d(J * w^+(x,t) - w^+(x,t)) - w^+(r(x-ct) - w^+)\\
=&w^+\left[\lambda_\epsilon^\delta(c^*(-\infty)+\epsilon) - d\left(\int_{\mathbb{R}} J(y)e^{-\lambda_\epsilon^\delta y}dy - 1\right) - r(x-ct) + w^+\right]\\
=&w^+\left(r(-\infty) + \delta - r(x-ct) + Be^{\lambda_\epsilon^\delta (x+(c^*(-\infty)+\epsilon)t)}\right)\\
\ge&0
\end{aligned}$$

for any $t > 0$ and $x < x_1 + ct$. On the other hand, for $x \ge x_1 + ct$, we know that

$$Be^{\lambda_\epsilon^\delta (x+(c^*(-\infty)+\epsilon)t)} \ge Be^{\lambda_\epsilon^\delta (x_1+ct+(c^*(-\infty)+\epsilon)t)} \ge Be^{\lambda_\epsilon^\delta x_1} \ge r(\infty).$$

Therefore, $w^+(x, t)$ is a super-solution of (1.1). It follows from the comparison principle that $u(x, t) \le w^+(x, t)$. The proof is complete. □

Now we start to verify Theorem 2.6.

(i). By Lemma 3.8(ii), we have $u(x, t) \le \breve{u}(x, t)$. Note that $\breve{u}(x, t) = K_1 e^{-\lambda_\epsilon(x-(c^*(\infty)+\frac{\epsilon}{2})t)}$ for $x \ge ct + s$ with s being sufficiently negative. This implies that $\lim_{t\to\infty} \sup_{x\ge(c^*(\infty)+\epsilon)t} u(x, t) = 0$. Since $c > c^*(\infty)$, it follows that $(c^*(\infty) + \epsilon)t \le ct + s$ for any $s \in \mathbb{R}$ and $t > T$, where $T > 0$ satisfies that $(c - c^*(\infty) - \epsilon)T > -s$. Thus, we have

$$\lim_{t\to\infty} \sup_{x\ge ct+s} u(x, t) = 0. \tag{3.14}$$

While for $x < ct + s$, we know that $\breve{u}(x, t) = K_2 e^{-\lambda'_\epsilon(x-(c^*(s)+\frac{\epsilon}{2})t)}$. Recall that s is sufficiently negative such that $x \ge (c^*(-\infty) + \frac{\epsilon}{2})t \ge (c^*(s) + \epsilon)\, t$ with $t \ge T$, it then follows that

$$\lim_{t\to\infty} \sup_{(c^*(-\infty)+\epsilon)t\le x<ct+s} u(x, t) = 0. \tag{3.15}$$

Combining (3.14) and (3.15), we have $\lim_{t\to\infty} \sup_{x\ge(c^*(-\infty)+\epsilon)t} u(x, t, u_0) = 0$.

(ii). Statement (ii) follows from Lemma 3.8(iii) directly.

(iii). We first claim that for any $\epsilon > 0$, there exists a sufficiently large $T > 0$ such that $u(x, t) \le r(-\infty) + \epsilon$ for all $x \in \mathbb{R}$ and $t > T$. Indeed, let $\tilde{u}(\xi, t)$ be the solution of

$$u_t(\xi, t) = d(J * u - u) + cu_\xi + u(r(\xi) - u)$$

with $\tilde{u}(\xi, 0) = r(\infty)$, where $\xi = x - ct$. It follows that $\tilde{u}(\xi, t)$ is nonincreasing with respect to t. Therefore, there exists a continuously function $\tilde{u}(\xi)$ such that $\lim_{t\to\infty} \tilde{u}(\xi, t) = \tilde{u}(\xi)$. Furthermore, it follows from [34, Lemmas 3.1 and 3.2] that $\tilde{u}(\xi)$ is continuously nondecreasing in $\xi \in \mathbb{R}$ and satisfies

$$\tilde{u}(\xi)=\int_0^\infty e^{-(d+\rho)s}\sum_{k=0}^\infty \frac{(ds)^k}{k!}a_k(\tilde{u}(\xi+cs)(\rho+r(\xi+cs)-\tilde{u}(\xi+cs)))ds.$$

It follows that

$$\tilde{u}(\pm\infty)=\frac{\tilde{u}(\pm\infty)}{\rho}(\rho+r(\pm\infty)-\tilde{u}(\pm\infty)).$$

This implies that $\tilde{u}(\pm\infty)=r(\pm\infty)$. The comparison principle yields that $u(x,t)\le\tilde{u}(\xi,t)$. Then for any $\epsilon>0$, there exist $T>0$ and sufficiently large $M>0$ such that for any $t>T$, $u(x,t)\le\tilde{u}(\xi,t)\le r(-\infty)+\epsilon$ for $x\le ct-M$. Additionally, since $c>c^*(\infty)$, we have $ct-M\ge(c^*(\infty)+\epsilon)t$ for $t>T$ (we can increase T if necessary). It follows from Lemma 3.8(i) that $u(x,t)\le\epsilon$ for $x\ge ct-M$ and $t>T$. Then we have that for any $\epsilon>0$, there exists a $T>0$ such that $u(x,t)\le r(-\infty)+\epsilon$ for any $t>T$ and $x\in\mathbb{R}$.

On the other hand, according to Lemma 3.7(i), for any $\epsilon>0$, there exist positive numbers α, γ, λ_3, l and t_0 such that $u(x,t)\ge \mathrm{w}(x-l,t-t_0;\alpha,\lambda_3,\lambda_3)$ for all $t>t_0$ and $x\in\mathbb{R}$, where $\varphi(\lambda_3,\gamma)=c^*(-\infty)-\epsilon$. Then define

$$\begin{aligned}r_-^*=\sup\Big\{&r'_-\in[0,\infty):\text{there exist } t'>0,\ \frac{\epsilon}{2}<\epsilon'<\epsilon \text{ with } u(x,t)\ge r'_-\\ &\text{for } t\ge t',\ -(c^*(-\infty)-\epsilon')t\le x\le(c^*(-\infty)-\epsilon')t\Big\},\end{aligned}$$

similarly to the proof of Theorem 2.5(iii), we can show that $r_-^*=r(-\infty)$. This ends the proof. □

Proof of Theorem 2.7

(i). By Lemma 3.8(i), for any $\epsilon>0$, there exist $A,\lambda_\epsilon>0$ such that $u(x,t)\le Ae^{-\lambda_\epsilon(x-(c^*(\infty)+\frac{\epsilon}{2})t)}$. It follows that

$$\lim_{t\to\infty}\sup_{x\ge(c^*(\infty)+\epsilon)t}u(x,t)\le\lim_{t\to\infty}\sup_{x\ge(c^*(\infty)+\epsilon)t}Ae^{-\lambda_\epsilon(x-(c^*(\infty)+\frac{\epsilon}{2})t)}=0.$$

This implies that $\lim\limits_{t\to\infty}\sup\limits_{x\ge(c^*(\infty)+\epsilon)t}u(x,t)=0$.

(ii). Statement (ii) follows directly from Lemma 3.8(iii).

(iii). We use a similar argument to the proof of Theorem 2.5(iii) to obtain

$$\lim_{t\to\infty}\inf_{t(c+\epsilon)\le x\le(c^*(\infty)-\epsilon)t}u(x,t)\ge r(\infty)$$

by virtue of the sub-solution $\mathrm{w}_r(x-l,t-t_0;\alpha,\lambda_4,\lambda_2)$ with $\varphi(\lambda_4,\gamma)=c+\epsilon$ and $\varphi(\lambda_2,\gamma)=c^*(\infty)-\epsilon$ defined in the Lemma 3.7(ii) for $c\ge0$, or the sub-solution $\mathrm{w}(x-l,t-t_0;\alpha,\lambda_1,\lambda_2)$ with $\varphi(\lambda_1,\gamma)=-c-\epsilon$ and $\varphi(\lambda_2.\gamma)=c^*(\infty)-\epsilon$ defined in Lemma 3.7(v) for $c<0$. This, together with $0\le u(x,t)\le r(\infty)$, leads to the first conclusion in Theorem 2.7(iii).

By using the sub-solution $\mathrm{w}(x-l,t-t_0;\alpha,\lambda_3,\lambda_3)$ with $\varphi(\lambda_3,\gamma)=c^*(-\infty)-\epsilon$ defined in Lemma 3.7(ii) for $c^*(-\infty)\le c<c^*(\infty)$, the sub-solution $\mathrm{w}(x-l,t-t_0;\alpha,\lambda_3,\lambda_6)$ with $\varphi(\lambda_3,\gamma)=c^*(-\infty)-\epsilon$ and $\varphi(\lambda_6,\gamma)=c-\epsilon$ defined in Lemma 3.7(iii) for $0\le c<c^*(-\infty)$ or the sub-solution $\mathrm{w}_l(x-l,t-t_0;\alpha,\lambda_3,\lambda_5)$ with $\varphi(\lambda_3,\gamma)=c^*(-\infty)-\epsilon$ and $\varphi(\lambda_5,\gamma)=-c+\epsilon$ defined in Lemma 3.7(iv) for $c<0$, and following a similar process as that for Theorem 2.6(iii), we can easily obtain the latter result in Theorem 2.7(iii). □

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Proof of Theorem 2.8

(i). Statement (i) can be shown as that for Theorem 2.7(i).

(ii). Since $-c > c^*(-\infty)$, there exist $\delta > 0$ and $0 < \lambda_\delta < \lambda^*(-\infty)$ such that

$$-c\lambda_\delta = d\left(\int_{\mathbb{R}} J(y)e^{-\lambda_\delta y}dy - 1\right) + r(-\infty) + \delta.$$

Let $u_\delta(x,t) = A_\delta e^{\lambda_\delta(x-ct)}$ and $\hat{x} < 0$ such that $r(x) < r(-\infty) + \delta$, $\forall x < \hat{x}$. Choose sufficiently large $A_\delta > 0$ such that $A_\delta e^{\lambda_\delta \hat{x}} \geq r(\infty)$ and $u(x,0) \leq u_\delta(x,0) = A_\delta e^{\lambda_\delta x}$ for all $x \in \mathbb{R}$. For $x \geq ct + \hat{x}$, $u_\delta(x,t) \geq A_\delta e^{\lambda_\delta \hat{x}} \geq r(\infty) \geq u(x,t)$. As in the proof of Lemma 3.8(ii), one can show that $u_\delta(x,t)$ is a super-solution of (1.1) with $u_\delta(x,t) \geq u(x,t)$ for $t \geq 0$ and $x < ct + S$. Thus, we have

$$0 \leq \lim_{t\to\infty} \sup_{x\leq -(-c+\epsilon)t} u(x,t) \leq \lim_{t\to\infty} \sup_{x\leq -(-c+\epsilon)t} A_\delta e^{\lambda_\delta(x-ct)} = 0.$$

This shows that (ii) holds true.

(iii). By Lemma 3.7(v), we see that when $t > t_0$, $u(x,t) \geq w(x-l, t-t_0; \alpha, \lambda_1, \lambda_2)$ with $\varphi(\lambda_1, \gamma) = -c - \epsilon$ and $\varphi(\lambda_2, \gamma) = c^*(\infty) - \epsilon$. It then follows from Theorem 2.5(iii) that

$$\lim_{t\to\infty} \inf_{(c+\epsilon)t\leq x\leq (c^*(\infty)-\epsilon)t} u(x,t) \geq r(\infty).$$

This, together with $0 \leq u(x,t) \leq r(\infty)$, implies that statement (iii) holds true. The proof is complete. □

Proof of Theorem 2.9

(i). Statement (i) can be shown as that for Theorem 2.7(i).

(ii). For any $\epsilon > 0$, there exist A, $\lambda_\epsilon > 0$ such that

$$u(x,t) \leq Ae^{\lambda_\epsilon(x+(c^*(\infty)+\frac{\epsilon}{2})t)} := v(x,t).$$

In fact, let λ_ϵ satisfy

$$\left(c^*(\infty) + \frac{\epsilon}{2}\right)\lambda_\epsilon = d\left(\int_{\mathbb{R}} J(y)e^{-\lambda_\epsilon y}dy - 1\right) + r(\infty).$$

It then follows that

$$\begin{aligned} &v_t - d(J * v - v) - v(r(x-ct) - v) \\ \geq &Ae^{\lambda_\epsilon(x+(c^*(\infty)+\frac{\epsilon}{2})t)}\left[\left(c^*(\infty) + \frac{\epsilon}{2}\right)\lambda_\epsilon - d\left(\int_{\mathbb{R}} J(y)e^{-\lambda_\epsilon y}dy - 1\right) - r(\infty)\right] = 0. \end{aligned}$$

We can choose $A > 0$ large enough such that $u(x,0) \leq Ae^{\lambda_\epsilon x}$ due to $u(x,0) = 0$ for all sufficiently negative x. By the comparison principle, we have

$$0 \leq \lim_{t\to\infty} \sup_{x\leq -(c^*(\infty)+\epsilon)t} u(x,t) \leq \lim_{t\to\infty} \sup_{x\leq -(c^*(\infty)+\epsilon)t} Ae^{\lambda_\epsilon\left(x+\left(c^*(\infty)+\frac{\epsilon}{2}\right)t\right)} = 0,$$

which leads to the desired conclusion.

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(iii). By using the sub-solution $\mathrm{w}(x-l, t-t_0; \alpha, \lambda_2, \lambda_2)$ with $\varphi(\lambda_2, \gamma) = c^*(\infty) - \epsilon$ defined in Lemma 3.7(vi) and following a similar argument in the proof of Theorem 2.5(iii), we can obtain

$$\lim_{t\to\infty} \inf_{-(c^*(\infty)-\epsilon)t \le x \le (c^*(\infty)-\epsilon)t} u(x,t) \ge r(\infty),$$

which implies that statement (iii) holds true since $0 \le u(x,t) \le r(\infty)$. The proof is complete. □

4 Simulations and discussions

4.1 Simulations

In this subsection, we present some numerical simulations for model (1.1) to demonstrate our analytic results. To be computable, we choose

$$d = 1, \quad J(x) = \frac{e^{-x^2}}{\sqrt{\pi}}, \quad r_i(x-ct) = \frac{2}{\pi}\arctan(x-ct) + \kappa_i, \ i = 1, 2, 3,$$

where $\kappa_1 = \frac{1}{2}$, $\kappa_2 = 1$ and $\kappa_3 = 2$. Clearly, we have $r_1(-\infty) = -\frac{1}{2} < 0$, $r_2(-\infty) = 0$ and $r_3(-\infty) = 1 > 0$, while $r_1(\infty) = \frac{3}{2} > 0$, $r_2(\infty) = 2 > 0$ and $r_3(\infty) = 3 > 0$. Recall that

$$c_i^*(\infty) = \frac{1}{\lambda_i^*(\infty)}\left[d\left(\int_{\mathbb{R}} J(y)e^{\lambda_i^*(\infty)y}dy - 1\right) + r_i(\infty)\right], \quad i = 1, 2, 3,$$

and

$$c_3^*(-\infty) = \frac{1}{\lambda_3^*(-\infty)}\left[d\left(\int_{\mathbb{R}} J(y)e^{\lambda_3^*(-\infty)y}dy - 1\right) + r_3(-\infty)\right].$$

Moreover, we can calculate to obtain

$$\lambda_1^*(\infty) = 1.5909, \quad \lambda_2^*(\infty) = 1.7191, \quad \lambda_3^*(\infty) = 1.9023, \quad \lambda_3^*(-\infty) = 1.4142.$$
$$c_1^*(\infty) = 1.49977, \quad c_2^*(\infty) = 1.7995, \quad c_3^*(\infty) = 2.3504, \quad c_3^*(-\infty) = 1.1658.$$

We use the following initial data

$$u(x,0) = \begin{cases} -0.05x + 1, & x \in (10, 20), \\ 0.5, & x \in [-10, 10], \\ 0.05x + 1, & x \in (-20, -10), \\ 0, & \text{elsewhere}. \end{cases}$$

Numerical simulations were conducted using MATLAB.

In the case $r_1(-\infty) < 0$, we first choose $c = 1.5$ such that $c > c_1^*(\infty)$, then Theorem 2.4 declares that the species eventually becomes extinct, as presented in Figure 1(a). Choose $c = 0.5 \in (0, c_1^*(\infty))$, then Figure 1(b) indicates the species will persist and spread rightward. Further, choose $c = -0.5 \in (-c_1^*(\infty), 0)$ and $c = -1.5 < -c_1^*(\infty)$, respectively, we then see from Figure 1(c) and (d) that the species invades not only rightward but also leftward, agreeing with Theorem 2.5.

In the case $r_2(-\infty) = 0$, we first set $c = 2 > c_2^*(\infty)$ and the numeric result presented in Figure 2(a) illustrates that the species will disappear in the whole habitat, as shown in Theorem 2.4. Then

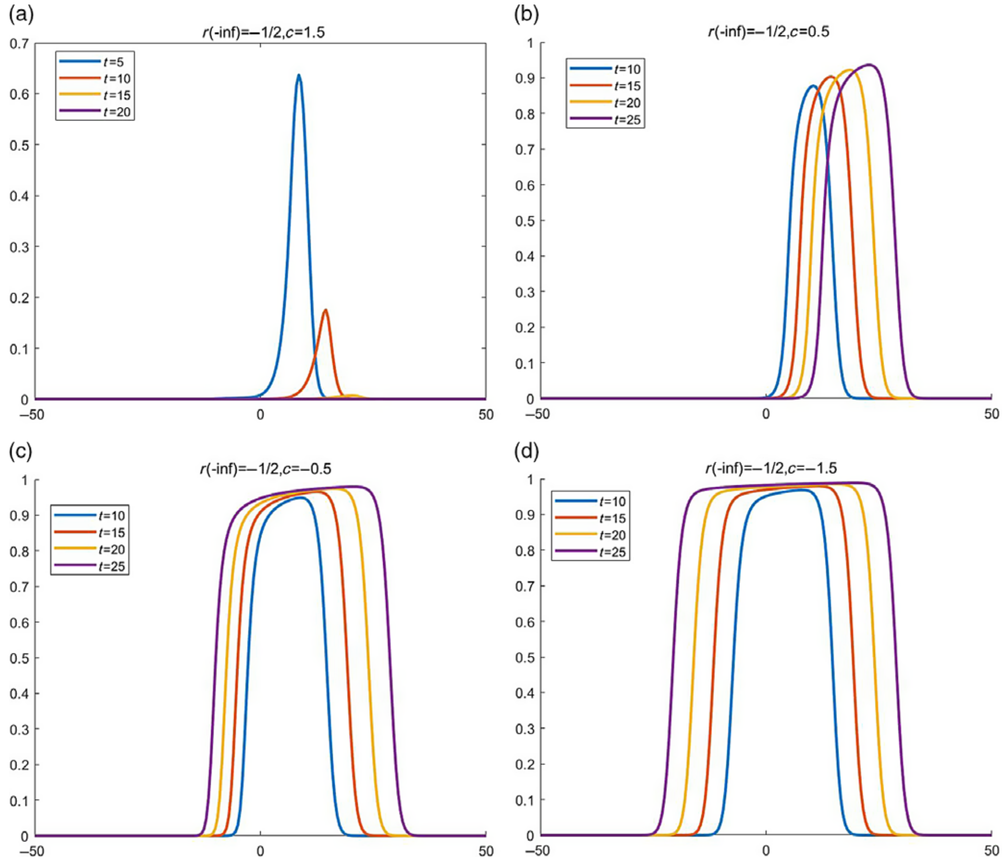

FIGURE 1. The case of $r(-\infty) < 0$. (a) $r(-\infty) < 0,\ \ c > c^*(\infty)$. (b) $r(-\infty) < 0,\ \ 0 < c < c^*(\infty)$. (c) $r(-\infty) < 0,\ \ -c^*(\infty) < c < 0$. (d) $r(-\infty) < 0,\ \ c < -c^*(\infty)$.

choose $c = 1 \in (0, c_2^*(\infty))$, Figure 2(b) indicates the species will persist and spread towards the better resource. Regarding the case $c < 0$, we set $c = -1 \in (-c_2^*(\infty), 0)$ and $c = -2 < -c_2^*(\infty)$, respectively, then Figure 2(c) and (d) show the species spreads towards both right and left, as described in Theorem 2.5.

In the case $r_3(-\infty) > 0$, we first choose $c = 6 > c_3^*(\infty) + \hat{c}$ with $\hat{c} = \frac{\lambda^*(-\infty)(c_3^*(\infty) - c^*(-\infty))}{\lambda_3^*(\infty) - \lambda^*(-\infty)} = 3.4322$, Figure 3(a) reveals that the species still spreads leftward and rightward with same speed $c^*(-\infty)$, agreeing with Theorem 2.6. Meanwhile, choose $c = 3 \in (c_3^*(\infty), c_3^*(\infty) + \hat{c})$, we find from Figure 3(b) that statement (i) of Theorem 2.6 still holds true. This indicates that Theorem 2.6 may be valid for all $c > c_3^*(\infty)$. Now set $c = 1.5 \in (c_3^*(-\infty), c_3^*(\infty))$, $c = 0.5 \in (0, c_3^*(-\infty))$ and $c = -0.5 \in (-c_3^*(-\infty), 0)$, respectively, then Theorem 2.7 demonstrates that the species spreads leftward and rightward with different speeds and the density of the species will eventually be different in the good-quality and poor-quality habitats, as presented in Figure 3(c), (d) and (e), which suggests that there may exist a two-layer wave solution. Next, we choose $c = -2 \in (-c_3^*(\infty), -c_3^*(-\infty))$ and $c = -3 < -c_3^*(\infty)$, respectively, that is the good-quality habitat expands leftward with a relatively fast speed, then by Theorems 2.8 and 2.9, we know that the species will spread both leftward (with different speeds) and rightward (with same

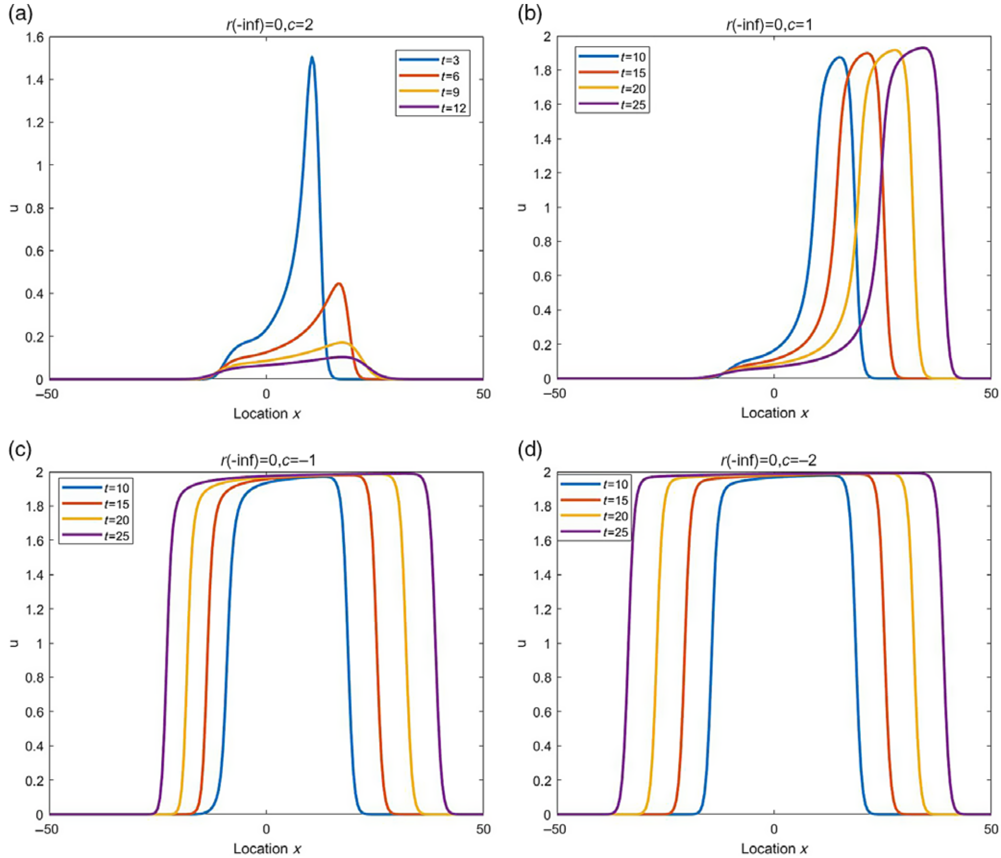

FIGURE 2. The case of $r(-\infty)=0$. (a) $r(-\infty)=0,\ c>c^*(\infty)$. (b) $r(-\infty)=0,\ 0<c<c^*(\infty)$. (c) $r(-\infty)=0,\ -c^*(\infty)<c<0$. (d) $r(-\infty)=0,\ c<-c^*(\infty)$.

speed) and grow to the capacity corresponding to the best-quality habitat, as presented in Figure 3(f) and (g).

4.2 Discussions

Nowadays, the so-called 'shifting environment' problem has become a hot topic, since its significant biological meanings and distinctive phenomena, brought by the 'shifting feature', about the persistence and spread of the species. To understand the effects of shifting habitats on spatial population dynamics, we investigated the spreading properties for solutions associated to the initial value problem of

$$u_t = d(J*u-u)+u(r(x-ct)-u).$$

Here, the convolution operator $J*u-u$ is adopted to describe the spatial dispersal of species. More specifically, if we use $J(x-y)$ to denote the probability distribution of the population jumping from location y to location x, then $\int_{\mathbb{R}} J(x-y)u(y,t)dy$ is the rate at which individuals are arriving to location x from all other places, while $\int_{\mathbb{R}} J(y-x)u(x,t)dy = u(x,t)$ is the rate at which they are leaving location x to all other sites. Obviously, migration of the species here is

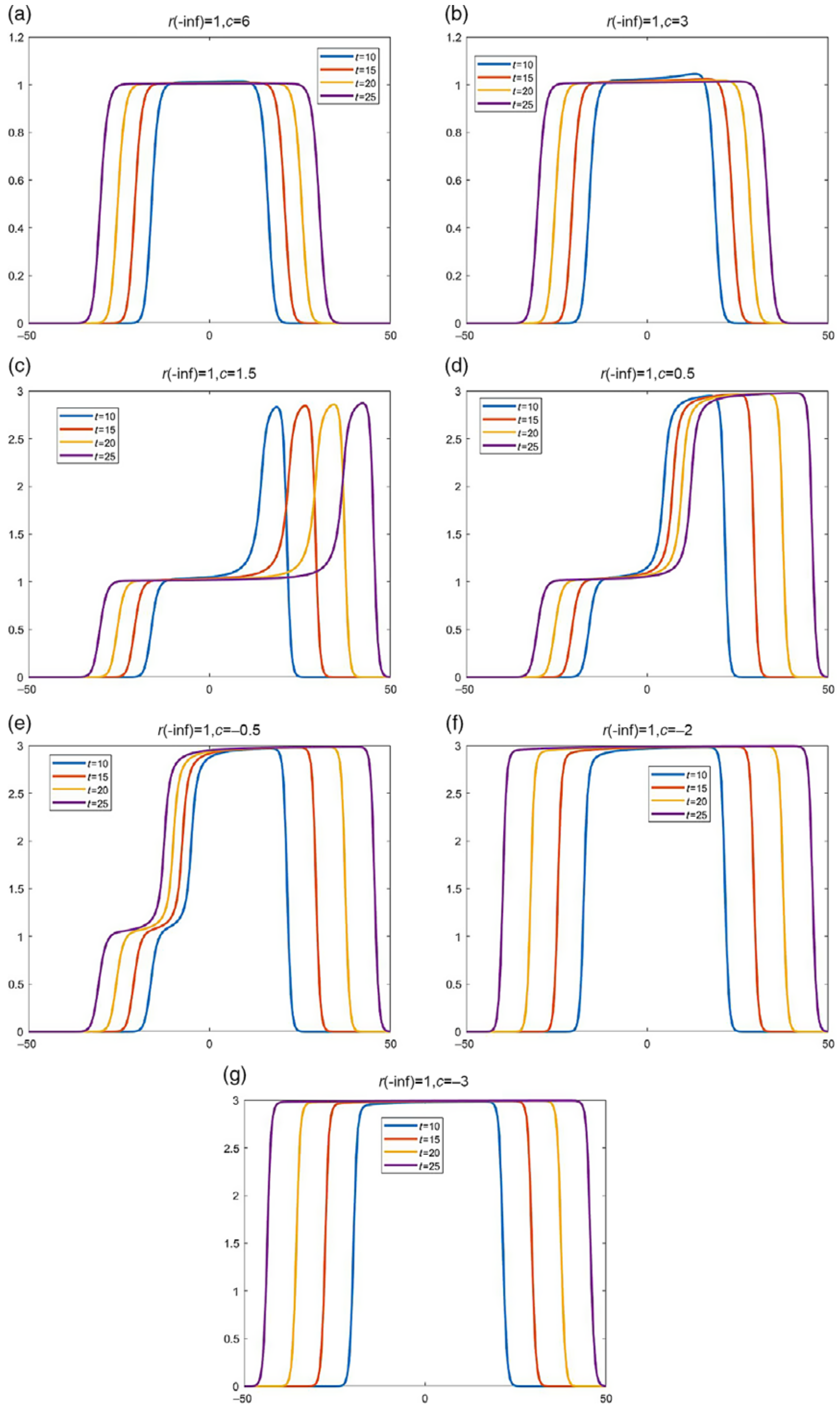

FIGURE 3. The case of $r(-\infty) > 0$. (a) $r(-\infty) > 0,\ c > c^*(\infty) + \hat{c}$. (b) $r(-\infty) > 0,\ c^*(\infty) < c < c^*(\infty) + \hat{c}$. (c) $r(-\infty) > 0,\ c^*(-\infty) < c < c^*(\infty)$. (d) $r(-\infty) > 0,\ 0 < c < c^*(-\infty)$. (e) $r(-\infty) > 0,\ -c^*(-\infty) < c < 0$. (f) $r(-\infty) > 0,\ -c^*(\infty) < c < -c^*(-\infty)$. (g) $r(-\infty) > 0,\ c < -c^*(\infty)$.

free and large-range with 'position-jump process' in contrast with the class of reaction-diffusion models with its fundamental assumption that motion is governed by a random walk. Meanwhile, the 'shifting habitat' is represented as $r(x-ct)$ by assuming that the habitat shifts with a constant speed c as time goes and becomes more favourable along the positive spatial direction. It is not a infrequent scenario that the unfavourable domain for a biological species is not hostile and the species is able to survive but grows relatively slow in contrast to favourable domain, as discussed in [15, 21, 42]. To this end, we do not ask for the sign of $r(-\infty)$. Besides, considering the threats or benefits associated with climate changes which induce the transition of the habitat, the speed c of shifting habitat edge can be any real number in our circumstances here.

Concretely speaking, when the habitat near the negative infinity is hostile for the species (i.e., $r(-\infty)\le 0$), our results indicate that the population will die out in the whole habitat if the habitat shifting speed c is lager than $c^*(\infty)$, where $c^*(\infty)$ denotes the minimum KPP travelling wave speed associated with the species' growth rate at the positive infinity. This means that a fast shrinking of favourable habitat leads to extinction. Conversely, if the shrinking speed is modest (i.e., $0<c<c^*(\infty)$), then the species can survive 'by moving' and spread towards new territory at speed $c^*(\infty)$, but lost the original domain at speed c as a result of the shifting habitat. However, if the favourable habitat is expanding with a modest speed $|c|$ (i.e., $-c^*(\infty)<c<0$), then the species can spread not only rightward with speed $c^*(\infty)$ but also leftward at speed $|c|$. And if the expanding speed is relatively fast (i.e., $c<-c^*(\infty)$), the leftward speed is also $c^*(\infty)$. In the current situation, the population will eventually approach the higher quality $r(\infty)$.

When the habitat near the negative infinity is not so harsh (i.e., $r(-\infty)>0$), no matter how fast (even $c>c^*(\infty)$) the good-quality habitat shrinks, the species can always persist 'by moving' and spread to both right and left at the asymptotic speed $c^*(-\infty)$ and will eventually approach the lower quality $r(-\infty)$. If the favourable zone shrinks or expands at a moderate speed (i.e., $-c^*(-\infty)<c<c^*(\infty)$), then the species spreads to the right at the asymptotic speed $c^*(\infty)$ and will eventually approach the higher quality $r(\infty)$ near positive infinity, while to the left at the asymptotic speed $c^*(-\infty)$ and will eventually approach the lower quality $r(-\infty)$ near negative infinity. If the favourable region expands with a relatively fast speed (i.e., $c<-c^*(-\infty)$), we see that the species can spread to the right at the asymptotic speed $c^*(\infty)$, and to the left at the asymptotic speed $-c$ when $-c^*(\infty)<c<-c^*(-\infty)$ and at the asymptotic speed $c^*(\infty)$ when $c<-c^*(\infty)$, and will eventually approach the higher quality $r(\infty)$.

Finally, we point out that the spreading properties of model (1.1) were obtained for a class of thin-tailed kernels $J(\cdot)$, which satisfies the so-called Mollison condition: $\int_{\mathbb{R}} J(y)e^{\lambda y}dy<+\infty$, $\forall\lambda>0$. When the dispersal kernel is fat-tailed in the sense that $|J'(y)|=o(J(y))$ as $|y|\to\infty$, the asymptotic propagation of model (1.1) remains an open problem. Here we refer the readers to the work of [12, 38, 39, 40, 41] and the reference therein for the study of nonlocal dispersal equations and systems with fat-tailed kernels and without shifting feature. To explore the asymptotic propagations of model (1.1) with fat-tailed kernels will be an interesting but challenging problem and we leave it as a future investigation.

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