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Classification of Certain Simple C^*-Algebras with Torsion in K_1

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Abstract. We show that the Elliott invariant is a classifying invariant for the class of C^*-algebras that are simple unital infinite dimensional inductive limits of finite direct sums of building blocks of the form

$$\{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_{d_i}, i = 1, 2, \ldots, N\},$$

where $x_1, x_2, \ldots, x_N \in \mathbb{T}$, $d_1, d_2, \ldots, d_N$ are integers dividing n, and M_{d_i} is embedded unitally into M_n. Furthermore we prove existence and uniqueness theorems for $*$-homomorphisms between such algebras and we identify the range of the invariant.

1 Introduction

During the last decade the Elliott invariant has been used with amazing success to classify simple unital C^*-algebras (see *e.g.* [8], [10], [20], [27], [15], [16]). This project is part of Elliott's program which has the ambitious goal of a classification result for all separable nuclear C^*-algebras by invariants of K-theoretical nature.

The goal of the present paper is to unify and generalize classification results due to Thomsen [27] and Jiang and Su [16]. In order to achieve this we will unfortunately have to consider the rather complicated building blocks defined in the abstract. Our main result (see Theorem 11.7) is the following:

Theorem 1.1 *The Elliott invariant is a classifying invariant for the class of unital simple infinite dimensional inductive limits of sequences of finite direct sums of building blocks.*

The main ideas of the proof are similar to those of Thomsen [27] who considers the simpler case $d_1 = d_2 = \cdots = d_N$. The technical problems are greater in our case, and in particular the possible lack of projections in our building blocks (see Lemma 3.8) means there is no straightforward generalization of Thomsen's proof.

Let us introduce the notation used in this paper before we describe our results in greater detail. Recall that for a unital C^*-algebra A the Elliott invariant consists of the ordered group $K_0(A)$ with order unit, the group $K_1(A)$, the compact convex set $T(A)$ of tracial states, and the restriction map $r_A\colon T(A) \to SK_0(A)$, where $SK_0(A)$ denotes the state space of $K_0(A)$.

Let A be a unital C^*-algebra. Let $\operatorname{Aff} T(A)$ denote the order unit space of all continuous real-valued affine functions on $T(A)$. Let $\rho_A\colon K_0(A) \to \operatorname{Aff} T(A)$ be the group

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homomorphism

$$\rho_A(x)(\omega) = r_A(\omega)(x), \quad \omega \in T(A), x \in K_0(A).$$

Let $U(A)$ denote the unitary group of A and let $DU(A)$ denote its commutator subgroup, *i.e.*, the group generated by all unitaries of the form uvu^*v^*, $u, v \in U(A)$. If A is a unital inductive limit of a sequence of finite direct sums of building blocks then there is a natural short exact sequence of abelian groups (see Section 5)

$$0 \longrightarrow \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} \xrightarrow{\lambda_A} U(A)/\overline{DU(A)} \xrightarrow{\pi_A} K_1(A) \longrightarrow 0$$

that splits (unnaturally). The group $U(\cdot)/\overline{DU(\cdot)}$ was introduced into the classification program by Nielsen and Thomsen [20].

Let A and B be unital C^*-algebras. An affine continuous map $\varphi_T\colon T(B) \to T(A)$ gives rise to a linear positive order unit preserving map $\varphi_{T*}\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ by setting $\varphi_{T*}(f) = f \circ \varphi_T$ for $f \in \operatorname{Aff} T(A)$. If furthermore $\varphi_{T*} \circ \rho_A = \rho_B \circ \varphi_0$ for some group homomorphism $\varphi_0\colon K_0(A) \to K_0(B)$ then φ_T induces a group homomorphism

$$\widetilde{\varphi_T}\colon \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)} \longrightarrow \operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}.$$

Let $\psi\colon A \to B$ be a unital $*$-homomorphism. Let $\psi^*\colon T(B) \to T(A)$ be the affine continuous map given by $\psi^*(\omega) = \omega \circ \psi$, $\omega \in T(B)$. Define $\widehat{\psi}\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ by $\widehat{\psi} = (\psi^*)_*$. Note that $\widehat{\psi}(f)(\omega) = f\big(\psi^*(\omega)\big)$. Since $\widehat{\psi} \circ \rho_A = \rho_B \circ \psi_*$ on $K_0(A)$, we see that ψ gives rise to a group homomorphism

$$\widetilde{\psi}\colon \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)} \longrightarrow \operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}.$$

Let $\psi^\#\colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be the homomorphism induced by ψ.

Besides the Elliott invariant, two other invariants will be crucial in the proof of the classification theorem, namely $U(\cdot)/\overline{DU(\cdot)}$ and Rørdam's KL-bifunctor [22]. These invariants are both determined by the Elliott invariant for the C^*-algebras under consideration, and are therefore useless as additional isomorphism invariants. They are, however, not determined canonically. This means that $*$-homomorphisms (or even automorphisms) between such C^*-algebras that agree on the Elliott invariant may fail to be approximately unitarily equivalent because they may act differently on these additional invariants. This was demonstrated by Nielsen and Thomsen [20, Section 5] for $U(\cdot)/\overline{DU(\cdot)}$ and by Dadarlat and Loring [6, pp. 375–376] for KL.

It is therefore necessary to include these invariants in the following uniqueness theorem (see Theorem 11.5):

Theorem 1.2 *Let A and B be unital inductive limits of sequences of finite direct sums of building blocks, with A simple. Two unital $*$-homomorphisms $\varphi, \psi\colon A \to B$ with $\varphi^* = \psi^*$ on $T(B)$, $\varphi^\# = \psi^\#$ on $U(A)/\overline{DU(A)}$, and $[\varphi] = [\psi]$ in $KL(A, B)$ are approximately unitarily equivalent.*

Let $KL(A,B)_T$ denote the set of elements $\kappa \in KL(A,B)$ for which the induced map $\kappa_*\colon K_0(A) \to K_0(B)$ preserves the order unit and for which there exists an affine continuous map $\varphi_T\colon T(B) \to T(A)$ such that $r_B(\omega)\big(\kappa_*(x)\big) = r_A\big(\varphi_T(\omega)\big)(x)$ for $x \in K_0(A), \omega \in T(B)$.

Let A and B be *e.g.* simple unital inductive limits of sequences of finite direct sums of building blocks. It turns out, perhaps surprisingly, that there is a connection between $KL(A,B)$ and the torsion subgroups of $U(A)/\overline{DU(A)}$ and $U(B)/\overline{DU(B)}$, see Section 10. If $\varphi, \psi\colon A \to B$ are unital $*$-homomorphisms with $[\varphi] = [\psi]$ in $KL(A,B)$ and if x is an element of finite order in the group $U(A)/\overline{DU(A)}$, then $\varphi^\#(x) = \psi^\#(x)$ in $U(B)/\overline{DU(B)}$. More generally, an element $\kappa \in KL(A,B)_T$ gives rise to a group homomorphism

$$s_\kappa\colon \operatorname{Tor}\big(U(A)/\overline{DU(A)}\big) \longrightarrow \operatorname{Tor}\big(U(B)/\overline{DU(B)}\big).$$

The map

$$KL(A,B)_T \longrightarrow \operatorname{Hom}\Big(\operatorname{Tor}\big(U(A)/\overline{DU(A)}\big), \operatorname{Tor}\big(U(B)/\overline{DU(B)}\big)\Big),$$

where $\kappa \mapsto s_\kappa$, is natural with respect to the Kasparov product and must be taken into account in the existence theorem:

Theorem 1.3 *Let A and B be simple unital inductive limits of sequences of finite direct sums of building blocks, with B infinite dimensional. Let $\varphi_T\colon T(B) \to T(A)$ be an affine continuous map, let $\kappa \in KL(A,B)_T$ be an element such that*

$$r_B(\omega)\big(\kappa_*(x)\big) = r_A\big(\varphi_T(\omega)\big)(x), \quad x \in K_0(A), \omega \in T(B),$$

and let $\Phi\colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be a homomorphism such that the diagram

$$\begin{array}{ccccc} \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\ \downarrow{\scriptstyle \widetilde{\varphi_T}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \kappa_*} \\ \operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B) \end{array}$$

commutes. Assume finally that

$$s_\kappa(y) = \Phi(y), \quad y \in \operatorname{Tor}\big(U(A)/\overline{DU(A)}\big).$$

There exists a unital $*$-homomorphism $\psi\colon A \to B$ such that $\psi^* = \varphi_T$ on $T(B)$, such that $\psi^\# = \Phi$ on $U(A)/\overline{DU(A)}$, and such that $[\psi] = \kappa$ in $KL(A,B)$.

The above theorem follows by combining the slightly more general Theorem 11.2 with Lemma 9.6, Lemma 10.3 and Theorem 9.9. It should be noted that it is possible to prove this existence theorem (and our classification theorem) for $K_0(A)$ non-cyclic without using the map s_κ, see Corollary 11.3 (or [27]).

Let us finally describe the range of the invariant for the C^*-algebras in our class. By combining Theorem 12.1 and Corollary 12.5 we have the following:

Theorem 1.4 *Let G be a countable simple dimension group with order unit, H a countable abelian group, Δ a compact metrizable Choquet simplex, and $\lambda\colon \Delta \to SG$ an affine continuous extreme point preserving surjection. There exists a simple unital inductive limit of a sequence of finite direct sums of building blocks A together with an isomorphism $\varphi_0\colon K_0(A) \to G$ of ordered groups with order unit, an isomorphism $\varphi_1\colon K_1(A) \to H$, and an affine homeomorphism $\varphi_T\colon \Delta \to T(A)$ such that*

$$r_A\big(\varphi_T(\omega)\big)(x) = \lambda(\omega)\big(\varphi_0(x)\big), \quad \omega \in \Delta, x \in K_0(A)$$

if and only if G is non-cyclic, or G is cyclic and H can be realized as an inductive limit of a sequence of the form

$$\mathbb{Z} \oplus H_1 \longrightarrow \mathbb{Z} \oplus H_2 \longrightarrow \mathbb{Z} \oplus H_3 \longrightarrow \cdots$$

where each H_k is a finite abelian group.

Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks. It is easy to see that A is unital projectionless if and only if $\big(K_0(A), K_0(A)^+, [1]\big) \cong (\mathbb{Z}, \mathbb{Z}^+, 1)$. Hence our classification theorem can be applied to a large class of simple unital projectionless C^*-algebras, including the C^*-algebra $\mathcal{Z}$ constructed by Jiang and Su [16].

It would be interesting if one could extend our classification result to a class that contains simple unital projectionless C^*-algebras with arbitrary countable abelian K_1-groups. This could probably be obtained by considering building blocks with $\mathbb{T}$ replaced by a general 1-dimensional compact Hausdorff space. It would also be interesting if one could include the class of C^*-algebras considered by Jiang and Su in [15].

Let A be a unital C^*-algebra. If $a \in A_{\mathrm{sa}}$ we define $\widehat{a} \in \operatorname{Aff} T(A)$ by $\widehat{a}(\omega) = \omega(a)$, $\omega \in T(A)$. It is well-known that $a \mapsto \widehat{a}$ is a surjective map from A_{sa} to $\operatorname{Aff} T(A)$. Let $q'_A\colon U(A) \to U(A)/\overline{DU(A)}$ be the canonical map. We equip the abelian group $U(A)/\overline{DU(A)}$ with the quotient metric

$$D_A\big(q'_A(u), q'_A(v)\big) = \inf\{\|uv^* - x\| : x \in \overline{DU(A)}\}.$$

Denote by d'_A the quotient metric on the group $\operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)}$. This group can be equipped with another metric which gives rise to the same topology, namely

$$d_A(f,g) = \begin{cases} 2 & d'_A(f,g) \geq \frac{1}{2}, \\ |e^{2\pi\,\mathrm{i} d'_A(f,g)} - 1| & d'_A(f,g) < \frac{1}{2}, \end{cases}$$

see [20, Chapter 3]. Let $q_A\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)}$ be the quotient map. Let finally $s(A)$ be the smallest positive integer n for which there exists a unital $*$-homomorphism $A \to M_n$ (we set $s(A) = \infty$ if A has no non-trivial finite dimensional representations).

Let gcd denote the greatest common divisor and lcm the least common multiple of a set of positive integers. Let Tr denote the (unnormalized) trace on a matrix algebra (*i.e.*, the number obtained by adding the diagonal entries). If

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

is a sequence of C^*-algebras and $*$-homomorphisms with inductive limit A, we let $\alpha_{n,m} = \alpha_{m-1} \circ \alpha_{m-2} \circ \cdots \circ \alpha_n \colon A_n \to A_m$ when $m > n$. We set $\alpha_{n,n} = \mathrm{id}$ and let $\alpha_{n,\infty} \colon A_n \to A$ denote the canonical map.

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2 Building Blocks

Let $\mathbb{T}$ denote the unit circle of the complex plane. We will equip $\mathbb{T}$ with the metric

$$\rho(e^{2\pi i s}, e^{2\pi i t}) = \min_{k \in \mathbb{Z}} |s - t + k|$$

which is easily seen to be equivalent to the usual metric on $\mathbb{T}$ inherited from $\mathbb{C}$.

As in [20] we say that a tuple $(a_1, a_2, \ldots, a_L)$ of elements from $\mathbb{T}$ is naturally numbered if there exist numbers $s_1, s_2, \ldots, s_L \in [0, 1[$ such that $s_1 \leq s_2 \leq \cdots \leq s_L$ and $a_j = e^{2\pi i s_j}$, $j = 1, 2, \ldots, L$.

We define a building block to be a C^*-algebra of the form

$$A(n, d_1, d_2, \ldots, d_N) = \{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_{d_i}, i = 1, 2, \ldots, N\},$$

where $(x_1, x_2, \ldots, x_N)$ is a naturally numbered tuple of (different) points in $\mathbb{T}$, $d_1, d_2, \ldots, d_N$ are integers dividing n, and M_{d_i} is embedded unitally into M_n, *e.g.* via the $*$-homomorphism

$$a \mapsto \mathrm{diag}(\underbrace{a, a, \ldots, a}_{\frac{n}{d_i} \text{ times}}).$$

The points $x_1, x_2, \ldots, x_N$ will be called the exceptional points of A. By allowing $d_i = n$ we may always assume that $N \geq 2$. It will also be convenient to always assume that 1 is not an exceptional point.

For every $i = 1, 2, \ldots, N$, evaluation at x_i gives rise to a unital $*$-homomorphism from A to M_{d_i} which will be denoted by Λ_i, or sometimes Λ_i^A. If s is a non-negative integer we define $\Lambda_i^s \colon A \to M_{sd_i}$ by

$$\Lambda_i^s(f) = \mathrm{diag}\big(\underbrace{\Lambda_i(f), \Lambda_i(f), \ldots, \Lambda_i(f)}_{s \text{ times}}\big).$$

Note that $\Lambda_i^{\frac{n}{d_i}}(f) = f(x_i)$ in M_n for $f \in A$ and $i = 1, 2, \ldots, N$.

The following lemmas are left as exercises.

Lemma 2.1 *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block. The irreducible representations (up to unitary equivalence) of A are $\Lambda_1, \Lambda_2, \ldots, \Lambda_N$, together with point evaluations at non-exceptional points.*

Lemma 2.2 *Let I be a closed two-sided ideal in A. There is a closed set $F \subseteq \mathbb{T}$ such that*

$$I = \{f \in A : f(x) = 0 \text{ for all } x \in F\}.$$

Lemma 2.3 *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block and let $\omega \in T(A)$. There exists a Borel probability measure μ on $\mathbb{T}$ such that*

$$\omega(f) = \frac{1}{n} \int_{\mathbb{T}} \operatorname{Tr}\big(f(x)\big)\, d\mu(x).$$

It follows that $C_{\mathbb{R}}(\mathbb{T})$ and $\operatorname{Aff} T(A)$ are isomorphic as order unit spaces via the map $f \mapsto \widehat{f \otimes 1}$, $f \in C_{\mathbb{R}}(\mathbb{T})$.

Theorem 2.4 *Let A be a finite direct sum of building blocks. Then A is finitely generated and semiprojective.*

Proof First note that A is a one-dimensional non-commutative CW complex, as defined in [7]. Hence A is semiprojective by [7, Theorem 6.2.2] and finitely generated by [7, Lemma 2.4.3]. ∎

Note that if $A = A(n, d_1, d_2, \ldots, d_N)$ then $s(A) = \min(d_1, d_2, \ldots, d_N)$.

Building blocks will sometimes be called circle building blocks in order to distinguish them from interval building blocks. An interval building block is a C^*-algebra A of the form

$$I(n, d_1, d_2, \ldots, d_N) = \{f \in C[0, 1] \otimes M_n : f(x_i) \in M_{d_i}, i = 1, 2, \ldots, N\},$$

where $0 = x_1 < x_2 < \cdots < x_N = 1$ and $d_1, d_2, \ldots, d_N$ are integers dividing n. We will call $x_1, x_2, \ldots, x_N$ the exceptional points of A.

3 K-Theory

The purpose of this section is to calculate and interpret the K-theory of a building block. We start out with the following lemma, which will be used to calculate the K_1-group.

Lemma 3.1 *Let $N \geq 2$ and let $a_1, a_2, \ldots, a_N$ be positive integers. Define a group homomorphism $\varphi\colon \mathbb{Z}^N \to \mathbb{Z}^N$ to be multiplication with the $N \times N$ matrix*

$$C = \begin{pmatrix} a_1 & -a_2 & & & \\ & a_2 & -a_3 & & \\ & & a_3 & \ddots & \\ & & & \ddots & -a_N \\ -a_1 & & & & a_N \end{pmatrix}.$$

For $k = 1, 2, \ldots, N-1$, set

$$s_k = \operatorname{lcm}(a_1, a_2, \ldots, a_k)$$

and

$$r_k = \gcd(s_k, a_{k+1}) = \gcd\bigl(\operatorname{lcm}(a_1, a_2, \ldots, a_k), a_{k+1}\bigr).$$

Choose integers α_k and β_k such that

$$r_k = \alpha_k s_k + \beta_k a_{k+1}, \quad k = 1, 2, \ldots, N-1.$$

Then

$$\operatorname{coker}(\varphi) \cong \mathbb{Z} \oplus \mathbb{Z}_{r_1} \oplus \mathbb{Z}_{r_2} \oplus \cdots \oplus \mathbb{Z}_{r_{N-1}}.$$

This isomorphism can be chosen such that for $k = 1, 2, \ldots, N-2$, a generator of the direct summand $\mathbb{Z}_{r_k}$ is mapped to the coset

$$\Bigl(\underbrace{0, 0, \ldots, 0}_{k-1 \text{ times}}, 1, -\frac{\beta_k a_{k+1}}{r_k}, \underbrace{0, 0, \ldots, 0}_{N-k-2 \text{ times}}, -\frac{\alpha_k s_k}{r_k}\Bigr) + \operatorname{im}(\varphi),$$

such that a generator of the direct summand $\mathbb{Z}_{r_{N-1}}$ is mapped to the coset

$$(0, 0, \ldots, 0, 1, -1) + \operatorname{im}(\varphi),$$

and such that a generator of the direct summand $\mathbb{Z}$ is mapped to the coset

$$(0, 0, \ldots, 0, 1) + \operatorname{im}(\varphi).$$

Proof Let I_j denote the $j \times j$ identity matrix for any non-negative integer j. For each $k = 1, 2, \ldots, N-2$, define an integer matrix of size $N \times N$ by

$$A_k = \begin{pmatrix} I_{k-1} & & & & & & \\ & 1 & & & & & \\ & -\frac{\alpha_k s_k}{r_k} & 1 & & & & \\ & -\frac{\alpha_k s_k}{r_k} & & 1 & & & \\ & \vdots & & & \ddots & & \\ & -\frac{\alpha_k s_k}{r_k} & & & & 1 & \\ & 0 & & & & & 1 \end{pmatrix}.$$

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Let D_k denote the 2×2 matrix

$$\begin{pmatrix} \alpha_k & \frac{a_{k+1}}{r_k} \\ -\beta_k & \frac{s_k}{r_k} \end{pmatrix},$$

and define for $k = 1, 2, \ldots, N-1$, an integer matrix of size $N \times N$ by

$$B_k = \begin{pmatrix} I_{k-1} & & \\ & D_k & \\ & & I_{N-k-1} \end{pmatrix}.$$

For $k = 0, 1, 2, \ldots, N-2$, define yet another $N \times N$ matrix by

$$X_k = \begin{pmatrix} r_1 & & & & & & & & \\ & r_2 & & & & & & & \\ & & \ddots & & & & & & \\ & & & r_k & & & & & \\ & & & & s_{k+1} & -a_{k+2} & & & \\ & & & & s_{k+1} & & -a_{k+3} & & \\ & & & & \vdots & & & \ddots & \\ & & & & s_{k+1} & & & & -a_N \\ & & & & 0 & & & & 0 \end{pmatrix}.$$

Finally, let P be the $N \times N$ matrix

$$\begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Note that for $k = 1, 2, \ldots, N-2$,

$$s_{k+1} = \operatorname{lcm}(s_k, a_{k+1}) = \frac{s_k a_{k+1}}{r_k}.$$

Using this, it is easily seen by induction that

$$A_k A_{k-1} \cdots A_1 P C B_1 B_2 \cdots B_k = X_k, \quad k = 0, 1, 2, \ldots, N-2.$$

It follows that

$$A_{N-2} A_{N-3} \cdots A_1 P C B_1 B_2 \cdots B_{N-2} B_{N-1} = \begin{pmatrix} r_1 & & & & \\ & r_2 & & & \\ & & \ddots & & \\ & & & r_{N-1} & \\ & & & & 0 \end{pmatrix}.$$

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Since all the matrices on the left-hand side, except C, are invertible in $M_N(\mathbb{Z})$, we obtain the desired calculation of $\operatorname{coker}(\varphi)$. Finally, it is easily verified that

$$(A_{N-2}A_{N-3}\cdots A_1P)^{-1} = \begin{pmatrix} 1 & & & & & & \\ -\frac{\beta_1 a_2}{r_1} & 1 & & & & & \\ 0 & -\frac{\beta_2 a_3}{r_2} & 1 & & & & \\ 0 & 0 & -\frac{\beta_3 a_4}{r_3} & 1 & & & \\ \vdots & \vdots & & \ddots & \ddots & & \\ 0 & 0 & & & -\frac{\beta_{N-2}a_{N-1}}{r_{N-2}} & 1 & \\ -\frac{\alpha_1 s_1}{r_1} & -\frac{\alpha_2 s_2}{r_2} & \cdots & \cdots & -\frac{\alpha_{N-2}s_{N-2}}{r_{N-2}} & -1 & 1 \end{pmatrix}.$$

The last part of the lemma follows from this. ■

Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block with exceptional points $e^{2\pi i t_k}$, $k = 1, 2, \dots, N$, where $0 < t_1 < t_2 < \cdots < t_N < 1$. Set $t_{N+1} = t_1 + 1$. Define continuous functions $\omega_k \colon \mathbb{T} \to \mathbb{T}$ for $k = 1, 2, \dots, N$, by

$$\omega_k(e^{2\pi i t}) = \begin{cases} \exp\bigl(2\pi i \frac{t - t_k}{t_{k+1} - t_k}\bigr) & t_k \le t \le t_{k+1}, \\ 1 & t_{k+1} \le t \le t_k + 1. \end{cases}$$

Let U_k^A be the unitary in A defined by

$$U_k^A(z) = \operatorname{diag}\bigl(\omega_k(z), 1, 1, \dots, 1\bigr), \quad z \in \mathbb{T}.$$

Theorem 3.2 *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block. Set for $k = 1, 2, \dots, N-1$,*

$$s_k = \operatorname{lcm}\left(\frac{n}{d_1}, \frac{n}{d_2}, \dots, \frac{n}{d_k}\right),$$

and

$$r_k = \gcd\left(s_k, \frac{n}{d_{k+1}}\right) = \gcd\left(\operatorname{lcm}\left(\frac{n}{d_1}, \frac{n}{d_2}, \dots, \frac{n}{d_k}\right), \frac{n}{d_{k+1}}\right).$$

Choose integers α_k and β_k such that

$$r_k = \alpha_k s_k + \beta_k \frac{n}{d_{k+1}}, \quad k = 1, 2, \dots, N-1.$$

Then

$$K_1(A) \cong \mathbb{Z} \oplus \mathbb{Z}_{r_1} \oplus \mathbb{Z}_{r_2} \oplus \cdots \oplus \mathbb{Z}_{r_{N-1}}.$$

This isomorphism can be chosen such that for $k = 1, 2, \dots, N-1$, a generator of the direct summand $\mathbb{Z}_{r_k}$ is mapped to

$$[U_k^A] - \frac{\beta_k n}{r_k d_{k+1}}[U_{k+1}^A] - \frac{\alpha_k s_k}{r_k}[U_N^A],$$

and such that a generator of the direct summand $\mathbb{Z}$ is mapped to $[U_N^A]$.

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Proof Define a $*$-homomorphism $\pi\colon A \to M_{d_1} \oplus M_{d_2} \oplus \cdots \oplus M_{d_N}$ by

$$\pi(f) = \big(\Lambda_1(f), \Lambda_2(f), \dots, \Lambda_N(f)\big).$$

Via the identification $SM_n \cong \{f \in C[0,1] \otimes M_n : f(0) = f(1) = 0\}$ we define a $*$-homomorphism $\iota\colon (SM_n)^N \to A$ by

$$\iota(f_1, f_2, \dots, f_N)(e^{2\pi it}) = f_k\left(\frac{t - t_k}{t_{k+1} - t_k}\right), \quad t_k \le t \le t_{k+1}.$$

The short exact sequence

$$0 \longrightarrow (SM_n)^N \stackrel{\iota}{\longrightarrow} A \stackrel{\pi}{\longrightarrow} M_{d_1} \oplus M_{d_2} \oplus \cdots \oplus M_{d_N} \longrightarrow 0$$

gives rise to a six-term exact sequence

$$\begin{array}{ccccc} K_0\big((SM_n)^N\big) & \xrightarrow{\iota_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(M_{d_1} \oplus \cdots \oplus M_{d_N}) \\ \uparrow & & & & \downarrow \delta \\ K_1(M_{d_1} \oplus \cdots \oplus M_{d_N}) & \xleftarrow[\pi_*]{} & K_1(A) & \xleftarrow[\iota_*]{} & K_1\big((SM_n)^N\big) \end{array}$$

where δ denotes the exponential map.

By Bott periodicity $K_1\big((SM_n)^N\big) \cong \mathbb{Z}^N$ is generated by $[V_1], [V_2], \dots, [V_N]$, where

$$V_k(t) = \big(1, 1, \dots, 1, \underbrace{\operatorname{diag}(e^{2\pi it}, 1, \dots, 1)}_{\text{coordinate } k}, 1, 1, \dots, 1\big), \quad t \in [0,1],$$

is a unitary in $(\widetilde{SM_n})^N$. Note that $\iota(V_k - 1) = U_k^A - 1$ and hence $\iota_*([V_k]) = [U_k^A]$ in $K_1(A)$. Since the map $\iota_*\colon K_1\big((SM_n)^N\big) \to K_1(A)$ is surjective it follows that $K_1(A)$ is generated by $[U_1^A], [U_2^A], \dots, [U_N^A]$, and that ι_* gives rise to an isomorphism between the cokernel of δ and $K_1(A)$.

Let $\{e_{ij}^k\}$ denote the standard matrix units in $M_{d_1} \oplus \cdots \oplus M_{d_N}$. Recall that $K_0(M_{d_1} \oplus \cdots \oplus M_{d_N}) \cong \mathbb{Z}^N$ is generated by $[e_{11}^1], [e_{11}^2], \dots, [e_{11}^N]$. We leave it with the reader to check that

$$\delta([e_{11}^1]) = -\frac{n}{d_1}[V_N] + \frac{n}{d_1}[V_1],$$

and for $k = 2, 3, \dots, N$,

$$\delta([e_{11}^k]) = -\frac{n}{d_k}[V_{k-1}] + \frac{n}{d_k}[V_k].$$

The conclusion follows from Lemma 3.1. ■

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Choose a continuous function $\gamma\colon \mathbb{T} \to \mathbb{R}$ such that

$$\operatorname{Det}\big(U_N^A(z)\big) = z\exp\big(2\pi i\gamma(z)\big), \quad z \in \mathbb{T}.$$

Define a unitary v^A in A by

$$v^A(z) = U_N^A(z)\exp\left(-2\pi i\frac{\gamma(z)}{n}\right), \quad z \in \mathbb{T}.$$

Note that $\operatorname{Det}\big(v^A(z)\big) = z, z \in \mathbb{T}$.

Lemma 3.3 *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block and let $u \in A$ be a unitary. If*

$$\operatorname{Det}\big(\Lambda_k(u)\big) = 1, \quad k = 1, 2, \ldots, N,$$
$$\operatorname{Det}\big(u(z)\big) = 1, \quad z \in \mathbb{T},$$

then u can be connected to 1 via a continuous path of unitaries in A.

Proof Let us start with a simple and well-known observation. Let v be a unitary in the C^*-algebra $B = \{f \in C[0,1] \otimes M_n : f(0) = f(1)\}$ such that the winding number of $\operatorname{Det}\big(v(\cdot)\big)$ is 0. Then v can be connected to 1 via a continuous path $(v_t)_{t\in[0,1]}$ in $U(B)$. If $v(0) = 1$ we may assume that $v_t(0) = 1$ for every $t \in [0, 1]$.

Let $e^{2\pi i t_1}, \ldots, e^{2\pi i t_N}$ be the exceptional points of A, where $t_1 < t_2 < \cdots < t_N$ are numbers in $]0, 1[$. Set $t_0 = t_N - 1$, $t_{N+1} = t_1 + 1$ and let $\iota_k\colon M_{d_k} \to M_n$ be the inclusion, $k = 1, 2, \ldots, N$. Since the group of unitaries in M_{d_k} with determinant 1 is path-connected there exists a continuous function $\gamma_k\colon [t_{k-1}, t_{k+1}] \to U(M_{d_k})$ such that $\gamma_k(t_{k-1}) = \gamma_k(t_{k+1}) = 1$, $\gamma_k(t_k) = \Lambda_k(u)$, and $\operatorname{Det}\big(\gamma_k(\cdot)\big) = 1$. Set

$$w_k(e^{2\pi i t}) = \begin{cases} \iota_k\big(\gamma_k(t)\big) & t \in [t_{k-1}, t_{k+1}], \\ 1 & t \in [t_{k+1}, t_{k-1} + 1]. \end{cases}$$

It follows from the above observation that w_k can be connected to 1 via a continuous path of unitaries in A. Upon replacing u with $uw_1^* w_2^* \cdots w_N^*$ we may thus assume that $u(e^{2\pi i t_k}) = 1$ for $k = 1, 2, \ldots, N$. Set

$$y_k(e^{2\pi i t}) = \begin{cases} u(e^{2\pi i t}) & t \in [t_k, t_{k+1}], \\ 1 & t \in [t_{k+1}, t_k + 1]. \end{cases}$$

Then $u = y_1 y_2 \cdots y_N$. Again by the above observation, y_k can be connected to 1 within $U(A)$ for $k = 1, 2, \ldots, N$. ∎

Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block and set $d = \gcd(d_1, d_2, \ldots, d_N)$. Since d divides d_i for every $i = 1, 2, \ldots, N$, there exists a unital and injective $*$-homomorphism $M_d \to A$ given by $f \mapsto \operatorname{diag}(f, f, \ldots, f)$.

Lemma 3.4 *Let p be a projection in $A = A(n, d_1, d_2, \dots, d_N)$. Then p is unitarily equivalent to a projection in $M_d \subseteq A$.*

Proof Let $r \in \mathbb{Z}$ denote the rank of p and let $e^{2\pi i t_1}, e^{2\pi i t_2}, \dots, e^{2\pi i t_N}$ be the exceptional points of A, where $0 < t_1 < t_2 < \cdots < t_N < 1$. Since $\frac{n}{d_k}$ divides r for $k = 1, 2, \dots, N$, it follows that $\frac{n}{d}$ also divides r. Hence there is a projection $e \in M_d \subseteq A$ with the same trace as p.

For each $t \in [0, 1]$ there is a unitary $u_t \in M_n$ such that

$$e = u_t p(e^{2\pi i t}) u_t^*.$$

We may assume that $u_{t_k} \in M_{d_k}$, $k = 1, 2, \dots, N$, and that $u_0 = u_1$. By compactness

$$[0, 1] = \bigcup_{j=1}^{L-1} [s_j, s_{j+1}],$$

where $0 = s_1 < s_2 < \cdots < s_L = 1$, $\{t_1, t_2, \dots, t_N\} \subseteq \{s_1, s_2, \dots, s_L\}$, and

$$t \in [s_j, s_{j+1}] \Longrightarrow \|u_{s_j} p(e^{2\pi i t}) u_{s_j}^* - e\| < 1.$$

Set $z_j(t) = v_j(t)|v_j(t)|^{-1}$ for $t \in [s_j, s_{j+1}]$, $j = 1, 2, \dots, L-1$, where

$$v_j(t) = 1 - u_{s_j} p(e^{2\pi i t}) u_{s_j}^* - e + 2e u_{s_j} p(e^{2\pi i t}) u_{s_j}^*.$$

Then $t \mapsto z_j(t)$, $t \in [s_j, s_{j+1}]$, is a continuous path of unitaries in M_n, and by [19, Lemma 6.2.1]

$$e = z_j(t) u_{s_j} p(e^{2\pi i t}) u_{s_j}^* z_j(t)^*, \quad t \in [s_j, s_{j+1}].$$

As $U(M_n) \cap \{e\}'$ is path-connected there is for each $k = 1, 2, \dots, L-1$ a continuous map $\gamma_j \colon [s_j, s_{j+1}] \to U(M_n) \cap \{e\}'$ such that

$$\gamma_j(s_j) = 1, \quad \gamma_j(s_{j+1}) = u_{s_{j+1}} u_{s_j}^* z_j(s_{j+1})^*.$$

Since $z_j(s_j) = 1$ for $j = 1, 2, \dots, L-1$, we can define a unitary $u \in A$ by

$$u(e^{2\pi i t}) = \gamma_j(t) z_j(t) u_{s_j}, \quad t \in [s_j, s_{j+1}].$$

Then $upu^* = e$. ■

Corollary 3.5 *If $p \in A = A(n, d_1, d_2, \dots, d_N)$ is a projection of rank $r \neq 0$ then*

$$pAp \cong A\left(r, \frac{r}{n} d_1, \frac{r}{n} d_2, \dots, \frac{r}{n} d_N\right).$$

Corollary 3.6 *The embedding $M_d \subseteq A$ gives rise to an isomorphism of ordered groups with order units between $K_0(M_d)$ and $K_0(A)$. In other words,*

$$\big(K_0(A), K_0(A)^+, [1]\big) \cong (\mathbb{Z}, \mathbb{Z}^+, d).$$

By Lemma 2.3 we have the following:

Corollary 3.7 *If $A = A(n, d_1, d_2, \dots, d_N)$ then $\rho_A\big(K_0(A)\big) = \mathbb{Z}\frac{1}{d}\widehat{1}$ in* Aff $T(A)$.

Lemma 3.8 *$A = A(n, d_1, d_2, \dots, d_N)$ is unital projectionless if and only if $d = 1$.*

Proof As in the proof of Lemma 3.4 we see that there exists a projection $p \in A$ of rank $r \leq n$ if and only if $\frac{n}{d}$ divides r. The conclusion follows. ■

Lemma 3.9 *Let K be a positive integer and let H be a finite abelian group. There exists a unital projectionless building block A with $s(A) \geq K$ such that $K_1(A) \cong \mathbb{Z} \oplus H$.*

Proof Let

$$H \cong \mathbb{Z}_{p_1^{k_1}} \oplus \mathbb{Z}_{p_2^{k_2}} \oplus \cdots \oplus \mathbb{Z}_{p_m^{k_m}},$$

where m is a positive integer, $k_1, \dots, k_m$ are non-negative integers, and $p_1, \dots, p_m$ are prime numbers. Let $q_1, q_2, \dots, q_{m+1} \geq K$ be prime numbers, mutually different as well as different from $p_1, p_2, \dots, p_m$. Define integers n and $d_1, d_2, \dots, d_{m+1}$ by

$$n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m} q_1 q_2 \cdots q_{m+1},$$
$$d_1 = q_2 q_3 \cdots q_{m+1},$$
$$d_i = \frac{p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}}{p_{i-1}^{k_{i-1}}} \frac{q_1 q_2 \cdots q_{m+1}}{q_i}, \quad 2 \leq i \leq m+1.$$

Set $A = A(n, d_1, d_2, \dots, d_{m+1})$. Then $K_1(A) \cong \mathbb{Z} \oplus H$ by Theorem 3.2. A is unital projectionless by Lemma 3.8. ■

4 KK-Theory

Recall a few facts about KK-theory that can be found in *e.g.* [2]. KK is a homotopy invariant bifunctor from the category of C^*-algebras to the category of abelian groups that is contravariant in the first variable and covariant in the second. A $*$-homomorphism $\varphi\colon A \to M_n(B)$ defines an element $[\varphi] \in KK(A, B)$. We have an associative map $KK(B, C) \times KK(A, B) \to KK(A, C)$, the Kasparov product, that generalizes composition of $*$-homomorphisms.

The purpose of this section is to analyze the KK-theory of our building blocks. Inspired by the work of Jiang and Su [16, Section 3], we will consider the K-homology groups $K^0(A) = KK(A, \mathbb{C})$. A $*$-homomorphism $\varphi\colon A \to M_n(B)$ induces a group homomorphism $\varphi^*\colon K^0(B) \to K^0(A)$ via the Kasparov product. $K^0(M_n) \cong \mathbb{Z}$ is generated by the class of the identity map on M_n.

If A and B are unital C^*-algebras we let $KK(A, B)_e$ be the set of elements $\kappa \in KK(A, B)$ such that $\kappa_*\colon K_0(A) \to K_0(B)$ preserves the order unit.

Lemma 4.1 *Let*

$$A = \{ f \in C[0,1] \otimes M_n : f(t_i) \in M_{d_i}, i = 1, 2, \ldots, N \}$$

where $N \geq 2$, $0 \leq t_1 < t_2 < \cdots < t_N \leq 1$, *and* $d_1, d_2, \ldots, d_N$ *are integers dividing* n. *Let* $\Lambda_i \colon A \to M_{d_i}$ *be the* $*$*-homomorphism induced by evaluation at* t_i, $i = 1, 2, \ldots, N$. *Then* $K^0(A)$ *is generated by* $[\Lambda_1], [\Lambda_2], \ldots, [\Lambda_N]$. *Furthermore, for* $a_1, a_2, \ldots, a_N \in \mathbb{Z}$ *we have that*

$$a_1[\Lambda_1] + a_2[\Lambda_2] + \cdots + a_N[\Lambda_N] = 0$$

if and only if there exist $b_1, b_2, \ldots, b_N \in \mathbb{Z}$ *such that* $\sum_{i=1}^N b_i = 0$ *and*

$$a_i = b_i \frac{n}{d_i}, \quad i = 1, 2, \ldots, N.$$

Proof Choose $y \in \,]0,1[$ such that $t_1 < y < t_2$. Set

$$B = \{ f \in C[0,y] \otimes M_n : f(t_1) \in M_{d_1} \},$$
$$C = \{ f \in C[y,1] \otimes M_n : f(t_i) \in M_{d_i}, i = 2, 3, \ldots, N \}.$$

We have a pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{g_1} & B \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ C & \xrightarrow[f_2]{} & M_n \end{array}$$

where g_1, g_2 are the restriction maps and f_1, f_2 evaluation at y. Apply the Mayer-Vietoris sequence [2, Theorem 21.5.1] to get a six-term exact sequence

$$\begin{array}{ccccc} K^0(M_n) & \xrightarrow{(-f_1^*, f_2^*)} & K^0(B) \oplus K^0(C) & \xrightarrow{g_1^* + g_2^*} & K^0(A) \\ \uparrow & & & & \downarrow \\ K^1(A) & \xleftarrow[g_1^* + g_2^*]{} & K^1(B) \oplus K^1(C) & \xleftarrow[(-f_1^*, f_2^*)]{} & K^1(M_n). \end{array}$$

Note that $K^1(M_n) = 0$ and $K^0(M_n) \cong \mathbb{Z}$. Thus the exact sequence becomes

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\varphi} & K^0(B) \oplus K^0(C) & \xrightarrow{\psi} & K^0(A) \\ \uparrow & & & & \downarrow \\ K^1(A) & \longleftarrow & K^1(B) \oplus K^1(C) & \longleftarrow & 0. \end{array}$$

Since f_1 is homotopic to evaluation at x_1 in B and f_2 is homotopic to evaluation at x_2 in C we see that

$$\varphi(k) = \left(-k \frac{n}{d_1} [\Lambda_1|_B], k \frac{n}{d_N} [\Lambda_N|_C] \right), \quad k \in \mathbb{Z}.$$

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B is homotopic to M_{d_1} via $\Lambda_1|_B$ and hence $K^0(B) \cong \mathbb{Z}$ is generated by $[\Lambda_1|_B]$.
For $N = 2$ we have that $K^0(C)$ is generated by $[\Lambda_2|_C]$ and that

$$\psi(a_1[\Lambda_1|_B], a_2[\Lambda_2|_C]) = a_1[\Lambda_1] + a_2[\Lambda_2].$$

Thus $K^0(A)$ is generated by $[\Lambda_1]$ and $[\Lambda_2]$ and

$$a_1[\Lambda_1] + a_2[\Lambda_2] = 0 \iff \exists b_1 \in \mathbb{Z} : a_1 = -b_1\frac{n}{d_1}, a_2 = b_1\frac{n}{d_2}.$$

Proceeding by induction, assume that the lemma holds for $N-1$. By the induction hypothesis $K^0(C)$ is generated by $[\Lambda_2|_C], [\Lambda_3|_C], \dots, [\Lambda_N|_C]$. Note that

$$\psi\big(a_1[\Lambda_1|_B], (a_2[\Lambda_2|_C] + \cdots + a_N[\Lambda_N|_C])\big) = \sum_{i=1}^{N} a_i[\Lambda_i],$$

such that A is generated by $[\Lambda_1], \dots, [\Lambda_N]$. It also follows that

$$a_1[\Lambda_1] + a_2[\Lambda_2] + \cdots + a_N[\Lambda_N] = 0$$

if and only if there exists $k \in \mathbb{Z}$ such that

$$-k\frac{n}{d_1}[\Lambda_1|_B] = a_1[\Lambda_1|_B], \quad k\frac{n}{d_N}[\Lambda_N|_C] = a_2[\Lambda_2|_C] + \cdots + a_N[\Lambda_N|_C].$$

By the induction hypothesis this happens if and only if there exist $k, c_2, \dots, c_N \in \mathbb{Z}$ such that $\sum_{i=2}^{N} c_i = 0$ and

$$a_1 = -k\frac{n}{d_1}, \quad a_i = c_i\frac{n}{d_i}, \quad i = 2, 3, \dots, N-1, \quad a_N - k\frac{n}{d_N} = c_N\frac{n}{d_N}.$$

The desired conclusion follows easily from these equations. ■

Proposition 4.2 *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block. Then $K^0(A)$ is generated by $[\Lambda_1], [\Lambda_2], \dots, [\Lambda_N]$. Furthermore, for $a_1, a_2, \dots, a_N \in \mathbb{Z}$ we have that*

$$a_1[\Lambda_1] + a_2[\Lambda_2] + \cdots + a_N[\Lambda_N] = 0$$

if and only if there exist $b_1, b_2, \dots, b_N \in \mathbb{Z}$ such that $\sum_{i=1}^{N} b_i = 0$ and

$$a_i = b_i\frac{n}{d_i}, \quad i = 1, 2, \dots, N.$$

Proof Choose $t_1, t_2, \dots, t_N \in\,]0,1[$ such that $e^{2\pi i t_k}$, $k = 1, 2, \dots, N$, are the exceptional points for A. Set

$$B = \{f \in C[0,1] \otimes M_n : f(t_k) \in M_{d_k}, k = 1, 2, \dots, N\}.$$

Define a $*$-homomorphism $\iota\colon A \to B$ by $\iota(f)(t) = f(e^{2\pi i t})$. Let $\pi\colon A \to M_n$ be evaluation at $1 \in \mathbb{T}$. Let $\alpha\colon M_n \to M_n \oplus M_n$ denote the map $\alpha(x) = (x, x)$. Let $\beta\colon B \to M_n \oplus M_n$ be the map $\beta(f) = \big(f(0), f(1)\big)$. We have a pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi} & M_n \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\alpha} \\ B & \xrightarrow[\beta]{} & M_n \oplus M_n \end{array}$$

and hence by [2, Theorem 21.5.1] a six-term exact sequence of the form

$$\begin{array}{ccccc} K^0(M_n \oplus M_n) & \xrightarrow{(-\alpha^*, \beta^*)} & K^0(M_n) \oplus K^0(B) & \xrightarrow{\pi^* + \iota^*} & K^0(A) \\ \uparrow & & & & \downarrow \\ K^1(A) & \xleftarrow[\pi^* + \iota^*]{} & K^1(M_n) \oplus K^1(B) & \xleftarrow[(-\alpha^*, \beta^*)]{} & K^1(M_n). \end{array}$$

$K^0(M_n \oplus M_n) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $[\pi_1]$ and $[\pi_2]$ where $\pi_1, \pi_2\colon M_n \oplus M_n \to M_n$ are the coordinate projections. $K^0(M_n) \cong \mathbb{Z}$ is generated by the class of the identity map id on M_n. Note that

$$\pi^*([\mathrm{id}]) = \frac{n}{d_1}[\Lambda_1^A],$$

$$\iota^*([\Lambda_i^B]) = [\Lambda_i^A], \quad i = 1, 2, \dots, N,$$

$$(-\alpha^*, \beta^*)(a[\pi_1] + b[\pi_2]) = \left(-(a+b)[\mathrm{id}], (a+b)\frac{n}{d_1}[\Lambda_1^B]\right).$$

As $\pi^* + \iota^*$ maps onto $K^0(A)$ (because $K^1(M_n) = 0$) and as $\operatorname{im}(\pi^*) \subseteq \operatorname{im}(\iota^*)$, we see that ι^* is surjective. Assume that $\iota^*(x) = 0$. Then $(0, x) \in \operatorname{im}(-\alpha^*, \beta^*)$ and hence $x = 0$ by the above. Thus ι^* is an isomorphism and the conclusion follows from Lemma 4.1. ■

Proposition 4.3 *Let $A = A(n, d_1, d_2, \dots, d_N)$ and $B = A(m, e_1, e_2, \dots, e_M)$ be building blocks and let $h\colon K^0(B) \to K^0(A)$ be a group homomorphism. For every $j = 1, 2, \dots, M$, $i = 1, 2, \dots, N$, there is a uniquely determined integer h_{ji}, with $0 \le h_{ji} < \frac{n}{d_i}$ for $i \neq N$, such that*

$$\begin{pmatrix} h([\Lambda_1^B]) \\ h([\Lambda_2^B]) \\ \vdots \\ h([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & h_{22} & \cdots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \cdots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}.$$

This will be called the standard form for h.

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The integers determined by h above satisfy the equations

$$\frac{m}{e_j}h_{ji} \equiv \frac{m}{e_M}h_{Mi} \mod \frac{n}{d_i}, \quad j = 1, 2, \ldots, M, i = 1, 2, \ldots, N,$$

$$\frac{m}{e_j}\sum_{i=1}^{N} h_{ji}d_i = \frac{m}{e_M}\sum_{i=1}^{N} h_{Mi}d_i, \quad j = 1, 2, \ldots, M.$$

Proof By Proposition 4.2, or simply because homotopic $*$-homomorphisms $A \to M_n$ define the same elements in $K^0(A)$, we have that

$$\frac{n}{d_N}[\Lambda_N^A] = \frac{n}{d_i}[\Lambda_i^A], \quad i = 1, 2, \ldots, N.$$

From this the existence follows.

To check uniqueness, assume

$$\begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & h_{22} & \cdots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \cdots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix} = 0$$

where

$$-\frac{n}{d_i} < h_{ji} < \frac{n}{d_i}, \quad i = 1, 2, \ldots, N-1, j = 1, 2, \ldots, M.$$

Fix some $j = 1, 2, \ldots, M$. By Proposition 4.2 there exist integers b_{ji} such that $h_{ji} = b_{ji}\frac{n}{d_i}$, $i = 1, 2, \ldots, N$. Therefore $h_{ji} = 0$ for $i = 1, 2, \ldots, N$.

Finally, to prove the equations above, fix again some $j = 1, 2, \ldots, M$. Note that

$$0 = h(0) = h\left(-\frac{m}{e_j}[\Lambda_j^B] + \frac{m}{e_M}[\Lambda_M^B]\right) = \sum_{i=1}^{N}\left(-\frac{m}{e_j}h_{ji} + \frac{m}{e_M}h_{Mi}\right)[\Lambda_i^A].$$

Hence there exist integers b_{ji}, $i = 1, 2, \ldots, N$, such that $\sum_{i=1}^{N} b_{ji} = 0$ and

$$-\frac{m}{e_j}h_{ji} + \frac{m}{e_M}h_{Mi} = b_{ji}\frac{n}{d_i}.$$

The desired conclusion follows easily from these equations. ■

From now on, let $A = A(n, d_1, d_2, \ldots, d_N)$ and $B = A(m, e_1, e_2, \ldots, e_M)$ be building blocks. Define a group homomorphism

$$\Gamma\colon KK(A, B) \longrightarrow \mathrm{Hom}\big(K^0(B), K^0(A)\big) \oplus K_1(B)$$

by

$$\Gamma(\kappa) = (\kappa^*, \kappa_*[\nu^A]).$$

We want to show that Γ is an isomorphism in certain cases.

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Proposition 4.4 *Let $h\colon K^0(B) \to K^0(A)$ be a group homomorphism with standard form*

$$\begin{pmatrix} h([\Lambda_1^B]) \\ h([\Lambda_2^B]) \\ \vdots \\ h([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & h_{22} & \cdots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \cdots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}$$

where $h_{jN} \geq \frac{n}{d_N}$ for $j = 1, 2, \ldots, M$, and $\sum_{i=1}^N h_{Mi} d_i = e_M$. Let $\chi \in K_1(B)$. There is a unital $*$-homomorphism $\varphi\colon A \to B$ such that $\Gamma([\varphi]) = (h, \chi)$.

Proof Let $1 \leq i \leq N$. By Proposition 4.3 there is an integer s_i, $0 \leq s_i < \frac{n}{d_i}$, and integers l_{ji}, $j = 1, 2, \ldots, M$, such that

$$\frac{m}{e_j} h_{ji} = l_{ji} \frac{n}{d_i} + s_i. \tag{1}$$

Note that $l_{ji} \geq 0$ for $i = 1, 2, \ldots, N-1$, and $l_{jN} \geq 1$. By Proposition 4.3 we see that for $j = 1, 2, \ldots, M$,

$$m = \frac{m}{e_M} \sum_{i=1}^N h_{Mi} d_i = \frac{m}{e_j} \sum_{i=1}^N h_{ji} d_i = \sum_{i=1}^N (l_{ji} n + s_i d_i).$$

By (1) there exists a unitary $V_j \in M_m$ such that the matrix

$$V_j \operatorname{diag}\big(\Lambda_1^{s_1}(f), \ldots, \Lambda_N^{s_N}(f), \underbrace{f(x_1), \ldots, f(x_1)}_{l_{j1} \text{ times}}, \ldots, \underbrace{f(x_N), \ldots, f(x_N)}_{l_{jN} \text{ times}}\big) V_j^*$$

belongs to $M_{e_j} \subseteq M_m$ for all $f \in A$.

Set

$$L = \frac{1}{n}\Big(m - \sum_{i=1}^N s_i d_i\Big) = \sum_{i=1}^N l_{ji}, \quad j = 1, 2, \ldots, M.$$

Let $x_1, x_2, \ldots, x_N$ denote the exceptional points of A and let $y_1, y_2, \ldots, y_M$ be those of B. Choose continuous functions $\lambda_1, \lambda_2, \ldots, \lambda_{L-1}\colon \mathbb{T} \to \mathbb{T}$ such that

$$\big(\lambda_1(y_j), \lambda_2(y_j), \ldots, \lambda_{L-1}(y_j)\big) = \big(\underbrace{x_1, \ldots, x_1}_{l_{j1} \text{ times}}, \ldots, \underbrace{x_{N-1}, \ldots, x_{N-1}}_{l_{j(N-1)} \text{ times}}, \underbrace{x_N, \ldots, x_N}_{l_{jN}-1 \text{ times}}\big)$$

as ordered tuples. Choose a unitary $U \in C(\mathbb{T}) \otimes M_m$ such that $U(y_j) = V_j$. Define a unital $*$-homomorphism $\psi\colon A \to B$ by

$$\begin{aligned} &\psi(f)(z) \\ &\quad = U(z) \operatorname{diag}\Big(\Lambda_1^{s_1}(f), \ldots, \Lambda_N^{s_N}(f), f\big(\lambda_1(z)\big), \ldots, f\big(\lambda_{L-1}(z)\big), f(x_N)\Big) U(z)^*. \end{aligned}$$

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By Theorem 3.2 we have that $\chi = \sum_{j=1}^{M} a_j[U_j^B]$ for some $a_1, a_2, \ldots, a_M \in \mathbb{Z}$. Let

$$\psi_*[v^A] = \sum_{i=1}^{M} b_j[U_j^B]$$

in $K_1(B)$. Define $\xi\colon \mathbb{T} \to \mathbb{T}$ by

$$\xi(z) = \prod_{j=1}^{M} \mathrm{Det}\big(U_j^B(z)\big)^{a_j - b_j}, \tag{2}$$

and define $\lambda_L\colon \mathbb{T} \to \mathbb{T}$ by $\lambda_L(z) = \xi(z)x_N$. Note that $\lambda_L(y_j) = x_N$, $j = 1, 2, \ldots, M$. Define $\varphi\colon A \to B$ by

$$\varphi(f)(z) = U(z)\,\mathrm{diag}\Big(\Lambda_1^{s_1}(f), \ldots, \Lambda_N^{s_N}(f), f\big(\lambda_1(z)\big), \ldots, f\big(\lambda_L(z)\big)\Big)\,U(z)^*.$$

By Lemma 3.3 and (2) we see that in $K_1(B)$,

$$\begin{aligned}\varphi_*[v^A] &= \psi_*[v^A] + \Big[z \mapsto U(z)\,\mathrm{diag}\Big(1, 1, \ldots, 1, v^A\big(\lambda_L(z)\big)\,v^A(x_N)^*\Big)\,U(z)^*\Big]\\ &= \psi_*[v^A] + \sum_{j=1}^{M}(a_j - b_j)[U_j^B] = \sum_{j=1}^{M} a_j[U_j^B].\end{aligned}$$

Since $\varphi(f)(y_j) = \psi(f)(y_j)$, $f \in A$, $j = 1, 2, \ldots, M$, we conclude that

$$\begin{aligned}\varphi^*([\Lambda_j^B]) = [\Lambda_j^B \circ \varphi] = [\Lambda_j^B \circ \psi] &= \sum_{i=1}^{N}\left(s_i + l_{ji}\frac{n}{d_i}\right)\frac{e_j}{m}[\Lambda_i^A]\\ &= \sum_{i=1}^{N} h_{ji}[\Lambda_i^A] = h([\Lambda_j^B]).\end{aligned}$$ ■

Lemma 4.5 *Let $h\colon K^0(B) \to K^0(A)$ be a group homomorphism and assume that there exists a homomorphism $h'\colon K^0(B) \to K^0(A)$ with standard form*

$$\begin{pmatrix} h'([\Lambda_1^B]) \\ h'([\Lambda_2^B]) \\ \vdots \\ h'([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h'_{11} & h'_{12} & \cdots & h'_{1N} \\ h'_{21} & h'_{22} & \cdots & h'_{2N} \\ \vdots & \vdots & & \vdots \\ h'_{M1} & h'_{M2} & \cdots & h'_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}$$

where $h'_{jN} \geq \frac{n}{d_N}$ for $j = 1, 2, \ldots, M$, and $\sum_{i=1}^{N} h'_{Mi} d_i = e_M$. Then there is a $\kappa \in KK(A, B)$ such that $\kappa^* = h$ in $\mathrm{Hom}\big(K^0(B), K^0(A)\big)$.

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Proof By Proposition 4.4 there exists an element $\nu \in KK(A,B)$ such that $\nu^* = h'$. Let $h \in \operatorname{Hom}\big(K^0(B), K^0(A)\big)$ have standard form

$$\begin{pmatrix} h([\Lambda_1^B]) \\ h([\Lambda_2^B]) \\ \vdots \\ h([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & h_{22} & \cdots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \cdots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}.$$

By adding an integer-multiple of h' we may assume that $h_{jN} \geq 0$ for $j = 1,2,\dots,M$. Define l_{ji} and s_i, $i = 1,2,\dots,N$, as in the proof of Proposition 4.4. Let

$$c = \frac{m}{e_M} \sum_{i=1}^{N} h_{Mi} d_i = \frac{m}{e_j} \sum_{i=1}^{N} h_{ji} d_i = \sum_{i=1}^{N} (l_{ji} n + s_i d_i), \quad j = 1,2,\dots,M.$$

Choose a positive integer d such that $c \leq dm$. Choose for each $j = 1,2,\dots,M$, a unitary $V_j \in M_{dm}$ such that the matrix

$$V_j \operatorname{diag}\big(\Lambda_1^{s_1}(f),\dots,\Lambda_N^{s_N}(f), \underbrace{f(x_1),\dots,f(x_1)}_{l_{j1}\text{ times}},\dots,\underbrace{f(x_N),\dots,f(x_N)}_{l_{jN}\text{ times}}, \underbrace{0,\dots,0}_{dm-c}\big) V_j^*$$

belongs to $M_{de_j} \subseteq M_{dm}$ for all $f \in A$.

As in the proof of Proposition 4.4 these matrices can be connected to define a $*$-homomorphism $\varphi\colon A \to M_d(B)$. We leave it with the reader to check that $\varphi^* = h$ on $K^0(B)$. Set $\kappa = [\varphi]$. ■

Proposition 4.6 *Assume that there exists a homomorphism $h'\colon K^0(B) \to K^0(A)$ with standard form*

$$\begin{pmatrix} h'([\Lambda_1^B]) \\ h'([\Lambda_2^B]) \\ \vdots \\ h'([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h'_{11} & h'_{12} & \cdots & h'_{1N} \\ h'_{21} & h'_{22} & \cdots & h'_{2N} \\ \vdots & \vdots & & \vdots \\ h'_{M1} & h'_{M2} & \cdots & h'_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}$$

where $h'_{jN} \geq \frac{n}{d_N}$ for $j = 1,2,\dots,M$, and $\sum_{i=1}^{N} h'_{Mi} d_i = e_M$. Then the map $\Gamma\colon KK(A,B) \to \operatorname{Hom}\big(K^0(B), K^0(A)\big) \oplus K_1(B)$ is an isomorphism.

Proof By Theorem 3.2 there exist finite abelian groups G and H such that $K_1(A) \cong \mathbb{Z} \oplus G$, $K_1(B) \cong \mathbb{Z} \oplus H$. By the universal coefficient theorem, [23, Theorem 1.17],

$$\begin{aligned} KK(A,B) &\cong \operatorname{Ext}\big(K_0(A), K_1(B)\big) \oplus \operatorname{Ext}\big(K_1(A), K_0(B)\big) \\ &\qquad \oplus \operatorname{Hom}\big(K_0(A), K_0(B)\big) \oplus \operatorname{Hom}\big(K_1(A), K_1(B)\big) \\ &\cong 0 \oplus G \oplus \mathbb{Z} \oplus \operatorname{Hom}(G,H) \oplus K_1(B). \end{aligned}$$

By the universal coefficient theorem again, $K^0(A) \cong K_1(A)$ and $K^0(B) \cong K_1(B)$. Hence

$$\operatorname{Hom}\big(K^0(B), K^0(A)\big) \oplus K_1(B) \cong K_1(A) \oplus \operatorname{Hom}(H, G) \oplus K_1(B).$$

Note that $\operatorname{Hom}(G, H) \cong \operatorname{Hom}(H, G)$. Thus $\operatorname{Hom}\big(K^0(B), K^0(A)\big) \oplus K_1(B)$ and $KK(A, B)$ are isomorphic groups. Since any surjective endomorphism of a finitely generated abelian group is an isomorphism, it suffices to show that Γ is surjective.

Let $(h, \chi) \in \operatorname{Hom}\big(K^0(B), K^0(A)\big) \oplus K_1(B)$. By Lemma 4.5 there exists an element $\kappa \in KK(A, B)$ such that $\Gamma(\kappa) = (h - h', \eta)$ for some $\eta \in K_1(B)$. Next, by Proposition 4.4 there exists a $\nu \in KK(A, B)$ such that $\Gamma(\nu) = (h', \chi - \eta)$. Thus $\Gamma(\kappa + \nu) = (h, \chi)$. ■

Theorem 4.7 *Let $A = A(n, d_1, d_2, \ldots, d_N)$ and $B = A(m, e_1, e_2, \ldots, e_M)$ be building blocks such that $s(B) \geq Nn$ and assume that there exists an element κ in $KK(A, B)_e$. Then the map $\Gamma\colon KK(A, B) \to \operatorname{Hom}\big(K^0(B), K^0(A)\big) \oplus K_1(B)$ is an isomorphism and there exists a unital $*$-homomorphism $\varphi\colon A \to B$ such that $[\varphi] = \kappa$.*

Proof Let $\kappa^*\colon K^0(B) \to K^0(A)$ have standard form

$$\begin{pmatrix} \kappa^*([\Lambda_1^B]) \\ \kappa^*([\Lambda_2^B]) \\ \vdots \\ \kappa^*([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & h_{22} & \cdots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \cdots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}.$$

Let $\cdot$ denote the Kasparov product. By assumption we have that $[1_A] \cdot \kappa = [1_B]$ in $KK(\mathbb{C}, B) \cong K_0(B)$. Thus

$$[1_B] \cdot [\Lambda_j^B] = [1_A] \cdot \kappa \cdot [\Lambda_j^B] = [1_A] \cdot \Big(\sum_{i=1}^{N} h_{ji} [\Lambda_i^A]\Big)$$

in $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$. Hence $e_j = \sum_{i=1}^{N} h_{ji} d_i$ for $j = 1, 2, \ldots, M$. This implies that $h_{jN} > \frac{n}{d_N}$ since

$$Nn \leq e_j = \sum_{i=1}^{N} h_{ji} d_i < \sum_{i=1}^{N-1} \frac{n}{d_i} d_i + h_{jN} d_N = (N-1)n + h_{jN} d_N.$$

Therefore Γ is an isomorphism by Proposition 4.6. By Proposition 4.4 there is a unital $*$-homomorphism $\varphi\colon A \to B$ such that $\Gamma([\varphi]) = \Gamma(\kappa)$. Thus $[\varphi] = \kappa$. ■

5 The Commutator Subgroup of the Unitary Group

In this section we analyze the unitary group modulo the closure of its commutator subgroup for building blocks.

Lemma 5.1 *Let A be a unital inductive limit of a sequence of finite direct sums of building blocks. Then the canonical maps $\pi_0\big(U(A)\big) \to K_1(A)$ and $\pi_1\big(U(A)\big) \to K_0(A)$ are isomorphisms.*

Proof Following [24] we let $k_n(\cdot) = \pi_{n+1}\big(U(\cdot)\big)$ for every integer $n \geq -1$. By [24, Proposition 2.6] it suffices to show that the canonical maps $k_{-1}(A) \to k_{-1}(A \otimes \mathcal{K}) \cong K_1(A)$ and $k_0(A) \to k_0(A \otimes \mathcal{K}) \cong K_0(A)$ are isomorphisms, where $\mathcal{K}$ denotes the set of compact operators on a separable infinite dimensional Hilbert-space. As noted in [24] it follows from [14, Proposition 4.4] that k_n is a continuous functor. Since it is obviously additive, we may assume that A is a building block.

As in the proof of Theorem 3.2 we see that there exists finite dimensional C^*-algebras F_1 and F_2 such that we have a short exact sequence of the form

$$0 \longrightarrow SF_1 \longrightarrow A \longrightarrow F_2 \longrightarrow 0.$$

Apply [24, Proposition 2.5] to this short exact sequence and the one obtained by tensoring with $\mathcal{K}$ to obtain two long exact sequences for k_n. It is well-known that the canonical maps $k_i(F_2) \to k_i(F_2 \otimes \mathcal{K})$ and $k_i(SF_1) \to k_i(SF_1 \otimes \mathcal{K})$ are isomorphisms for $i = -1, 0$ (*cf.* [24, Lemma 2.3]), so the theorem follows from the five lemma in algebra. ∎

Let A be a unital C^*-algebra. Let $\pi_A \colon U(A)/\overline{DU(A)} \to K_1(A)$ denote the group homomorphism $\pi_A\big(q'_A(u)\big) = [u]$.

Proposition 5.2 *Let A be a unital inductive limit of a sequence of finite direct sums of building blocks. There exists a group homomorphism*

$$\lambda_A \colon \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)} \longrightarrow U(A)/\overline{DU(A)},$$
$$\lambda_A\big(q_A(\widehat{a})\big) = q'_A(e^{2\pi i a}), \quad a \in A_{\mathrm{sa}}.$$

This map is an isometry when $\operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)}$ is equipped with the metric d_A, and it gives rise to a short exact sequence of abelian groups

$$0 \longrightarrow \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)} \xrightarrow{\lambda_A} U(A)/\overline{DU(A)} \xrightarrow{\pi_A} K_1(A) \longrightarrow 0.$$

This sequence is natural in A and splits unnaturally.

Proof Combine Lemma 5.1 with [27, Lemma 6.4]. ∎

Proposition 5.3 *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block. Let $u \in A$ be a unitary. Assume that*

$$\operatorname{Det}\big(u(z)\big) = 1, \quad z \in \mathbb{T},$$
$$\operatorname{Det}\big(\Lambda_i(u)\big) = 1, \quad i = 1, 2, \ldots, N.$$

Then $u \in \overline{DU(A)}$.

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Proof First note that $[u] = 0$ in $K_1(A)$ by Lemma 3.3. Hence $q_A'(u) = q_A'(e^{2\pi i a})$ by Proposition 5.2 for some self-adjoint element $a \in A$. Since

$$\operatorname{Det}\big(u(z)\big) = \operatorname{Det}(e^{2\pi i a(z)}) = e^{2\pi i \operatorname{Tr}(a(z))}$$

it follows that $\operatorname{Tr}\big(a(z)\big) = k$ for some $k \in \mathbb{Z}$ and all $z \in \mathbb{T}$. Hence $\widehat{a} = \frac{k}{n}\widehat{1}$ in $\operatorname{Aff} T(A)$ by Lemma 2.3. By applying λ_A we get that $q_A'(u) = q_A'(e^{2\pi i a}) = q_A'(\lambda 1)$, where $\lambda = e^{2\pi i \frac{k}{n}}$. Since $\operatorname{Det}\big(\Lambda_i(u)\big) = 1$ we see that $\lambda^{d_i} = 1$, $i = 1, 2, \dots, N$. Thus $\lambda^d = 1$ where $d = \gcd(d_1, d_2, \dots, d_N)$. But then $\frac{k}{n} = \frac{l}{d}$ for some $l \in \mathbb{Z}$. It follows by Corollary 3.7 that $\frac{k}{n}\widehat{1} \in \rho_A\big(K_0(A)\big)$ and hence by Proposition 5.2 we get that $\lambda 1 \in \overline{DU(A)}$. ■

Lemma 5.4 *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block with exceptional points $x_1, x_2, \dots, x_N$. Let $g\colon \mathbb{T} \to \mathbb{T}$ be a continuous function and let $h_i \in \mathbb{T}$ be such that $h_i^{\frac{n}{d_i}} = g(x_i)$, $i = 1, 2, \dots, N$. There exists a unitary $u \in A$ such that*

$$\operatorname{Det}\big(u(z)\big) = g(z), \quad z \in \mathbb{T},$$
$$\operatorname{Det}\big(\Lambda_i(u)\big) = h_i, \quad i = 1, 2, \dots, N.$$

Proof Choose a continuous function $f\colon \mathbb{T} \to \mathbb{T}$ such that $f(x_i)^{d_i} = h_i$. Define a unitary $v \in A$ by $v = f \otimes 1$. Since

$$f(x_i)^n = h_i^{\frac{n}{d_i}} = g(x_i),$$

we can define a unitary $w \in A$ by

$$w(z) = \operatorname{diag}\big(g(z)f(z)^{-n}, 1, 1, \dots, 1\big), \quad z \in \mathbb{T}.$$

Set $u = wv$. ■

Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block. By Lemma 5.4 there exist unitaries $w_1^A, w_2^A, \dots, w_N^A \in A$ such that $\operatorname{Det}\big(w_k^A(z)\big) = 1$, $z \in \mathbb{T}$, $k = 1, 2, \dots, N$, and such that

$$\operatorname{Det}\big(\Lambda_l(w_k^A)\big) = \begin{cases} 1 & l \neq k, \\ \exp(2\pi i \frac{d_l}{n}) & l = k. \end{cases}$$

Let $A = A(n, d_1, d_2, \dots, d_N)$ and $B = A(m, e_1, e_2, \dots, e_M)$ be building blocks. Let $\varphi\colon A \to B$ be a unital $*$-homomorphism. As in [27, Chapter 1] we define $s^\varphi(j, i)$ to be the multiplicity of the representation Λ_i^A in the representation $\Lambda_j^B \circ \varphi$ for $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$.

The following theorem shows that there is a connection between $KK(A, B)$ and the torsion subgroups of $U(A)/\overline{DU(A)}$ and $U(B)/\overline{DU(B)}$ when A and B are building blocks.

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Theorem 5.5 *Let $A = A(n, d_1, d_2, \dots, d_N)$ and $B = A(m, e_1, e_2, \dots, e_M)$ be building blocks and let $\varphi, \psi\colon A \to B$ be unital $*$-homomorphisms. The following are equivalent.*

(i) $\varphi^* = \psi^*$ *in* $\operatorname{Hom}\big(K^0(B), K^0(A)\big)$,
(ii) $s^{\varphi}(j,i) \equiv s^{\psi}(j,i) \mod \frac{n}{d_i}$, $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$,
(iii) $\varphi^{\#}(x) = \psi^{\#}(x)$, $x \in \operatorname{Tor}\big(U(A)/\overline{DU(A)}\big)$,
(iv) $\varphi^{\#}\big(q'_A(w_k^A)\big) = \psi^{\#}\big(q'_A(w_k^A)\big)$, $k = 1, 2, \dots, N$.

Proof For each $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$, let r_i^j and s_i^j be the integers with $0 \le r_i^j, s_i^j < \frac{n}{d_i}$, and $r_i^j \equiv s^{\varphi}(j,i) \mod \frac{n}{d_i}$, $s_i^j \equiv s^{\psi}(j,i) \mod \frac{n}{d_i}$. By Lemma 2.1 there exist $a_1^j, \dots, a_{K_j}^j, b_1^j, \dots, b_{L_j}^j \in \mathbb{T}$ and unitaries $u_j, v_j \in M_{e_j}$ such that

(3)
$$\Lambda_j^B \circ \varphi(f) = u_j \operatorname{diag}\big(\Lambda_1^{r_1^j}(f), \Lambda_2^{r_2^j}(f), \dots, \Lambda_N^{r_N^j}(f), f(a_1^j), f(a_2^j), \dots, f(a_{K_j}^j)\big)\, u_j^*,$$

(4)
$$\Lambda_j^B \circ \psi(f) = v_j \operatorname{diag}\big(\Lambda_1^{s_1^j}(f), \Lambda_2^{s_2^j}(f), \dots, \Lambda_N^{s_N^j}(f), f(b_1^j), f(b_2^j), \dots, f(b_{L_j}^j)\big)\, v_j^*.$$

Since
$$e_j = \sum_{i=1}^N r_i^j d_i + K_j n = \sum_{i=1}^N s_i^j d_i + L_j n,$$
we remark that if (ii) holds then $K_j = L_j$, $j = 1, 2, \dots, M$.

Note that

$$\varphi^*([\Lambda_j^B]) = \sum_{i=1}^N r_i^j [\Lambda_i^A] + K_j \frac{n}{d_N} [\Lambda_N^A] = \sum_{i=1}^{N-1} r_i^j [\Lambda_i^A] + \left(K_j \frac{n}{d_N} + r_N^j\right) [\Lambda_N^A],$$
$$\psi^*([\Lambda_j^B]) = \sum_{i=1}^N s_i^j [\Lambda_i^A] + L_j \frac{n}{d_N} [\Lambda_N^A] = \sum_{i=1}^{N-1} s_i^j [\Lambda_i^A] + \left(L_j \frac{n}{d_N} + s_N^j\right) [\Lambda_N^A].$$

By Proposition 4.3 we see that $\varphi^* = \psi^*$ if and only if for every $j = 1, 2, \dots, M$,

$$K_j \frac{n}{d_N} + r_N^j = L_j \frac{n}{d_N} + s_N^j, \quad \text{and} \quad r_i^j = s_i^j, \quad i = 1, 2, \dots, N-1.$$

It follows that (i) holds if and only if $r_i^j = s_i^j$ and $K_j = L_j$ for every i, j. But this statement is equivalent to (ii) by the remark above.

Assume (ii) holds. To prove (iii), let $u \in A$ be a unitary such that $q_A(u)$ has finite order in the group $U(A)/\overline{DU(A)}$. Then $\operatorname{Det}\big(u(\cdot)\big)$ is constant. By (3), (4), and since $K_j = L_j$, $j = 1, 2, \dots, M$, it follows that

$$\operatorname{Det}\Big(\Lambda_j\big(\varphi(u)\big)\Big) = \operatorname{Det}\Big(\Lambda_j\big(\psi(u)\big)\Big), \quad j = 1, 2, \dots, M.$$

In particular, $\operatorname{Det}\big(\varphi(u)(\cdot)\big)$ equals $\operatorname{Det}\big(\psi(u)(\cdot)\big)$ at the exceptional points of B. On the other hand, $\operatorname{Det}\big(\varphi(u)(\cdot)\big)$ and $\operatorname{Det}\big(\psi(u)(\cdot)\big)$ are constant functions on $\mathbb{T}$ and are hence equal everywhere. We may therefore use Proposition 5.3 to conclude that $\varphi^{\#}\big(q_A(u)\big) = \psi^{\#}\big(q_A(u)\big)$. (iii) ⇒ (iv) is trivial. Assume (iv). By (3) and (4),

$$\exp\left(2\pi i\frac{d_k}{n}r_k^j\right) = \operatorname{Det}\big(\Lambda_j^B\circ\varphi(w_k^A)\big) = \operatorname{Det}\big(\Lambda_j^B\circ\psi(w_k^A)\big) = \exp\left(2\pi i\frac{d_k}{n}s_k^j\right).$$

Hence $r_k^j = s_k^j$ for $k = 1,2,\dots,N$, $j = 1,2,\dots,M$, and we have (ii). ■

Proposition 5.6 *Let A and B be finite direct sums of building blocks, let $\varphi,\psi\colon A\to B$ be unital $*$-homomorphisms, and let x be an element of finite order in the group $U(A)/\overline{DU(A)}$. If $[\varphi]=[\psi]$ in $KK(A,B)$ then $\varphi^{\#}(x)=\psi^{\#}(x)$.*

Proof We may assume that B is a building block rather than a finite direct sum of building blocks. Let $A = A_1\oplus A_2\oplus\cdots\oplus A_R$ where each A_i is a building block, and let $\iota_i\colon A_i\to A$ denote the inclusion. Let $p_1,p_2,\dots,p_R$ be the minimal non-zero central projections in A. Since $\varphi_*[p_i]=\psi_*[p_i]$ in $K_0(B)$, it follows from Lemma 3.4 that there is a unitary $u\in B$ such that $u\varphi(p_i)u^*=\psi(p_i)$, $i=1,2,\dots,R$. Hence we may assume that $\varphi(p_i)=\psi(p_i)$, $i=1,2,\dots,R$. Set $q_i=\varphi(p_i)$.

Let $\varphi_i,\psi_i\colon A_i\to q_iBq_i$ be the induced maps and let $\epsilon_i\colon q_iBq_i\to B$ be the inclusion, $i=1,2,\dots,R$. If $q_i\neq 0$ then $[\epsilon_i]\in KK(q_iBq_i,B)$ is a KK-equivalence by [23, Theorem 7.3]. Thus

$$[\varphi_i] = [\epsilon_i]^{-1}\cdot[\varphi]\cdot[\iota_i] = [\epsilon_i]^{-1}\cdot[\psi]\cdot[\iota_i] = [\psi_i]$$

in $KK(A_i,q_iBq_i)$. Let $x = q_A'(u)$ where $u\in A$ is a unitary. Let $u=\sum_{i=1}^R \iota_i(u_i)$ where $u_i\in A_i$. By Theorem 5.5 and Corollary 3.5 we see that $\varphi_i(u_i)=\psi_i(u_i)$ mod $\overline{DU(q_iBq_i)}$ and thus $\epsilon_i\circ\varphi_i(u_i)+(1-q_i)=\epsilon_i\circ\psi_i(u_i)+(1-q_i)$ mod $\overline{DU(B)}$. Hence

$$\begin{aligned}\varphi(u) &= \prod_{i=1}^R \varphi\big(\iota_i(u_i)+(1-p_i)\big) = \prod_{i=1}^R\big(\epsilon_i\circ\varphi_i(u_i)+(1-q_i)\big)\\ &= \prod_{i=1}^R\big(\epsilon_i\circ\psi_i(u_i)+(1-q_i)\big) = \prod_{i=1}^R\psi\big(\iota_i(u_i)+(1-p_i)\big) = \psi(u)\end{aligned}$$

modulo $\overline{DU(B)}$. ■

6 Homomorphisms Between Building Blocks

In this section we improve a result of Thomsen on $*$-homomorphisms between building blocks that will be needed in the next section.

Whenever $\theta_1,\theta_2,\dots,\theta_L$ are real numbers such that

$$\theta_1\le\theta_2\le\cdots\le\theta_L\le\theta_1+1,$$

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it will be convenient for us in the following to define θ_n for every $n \in \mathbb{Z}$ by the formula $\theta_{pL+r} = \theta_r + p$, where $p \in \mathbb{Z}$, $r = 1, 2, \dots, L$. Note that for every $n \in \mathbb{Z}$,

$$\theta_n \leq \theta_{n+1} \leq \cdots \leq \theta_{n+L} \leq \theta_n + 1,$$

and

$$(e^{2\pi i\theta_1}, e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_L}) = (e^{2\pi i\theta_n}, e^{2\pi i\theta_{n+1}}, \dots, e^{2\pi i\theta_{n+L}})$$

as unordered L-tuples.

Lemma 6.1 *Let $a_1, a_2, \dots, a_L \in \mathbb{T}$ and let k be an integer. There exist real numbers $\theta_1, \theta_2, \dots, \theta_L$ such that*

$$\theta_1 \leq \theta_2 \leq \cdots \leq \theta_L \leq \theta_1 + 1,$$

such that $\sum_{r=1}^{L} \theta_r \in [k, k+1[$ and such that

$$(a_1, a_2, \dots, a_L) = (e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, \dots, e^{2\pi i\theta_L})$$

as unordered L-tuples.

Proof Choose $\omega_1, \omega_2, \dots, \omega_L \in [0, 1[$ such that $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_L$ and such that

$$(a_1, a_2, \dots, a_L) = (e^{2\pi i\omega_1}, e^{2\pi i\omega_2}, \dots, e^{2\pi i\omega_L})$$

as unordered L-tuples. Let l be the integer such that $\sum_{r=1}^{L} \omega_r \in [l, l+1[$. Set $\theta_r = \omega_{r+k-l}$. ■

Lemma 6.2 *Assume that*

$$\big(\exp(2\pi i\theta_1), \dots, \exp(2\pi i\theta_L)\big) = \big(\exp(2\pi i\omega_1), \dots, \exp(2\pi i\omega_L)\big)$$

as unordered L-tuples, where $\theta_1, \theta_2, \dots, \theta_L$ and $\omega_1, \omega_2, \dots, \omega_L$ are real numbers such that

$$\theta_1 \leq \theta_2 \leq \cdots \leq \theta_L \leq \theta_1 + 1,$$
$$\omega_1 \leq \omega_2 \leq \cdots \leq \omega_L \leq \omega_1 + 1.$$

Then $\theta_j = \omega_{r+j}$, $j = 1, 2, \dots, L$, where $r = \sum_{j=1}^{L} (\theta_j - \omega_j)$.

Proof Choose $m \in \mathbb{Z}$ such that $\theta_m < \theta_{m+1}$ and choose $n \in \mathbb{Z}$ such that

$$\theta_{m+1} = \omega_{n+1} > \omega_n.$$

Assume that

$$\theta_{m+p} = \omega_{n+q} + k.$$

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for some integers p, q with $1 \le p \le L$, $1 \le q \le L$, and an integer k. Then

$$-1 < \theta_{m+1} - \theta_{m+L} \le \theta_{m+1} - \theta_{m+p} = \omega_{n+1} - \omega_{n+q} - k \le -k,$$
$$0 \le \theta_{m+p} - \theta_{m+1} = \omega_{n+q} + k - \omega_{n+1} < \omega_{n+q} + k - \omega_n \le 1 + k.$$

Hence $k = 0$. By assumption it follows that for every $x \in \mathbb{R}$,

$$\#\{j = 1, 2, \dots, L : \theta_{m+j} = x\} = \#\{j = 1, 2, \dots, L : \omega_{n+j} = x\}.$$

Thus

$$(\theta_{m+1}, \theta_{m+2}, \dots, \theta_{m+L}) = (\omega_{n+1}, \omega_{n+2}, \dots, \omega_{n+L})$$

as unordered L-tuples. Therefore

$$\theta_{m+j} = \omega_{n+j}, \quad j = 1, 2, \dots, L.$$

Hence $\theta_j = \omega_{n-m+j}$ for $j = 1, 2, \dots, L$. From this it follows that $r = n - m$. ■

Proposition 6.3 *Let $\lambda_1, \lambda_2, \dots, \lambda_L \colon [0, 1] \to \mathbb{T}$ be continuous functions and let k be an integer. There exist continuous functions $F_1, F_2, \dots, F_L \colon [0, 1] \to \mathbb{R}$ such that $\sum_{j=1}^{L} F_j(0) \in [k, k+1[$ and such that for each $t \in [0, 1]$,*

$$F_1(t) \le F_2(t) \le \cdots \le F_L(t) \le F_1(t) + 1,$$

and

$$\big(\lambda_1(t), \lambda_2(t), \dots, \lambda_L(t)\big) = \Big(\exp\big(2\pi i F_1(t)\big), \exp\big(2\pi i F_2(t)\big), \dots, \exp\big(2\pi i F_L(t)\big)\Big)$$

as unordered L-tuples.

Proof Choose a positive integer n such that

$$|s - t| \le \frac{1}{n} \Longrightarrow \rho\big(\lambda_j(s), \lambda_j(t)\big) < \frac{1}{2L}, \quad s, t \in [0, 1], j = 1, 2, \dots, L.$$

We will prove by induction in m that there exist continuous functions $F_1, \dots, F_L$ that satisfy the above for $t \in [0, \frac{m}{n}]$. The case $m = 0$ follows from Lemma 6.1.

Now assume that for some $m = 0, 1, \dots, n-1$ we have constructed continuous functions $F_1, F_2, \dots, F_L \colon [0, \frac{m}{n}] \to \mathbb{R}$ such that $\sum_{j=1}^{L} F_j(0) \in [k, k+1[$, and such that for each $t \in [0, \frac{m}{n}]$, $F_1(t) \le F_2(t) \le \cdots \le F_L(t) \le F_1(t) + 1$, and

$$\big(\lambda_1(t), \lambda_2(t), \dots, \lambda_L(t)\big) = \Big(\exp\big(2\pi i F_1(t)\big), \exp\big(2\pi i F_2(t)\big), \dots, \exp\big(2\pi i F_L(t)\big)\Big)$$

as unordered L-tuples. Choose $\alpha_m \in \mathbb{R}$ such that $\rho\big(e^{2\pi i \alpha_m}, \lambda_j(\frac{m}{n})\big) \ge \frac{1}{2L}$ for $j = 1, 2, \dots, L$. Choose continuous functions $G_j \colon [\frac{m}{n}, \frac{m+1}{n}] \to]\alpha_m, \alpha_m + 1[$ such that for each $t \in [\frac{m}{n}, \frac{m+1}{n}]$,

$$G_1(t) \le G_2(t) \le \cdots \le G_L(t)$$

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and

$$\big(\lambda_1(t), \lambda_2(t), \dots, \lambda_L(t)\big) = \Big(\exp\big(2\pi i G_1(t)\big), \exp\big(2\pi i G_2(t)\big), \dots, \exp\big(2\pi i G_L(t)\big)\Big)$$

as unordered L-tuples. Set for $j = 1, 2, \dots, L$, $p \in \mathbb{Z}$,

$$G_{pL+j}(t) = G_j(t) + p, \quad t \in \left[\frac{m}{n}, \frac{m+1}{n}\right].$$

By Lemma 6.2 there exists an integer r such that for $j = 1, 2, \dots, L$,

$$F_j\left(\frac{m}{n}\right) = G_{r+j}\left(\frac{m}{n}\right).$$

Define for $j = 1, 2, \dots, L$, a continuous function $F'_j \colon [0, \frac{m+1}{n}] \to \mathbb{R}$ by

$$F'_j(t) = \begin{cases} F_j(t) & t \in [0, \frac{m}{n}], \\ G_{r+j}(t) & t \in [\frac{m}{n}, \frac{m+1}{n}]. \end{cases}$$

$F'_1, F'_2, \dots, F'_L$ satisfy the conclusion of the lemma for $t \in [0, \frac{m+1}{n}]$. ■

Proposition 6.4 *Let $A = A(n, d_1, d_2, \dots, d_N)$ and $B = A(m, e_1, e_2, \dots, e_M)$ be building blocks and let $\varphi \colon A \to B$ be a unital $*$-homomorphism. There exist integers $r_1, r_2, \dots, r_N$ with $0 \le r_i < \frac{n}{d_i}$, an integer $L \ge 0$, and a unitary $w \in M_m$ such that if $\psi \colon A \to B$ is a unital $*$-homomorphism with $\varphi^\#\big(q'_A(\omega_k^A)\big) = \psi^\#\big(q'_A(\omega_k^A)\big)$, $k = 1, 2, \dots, N$, and if $\gamma \colon \mathbb{T} \to \mathbb{R}$ is a continuous function such that*

$$\operatorname{Det}\big(\psi(v^A)(z)\big) = \operatorname{Det}\big(\varphi(v^A)(z)\big) \exp\big(2\pi i \gamma(z)\big), \quad z \in \mathbb{T},$$

then φ and ψ are approximately unitarily equivalent to $*$-homomorphisms of the form

$$\varphi'(f)(e^{2\pi i t}) = u(t) \operatorname{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i F_1(t)}), \dots, f(e^{2\pi i F_L(t)})\big)\, u(t)^*,$$
$$\psi'(f)(e^{2\pi i t}) = v(t) \operatorname{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i G_1(t)}), \dots, f(e^{2\pi i G_L(t)})\big)\, v(t)^*,$$

where $u, v \in C[0,1] \otimes M_m$ are unitaries with $u(0) = v(0) = 1$, $u(1) = v(1) = w$, and $F_1, F_2, \dots, F_L \colon [0,1] \to \mathbb{R}$ and $G_1, G_2, \dots, G_L \colon [0,1] \to \mathbb{R}$ are continuous functions such that for every $t \in [0,1]$,

$$F_1(t) \le F_2(t) \le \cdots \le F_L(t) \le F_1(t) + 1,$$
$$G_1(t) \le G_2(t) \le \cdots \le G_L(t) \le G_1(t) + 1,$$

and such that $\gamma(e^{2\pi i t}) = \sum_{r=1}^{L}\big(G_r(t) - F_r(t)\big)$ for every $t \in [0,1]$.

Proof By [27, Chapter 1] it follows that φ is approximately unitarily equivalent to a $*$-homomorphism $\varphi_1\colon A \to B$ of the form

(5)
$$\varphi_1(f)(e^{2\pi it}) = u_0(t)\,\mathrm{diag}\Big(\Lambda_1^{r_1}(f),\dots,\Lambda_N^{r_N}(f), f\big(\lambda_1(t)\big),\dots,f\big(\lambda_L(t)\big)\Big)\,u_0(t)^*$$

for $t \in [0,1]$, where $r_1, r_2, \dots, r_N$ are integers, $0 \le r_i < \frac{n}{d_i}$, $i = 1,2,\dots,N$, $\lambda_1,\lambda_2,\dots,\lambda_L\colon [0,1] \to \mathbb{T}$ are continuous functions, and $u_0 \in C[0,1] \otimes M_m$ is a unitary. Let l denote the winding number of $\mathrm{Det}\big(\varphi(v^A)(\cdot)\big)$. Let y be a unitary $(Ln)\times(Ln)$ matrix such that

$$y\,\mathrm{diag}(a_1,a_2,\dots,a_L)y^* = \mathrm{diag}(a_L,a_1,a_2,\dots,a_{L-1})$$

for all $a_1,a_2,\dots,a_L \in M_m$. Set

$$w = \mathrm{diag}(\underbrace{1,1,\dots,1}_{m-Ln\text{ times}}, y^l).$$

Let now $\psi\colon A \to B$ be given. As above ψ is approximately unitarily equivalent to a $*$-homomorphism $\psi_1\colon A \to B$ of the form

$$\psi_1(f)(e^{2\pi it}) = v_0(t)\,\mathrm{diag}\Big(\Lambda_1^{s_1}(f),\dots,\Lambda_N^{s_N}(f), f\big(\mu_1(t)\big),\dots,f\big(\mu_K(t)\big)\Big)\,v_0(t)^*.$$

Note that

$$s^{\varphi}(j,i)\frac{m}{e_j} = s^{\varphi_1}(j,i)\frac{m}{e_j} = r_i + \#\{r = 1,2,\dots,L : \lambda_r(y_j) = x_i\}\frac{n}{d_i},$$
$$s^{\psi}(j,i)\frac{m}{e_j} = s^{\psi_1}(j,i)\frac{m}{e_j} = s_i + \#\{r = 1,2,\dots,K : \mu_r(y_j) = x_i\}\frac{n}{d_i}.$$

By Theorem 5.5 it follows that $r_i = s_i$, $i = 1,2,\dots,N$. And since

$$m = Kn + \sum_{i=1}^{N} s_i d_i = Ln + \sum_{i=1}^{N} r_i d_i$$

we see that $K = L$.

By Proposition 6.3 choose continuous functions $F_1, F_2, \dots, F_L : [0,1] \to \mathbb{R}$ such that for every $t \in [0,1]$,

$$F_1(t) \le F_2(t) \le \cdots \le F_L(t) \le F_1(t) + 1,$$

and such that

$$\big(\lambda_1(t),\lambda_2(t),\dots,\lambda_L(t)\big) = (e^{2\pi iF_1(t)}, e^{2\pi iF_2(t)},\dots,e^{2\pi iF_L(t)})$$

as unordered L-tuples for each $t \in [0,1]$. Again by Proposition 6.3 there exist continuous functions $G_1, G_2, \dots, G_L \colon [0,1] \to \mathbb{R}$ such that for every $t \in [0,1]$,

$$G_1(t) \le G_2(t) \le \dots \le G_L(t) \le G_1(t) + 1,$$

such that

$$\big(\mu_1(t), \mu_2(t), \dots, \mu_L(t)\big) = (e^{2\pi i G_1(t)}, e^{2\pi i G_2(t)}, \dots, e^{2\pi i G_L(t)})$$

as unordered L-tuples for each $t \in [0,1]$, and such that

$$\Big|\sum_{r=1}^{L} G_r(0) - \sum_{r=1}^{L} F_r(0) + \gamma(1)\Big| < 1. \tag{6}$$

It follows from (5) that

$$\Big(\exp\big(2\pi i F_1(0)\big), \dots, \exp\big(2\pi i F_L(0)\big)\Big) = \Big(\exp\big(2\pi i F_1(1)\big), \dots, \exp\big(2\pi i F_L(1)\big)\Big)$$

as unordered L-tuples. Since $l = \sum_{r=1}^{L}\big(F_r(1) - F_r(0)\big)$ we see by Lemma 6.2 that $F_r(1) = F_{r+l}(0)$ for each $r = 1, 2, \dots, L$. Similarly, as $\mathrm{Det}\big(\varphi(v^A)(\cdot)\big)$ and $\mathrm{Det}\big(\psi(v^A)(\cdot)\big)$ have the same winding number, $G_r(1) = G_{r+l}(0)$, $r = 1, 2, \dots, L$.

Let $t_1, t_2, \dots, t_M \in \,]0,1[$ be numbers such that $e^{2\pi i t_j}$, $j = 1, 2, \dots, M$, are the exceptional points of B. By (5) there exist a unitary $u_j \in M_m$ such that

$$u_j \,\mathrm{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i F_1(t_j)}), \dots, f(e^{2\pi i F_L(t_j)})\big)\, u_j^* \in M_{e_j} \subseteq M_m$$

for every $f \in A$. Choose a unitary $u \in C[0,1] \otimes M_m$ such that $u(t_j) = u_j$, $j = 1, 2, \dots, M$, $u(0) = 1$ and $u(1) = w$. Note that for every $f \in A$,

$$\begin{aligned} &u(0)\,\mathrm{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i F_1(0)}), \dots, f(e^{2\pi i F_L(0)})\big)\, u(0)^* \\ &\quad = u(1)\,\mathrm{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i F_1(1)}), \dots, f(e^{2\pi i F_L(1)})\big)\, u(1)^*. \end{aligned}$$

It follows that we may define a unital $*$-homomorphism $\varphi' \colon A \to B$ by

$$\varphi'(f)(e^{2\pi i t}) = u(t)\,\mathrm{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i F_1(t)}), \dots, f(e^{2\pi i F_L(t)})\big)\, u(t)^*,$$

for $f \in A$, $t \in [0,1]$. Then for every $f \in A$, $z \in \mathbb{T}$,

$$\mathrm{Tr}\big(\varphi(f)(z)\big) = \mathrm{Tr}\big(\varphi_1(f)(z)\big) = \mathrm{Tr}\big(\varphi'(f)(z)\big).$$

Hence φ and φ' are approximately unitarily equivalent by [27, Theorem 1.4].

Similarly we see that there exists a unitary $v \in C[0,1] \otimes M_m$ such that $v(0) = 1$ and $v(1) = w$, and such that

$$\psi'(f)(e^{2\pi i t}) = v(t)\,\mathrm{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i G_1(t)}), \dots, f(e^{2\pi i G_L(t)})\big)\, v(t)^*$$

defines a $*$-homomorphism that is approximately unitarily equivalent to ψ. Finally note that by (6) we have that $\gamma(e^{2\pi i t}) = \sum_{r=1}^{L}\big(G_r(t) - F_r(t)\big)$ for every $t \in [0,1]$. ■

7 Uniqueness

The purpose of this section is to prove a uniqueness theorem, *i.e.*, a theorem saying that two unital $*$-homomorphisms between (finite direct sums of) building blocks are close in a suitable sense if they approximately agree on the invariant. Many of the arguments here are inspired by similar arguments in [8], [10], [20], [27], and [16].

We start out with some definitions. Let k be a positive integer. A k-arc is an arc-segment of the form

$$I = \left\{e^{2\pi it} : t \in \left[\frac{m}{k}, \frac{n}{k}\right]\right\}$$

where m and n are integers, $m < n$. We set

$$I \pm \epsilon = \left\{e^{2\pi it} : t \in \left[\frac{m}{k} - \epsilon, \frac{n}{k} + \epsilon\right]\right\}.$$

Define a metric on the set of unordered L-tuples consisting of elements from $\mathbb{T}$ by

$$R_L\big((a_1, a_2, \dots, a_L), (b_1, b_2, \dots, b_L)\big) = \min_{\sigma\in\Sigma_L}\big(\max_{1\le i\le L} \rho(a_i, b_{\sigma(i)})\big),$$

where Σ_L denotes the group of permutations of the set $\{1, 2, \dots, L\}$. It follows from Lemma 7.3 below that it suffices to take the minimum over a certain subset of Σ_L.

Lemma 7.1 *Let $a_1, a_2, \dots, a_L, b_1, b_2, \dots, b_L \in \mathbb{T}$ and let $\epsilon > 0$. Assume that there is a positive integer k such that*

$$\#\{r : a_r \in I\} \le \#\{r : b_r \in I \pm \epsilon\}$$

for all k-arcs I. Then

$$R_L\big((a_1, a_2, \dots, a_L), (b_1, b_2, \dots, b_L)\big) \le \epsilon + \frac{1}{k}.$$

Proof For $j = 1, 2, \dots, L$, set

$$X_j = \left\{x \in \mathbb{T} : \rho(x, a_j) \le \epsilon + \frac{1}{k}\right\}$$

and

$$C_j = \{r = 1, 2, \dots, L : b_r \in X_j\}.$$

Let $S \subseteq \{1, 2, \dots, L\}$ be arbitrary. We will show that $\#S \le \#\bigcup_{j\in S} C_j$.

Let $Y_1, Y_2, \dots, Y_m$ be the connected components of $\bigcup_{j\in S} X_j$. Choose for each $n = 1, 2, \dots, m$ a k-arc I_n such that $I_n \pm \epsilon \subseteq Y_n$ and $\{a_j : j \in S\} \cap Y_n \subseteq I_n$. Then

$$\#\{r : a_r \in I_n\} \le \#\{r : b_r \in I_n \pm \epsilon\} \le \#\{r : b_r \in Y_n\}.$$

If $r \in S$ then $a_r \in I_n$ for some n. Hence

$$\#S \le \#\Big\{r : a_r \in \bigcup_{n=1}^{m} I_n\Big\} \le \#\Big\{r : b_r \in \bigcup_{n=1}^{m} Y_n\Big\} = \#\Big\{r : b_r \in \bigcup_{j\in S} X_j\Big\} = \#\bigcup_{j\in S} C_j.$$

By Hall's marriage lemma, see *e.g.* [4, Theorem 2.2], the sets C_j, $j = 1, 2, \dots, L$, have distinct representatives. In other words, there exists a permutation σ of $\{1, 2, \dots, L\}$ such that $\rho(a_j, b_{\sigma(j)}) \le \epsilon + \frac{1}{k}$ for $j = 1, 2, \dots, L$. ■

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Lemma 7.2 *Let $a_1 \le a_2 \le \cdots \le a_L$ and $b_1 \le b_2 \le \cdots \le b_L$ be real numbers and let σ be a permutation of $\{1, 2, \ldots, L\}$. Then*

$$\max_{1 \le j \le L} |a_j - b_j| \le \max_{1 \le j \le L} |a_j - b_{\sigma(j)}|.$$

Proof Let $\epsilon = \max |a_j - b_{\sigma(j)}|$. If *e.g.* $b_j < a_j - \epsilon$ for some j then σ must map the set $\{1, 2, \ldots, j\}$ into $\{1, 2, \ldots, j - 1\}$. Contradiction. ■

The corresponding statement for the circle is slightly more complicated. It can be viewed as a generalization of Lemma 6.2.

Lemma 7.3 *Let $\theta_1, \theta_2, \ldots, \theta_L$ and $\omega_1, \omega_2, \ldots, \omega_L$ be real numbers such that*

$$\theta_1 \le \theta_2 \le \cdots \le \theta_L \le \theta_1 + 1,$$
$$\omega_1 \le \omega_2 \le \cdots \le \omega_L \le \omega_1 + 1.$$

There exists an integer p such that

$$\max_{1 \le j \le L} |\theta_j - \omega_{j+p}| = R_L\big((e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_L}), (e^{2\pi i\omega_1}, \ldots, e^{2\pi i\omega_L})\big).$$

Proof Let $\epsilon = R_L\big((e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_L}), (e^{2\pi i\omega_1}, \ldots, e^{2\pi i\omega_L})\big)$. Note that $0 \le \epsilon \le \frac{1}{2}$. There exist $y_1, y_2, \ldots, y_L \in \mathbb{R}$ such that

$$(e^{2\pi i\omega_1}, e^{2\pi i\omega_2}, \ldots, e^{2\pi i\omega_L}) = (e^{2\pi i y_1}, e^{2\pi i y_2}, \ldots, e^{2\pi i y_L}) \tag{7}$$

as unordered tuples, and such that

$$|\theta_j - y_j| \le \epsilon, \quad j = 1, 2, \ldots, L. \tag{8}$$

By Lemma 7.2 we may assume that $y_1 \le y_2 \le \cdots \le y_L$ and still have that (7) and (8) hold.

Choose an integer n, $0 \le n \le L - 1$, such that $y_1, y_2, \ldots, y_n < y_L - 1$ and $y_L - 1 \le y_{n+1}, y_{n+2}, \ldots, y_L$. Then $y_1 + 1, \ldots, y_n + 1 \in [y_L - 1, y_L]$ since $y_L \le y_1 + 2$ by (8). Choose $z_1, z_2, \ldots, z_L \in [y_L - 1, y_L]$ such that $z_1 \le z_2 \le \cdots \le z_L$ and

$$(z_1, z_2, \ldots, z_L) = (y_1 + 1, \ldots, y_n + 1, y_{n+1}, \ldots, y_L)$$

as unordered L-tuples. By (8) and Lemma 7.2 we see that $\max |z_j - \theta_{n+j}| \le \epsilon$. By (7) and Lemma 6.2 we have that $z_j = \omega_{j+m}$ for some integer m. Hence

$$\max_j |\theta_j - \omega_{j+m-n}| = \max_j |\theta_{n+j} - \omega_{j+m}| \le \epsilon.$$

The reversed inequality is trivial. ■

The following lemma is fundamental in the proof of Theorem 7.5.

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Lemma 7.4 *Let $\theta_1, \theta_2, \dots, \theta_L$ and $\omega_1, \omega_2, \dots, \omega_L$ be real numbers such that*

$$\theta_1 \le \theta_2 \le \cdots \le \theta_L \le \theta_1 + 1,$$
$$\omega_1 \le \omega_2 \le \cdots \le \omega_L \le \omega_1 + 1,$$

and $|\sum_{j=1}^{L}(\theta_j - \omega_j)| < \delta$ for some $\delta > 0$. Let $\epsilon > 0$ satisfy that $L\epsilon \le \delta$ and

$$R_L\big((e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, \dots, e^{2\pi i\theta_L}), (e^{2\pi i\omega_1}, e^{2\pi i\omega_2}, \dots, e^{2\pi i\omega_L})\big) \le \epsilon.$$

Assume finally that for some positive integer s,

$$\#\{j : e^{2\pi i\omega_j} \in I\} \ge 2\delta, \quad j = 1, 2, \dots, L, \tag{9}$$

for every s-arc I. Then

$$|\theta_j - \omega_j| < \epsilon + \frac{2}{s}, \quad j = 1, 2, \dots, L.$$

Proof By Lemma 7.3 there exists an integer p such that

$$|\theta_j - \omega_{j+p}| \le \epsilon, \quad j = 1, 2, \dots, L.$$

Note that

$$|p| = \Big|\sum_{j=1}^{L}(\omega_{j+p} - \omega_j)\Big| \le \Big|\sum_{j=1}^{L}(\omega_{j+p} - \theta_j)\Big| + \Big|\sum_{j=1}^{L}(\theta_j - \omega_j)\Big| < L\epsilon + \delta \le 2\delta.$$

Fix some $j = 1, 2, \dots, L$. Set

$$J = \begin{cases} \{e^{2\pi it} : \omega_j < t < \omega_{j+p}\} & \text{if } p \ge 0 \\ \{e^{2\pi it} : \omega_{j+p} < t < \omega_j\} & \text{if } p < 0. \end{cases}$$

Since $\#\{j : e^{2\pi i\omega_j} \in J\} < |p|$ we see by (9) that J cannot contain an s-arc. Thus $|\omega_j - \omega_{j+p}| < \frac{2}{s}$. It follows that $|\theta_j - \omega_j| < \epsilon + \frac{2}{s}$, $j = 1, 2, \dots, L$. ∎

Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block and p a positive integer. Let I be a p-arc. Choose a continuous function $f_A^I \colon \mathbb{T} \to [0, \frac{1}{n}]$ such that $\varnothing \ne \operatorname{supp} f_A^I \subseteq I$ and such that f_A^I equals 0 at all the exceptional points of A. Choose a continuous function $g_A^I \colon \mathbb{T} \to [0, 1]$ such that g_A^I equals 1 on I, such that $\operatorname{supp} g_A^I \subseteq I \pm \frac{1}{2p}$, and such that $\operatorname{supp} g_A^I \setminus I$ contains no exceptional points of A. Set

$$H(A, p) = \{f_A^I \otimes 1 : I \ p\text{-arc}\},$$
$$\widetilde{H}(A, p) = \{g_A^I \otimes 1 : I \ p\text{-arc}\}.$$

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Theorem 7.5 *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block. Let $\epsilon > 0$ and let $F \subseteq A$ be a finite set. There exists a positive integer l_0 such that if l, p and q are positive integers with $l_0 \le l \le p \le q$, if $B = A(m, e_1, e_2, \dots, e_M)$ is a building block, if $\varphi, \psi\colon A \to B$ are unital $*$-homomorphisms, and if $\delta > 0$, such that*

(i) $\widehat{\psi}(\widehat{h}) > \frac{8}{p}$, $h \in H(A, l)$,
(ii) $\widehat{\psi}(\widehat{h}) > \frac{2}{q}$, $h \in H(A, p)$,
(iii) $\|\widehat{\varphi}(\widehat{h}) - \widehat{\psi}(\widehat{h})\| < \delta$, $h \in \widetilde{H}(A, 2q)$,
(iv) $\widehat{\psi}(\widehat{h}) > \delta$, $h \in H(A, 4q)$,
(v) $\varphi^{\#}\big(q'_A(\omega_k^A)\big) = \psi^{\#}\big(q'_A(\omega_k^A)\big)$, $k = 1, 2, \dots, N$,
(vi) $D_B\Big(\varphi^{\#}\big(q'_A(v^A)\big), \psi^{\#}\big(q'_A(v^A)\big)\Big) < \frac{1}{q}$;

then there exists a unitary $W \in B$ such that

$$\|\varphi(f) - W\psi(f)W^*\| < \epsilon, \quad f \in F.$$

Proof Choose l_0 such that for $x, y \in \mathbb{T}$,

$$\rho(x, y) \le \frac{6}{l_0} \Longrightarrow \|f(x) - f(y)\| < \frac{\epsilon}{6}, \quad f \in F.$$

Let integers $q \ge p \ge l \ge l_0$, a building block $B = A(m, e_1, e_2, \dots, e_M)$, and unital $*$-homomorphisms $\varphi, \psi\colon A \to B$ be given such that (i)–(vi) are satisfied. Choose $c > 0$ such that for $x, y \in \mathbb{T}$,

$$\rho(x, y) < c \Longrightarrow \|\varphi(f)(x) - \varphi(f)(y)\| < \frac{\epsilon}{6}, \quad f \in F,$$

$$\rho(x, y) < c \Longrightarrow \|\psi(f)(x) - \psi(f)(y)\| < \frac{\epsilon}{6}, \quad f \in F.$$

Let $x_1, x_2, \dots, x_N$ denote the exceptional points of A and let $y_1, y_2, \dots, y_M$ be those of B. Let for each $j = 1, 2, \dots, M$, $t_j \in]0, 1[$ be the number such that $e^{2\pi i t_j} = y_j$. Let $\tau\colon \mathbb{T} \to \mathbb{T}$ be a continuous function such that $\rho\big(\tau(z), z\big) < c$ for every $z \in \mathbb{T}$, and such that for each $j = 1, 2, \dots, M$, τ is constantly equal to y_j on some arc

$$I_j = \{e^{2\pi i t} : t \in [a_j, b_j]\},$$

where $0 < a_j < t_j < b_j < 1$. Define a unital $*$-homomorphism $\chi\colon B \to B$ by $\chi(f) = f \circ \tau$. Set $\varphi_1 = \chi \circ \varphi$ and $\psi_1 = \chi \circ \psi$. Then

$$\|\varphi(f) - \varphi_1(f)\| < \frac{\epsilon}{6}, \quad f \in F,$$

$$\|\psi(f) - \psi_1(f)\| < \frac{\epsilon}{6}, \quad f \in F.$$

φ_1 and ψ_1 satisfy (i)–(vi). Let

$$\varphi_1(v^A) = c\psi_1(v^A)e^{2\pi i b},$$

where $c \in \overline{DU(B)}$ and $b \in B$ is a self-adjoint element with $\|b\| < \frac{1}{q}$. Note that

$$\text{(10)} \qquad \operatorname{Det}\big(\varphi_1(v^A)(z)\big) = \operatorname{Det}\big(\psi_1(v^A)(z)\big)\exp\Big(2\pi i \operatorname{Tr}\big(b(z)\big)\Big), \quad z \in \mathbb{T},$$

(11)

$$\operatorname{Det}\big(\Lambda_j \circ \varphi_1(v^A)\big) = \operatorname{Det}\big(\Lambda_j \circ \psi_1(v^A)\big)\exp\Big(2\pi i \operatorname{Tr}\big(\Lambda_j(b)\big)\Big), \quad j = 1, 2, \ldots, M.$$

Fix some $j = 1, 2, \ldots, M$. Let $\iota_j \colon M_{e_j} \to M_m$ denote the (unital) inclusion. By Theorem 5.5 and (v) we have that $s^{\varphi_1}(j, i) \equiv s^{\psi_1}(j, i) \mod \frac{n}{d_i}$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Choose s_i^j, $0 \le s_i^j < \frac{n}{d_i}$, such that $s_i^j \equiv s^{\varphi_1}(j, i) \mod \frac{n}{d_i}$. By Lemma 2.1 we see that for each $z \in I_j$,

$$\varphi_1(f)(z) = \iota_j\Big(y_1^j \operatorname{diag}\big(\Lambda_1^{s_1^j}(f), \ldots, \Lambda_N^{s_N^j}(f), f(e^{2\pi i\theta_1^j}), \ldots, f(e^{2\pi i\theta_{D_j}^j})\big)\, y_1^{j*}\Big),$$

$$\psi_1(f)(z) = \iota_j\Big(y_2^j \operatorname{diag}\big(\Lambda_1^{s_1^j}(f), \ldots, \Lambda_N^{s_N^j}(f), f(e^{2\pi i\omega_1^j}), \ldots, f(e^{2\pi i\omega_{D_j}^j})\big)\, y_2^{j*}\Big),$$

for some unitaries $y_1^j, y_2^j \in M_{e_j}$ and numbers $\theta_1^j, \ldots, \theta_{D_j}^j, \omega_1^j, \ldots, \omega_{D_j}^j \in \mathbb{R}$. By changing y_1^j and y_2^j we may by (11) and Lemma 6.1 assume that

$$\theta_1^j \le \theta_2^j \le \cdots \le \theta_{D_j}^j \le \theta_1^j + 1,$$

$$\omega_1^j \le \omega_2^j \le \cdots \le \omega_{D_j}^j \le \omega_1^j + 1,$$

and

$$\text{(12)} \qquad \sum_{r=1}^{D_j} (\theta_r^j - \omega_r^j) = \operatorname{Tr}\big(\Lambda_j(b)\big).$$

Let I be a $2q$-arc. By (iii),

$$\begin{aligned}
\#\{r : e^{2\pi i\theta_r^j} \in I\}n + \sum_{\{i : x_i \in I\}} s_i^j d_i &\le \operatorname{Tr}\big(\Lambda_j \circ \varphi_1(g_I^A \otimes 1)\big) \\
&< e_j\delta + \operatorname{Tr}\big(\Lambda_j \circ \psi_1(g_I^A \otimes 1)\big) \\
&\le e_j\delta + \#\left\{r : e^{2\pi i\omega_r^j} \in I \pm \frac{1}{4q}\right\} n + \sum_{\{i : x_i \in \operatorname{supp} g_I^A\}} s_i^j d_i \\
&\le \#\left\{r : e^{2\pi i\omega_r^j} \in I \pm \frac{1}{2q}\right\} n + \sum_{\{i : x_i \in \operatorname{supp} g_I^A\}} s_i^j d_i.
\end{aligned}$$

The last inequality uses (iv) and that $\|f_A^K\|_\infty \le 1$ for some $4q$-arc K. Hence

$$\#\{r : e^{2\pi i\theta_r^j} \in I\} \le \#\left\{r : e^{2\pi i\omega_r^j} \in I \pm \frac{1}{2q}\right\}.$$

Therefore by Lemma 7.1,

$$R_{D_j}\big((e^{2\pi i\theta_1^j}, e^{2\pi i\theta_2^j}, \dots, e^{2\pi i\theta_{D_j}^j}), (e^{2\pi i\omega_1^j}, e^{2\pi i\omega_2^j}, \dots, e^{2\pi i\omega_{D_j}^j})\big) \le \frac{1}{2q} + \frac{1}{2q} \le \frac{1}{q}.$$

By (ii), if J is a p-arc then

$$\#\{r : e^{2\pi i\omega_r^j} \in J\} \ge 2\frac{e_j}{q},$$

since $\|f_A^J\|_\infty \le \frac{1}{n}$. Clearly $\frac{D_j}{q} \le \frac{e_j}{q}$. Furthermore,

$$\Big|\sum_{r=1}^{D_j}(\theta_r^j - \omega_r^j)\Big| = \big|\operatorname{Tr}\big(\Lambda_j(b)\big)\big| \le e_j\|b\| < \frac{e_j}{q}.$$

By Lemma 7.4 it follows that

$$|\theta_r^j - \omega_r^j| \le \frac{1}{q} + \frac{2}{p} \le \frac{3}{p}, \quad r = 1, 2, \dots, D_j.$$

Let $g_r^j\colon [a_j, b_j] \to \mathbb{R}$ be the continuous function such that $g_r^j(a_j) = g_r^j(b_j) = \theta_r^j$, $g_r^j(t_j) = \omega_r^j$, and such that g_r^j is linear when restricted to each of the two intervals $[a_j, t_j]$ and $[t_j, b_j]$. Note that

$$|g_r^j(t) - \theta_r^j| \le \frac{3}{p}, \quad r = 1, 2, \dots, D_j. \tag{13}$$

Finally, define a $*$-homomorphism $\xi_j\colon A \to C(I_j) \otimes M_m$ by

$$\xi_j(f)(e^{2\pi it}) = \iota_j\Big(y_1^j \operatorname{diag}\big(\Lambda_1^{s_1^j}(f), \dots, \Lambda_N^{s_N^j}(f), f(e^{2\pi i g_1^j(t)}), \dots, f(e^{2\pi i g_{D_j}^j(t)})\big)\, y_1^{j*}\Big),$$

for $t \in [a_j, b_j]$, $f \in A$.

Define a unital $*$-homomorphism $\xi\colon A \to B$ by

$$\xi(f)(z) = \begin{cases} \xi_j(f)(z), & z \in I_j,\ j = 1, 2, \dots, M, \\ \varphi_1(f)(z), & z \in \mathbb{T} \setminus \bigcup_{j=1}^M I_j. \end{cases}$$

Then by (13)

$$\|\varphi_1(f) - \xi(f)\| < \frac{\epsilon}{6}, \quad f \in F.$$

Note that $f \mapsto \Lambda_j \circ \xi(f)$ and $f \mapsto \Lambda_j \circ \psi_1(f)$ are equivalent representations of A on M_{e_j}, $j = 1, 2, \dots, M$. In particular, $s^\xi(j, i) = s^{\psi_1}(j, i)$, $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$, and hence $\xi^\#\big(q_A'(\omega_k^A)\big) = \psi_1^\#\big(q_A'(\omega_k^A)\big)$, $k = 1, 2, \dots, N$ by Theorem 5.5. Let $\eta\colon \mathbb{T} \to \mathbb{R}$ be the continuous function

$$\eta(e^{2\pi it}) = \begin{cases} \frac{m}{e_j}\sum_{r=1}^{D_j}\big(g_r^j(t) - \theta_r^j\big) & t \in [a_j, b_j],\ j = 1, 2, \dots, M, \\ 0 & \text{otherwise.} \end{cases}$$

For $z \in \mathbb{T}$,

$$\begin{aligned}\operatorname{Det}\big(\xi(v^A)(z)\big) &= \operatorname{Det}\big(\varphi_1(v^A)(z)\big)\ \exp\big(2\pi i\eta(z)\big)\\ &= \operatorname{Det}\big(\psi_1(v^A)(z)\big)\ \exp\Big(2\pi i \operatorname{Tr}\big(b(z)\big)\Big)\ \exp\big(2\pi i\eta(z)\big)\\ &= \operatorname{Det}\big(\psi_1(v^A)(z)\big)\ \exp\big(2\pi i\gamma(z)\big),\end{aligned}$$

where $\gamma\colon \mathbb{T} \to \mathbb{R}$ is defined by $\gamma(z) = \eta(z) + \operatorname{Tr}\big(b(z)\big)$. Note that by (12)

$$\begin{aligned}\gamma(y_j) &= \eta(y_j) + \operatorname{Tr}\big(b(y_j)\big)\\ &= \frac{m}{e_j}\sum_{r=1}^{D_j}(\omega_r^j - \theta_r^j) + \frac{m}{e_j}\operatorname{Tr}\big(\Lambda_j(b)\big) = 0, \quad j = 1, 2, \dots, M,\end{aligned}$$

and

$$\|\gamma\|_\infty \le \|\eta\|_\infty + \big\|\operatorname{Tr}\big(b(\cdot)\big)\big\|_\infty < \frac{m}{e_j}\frac{3}{p}D_j + \frac{m}{q} \le 3\frac{m}{p} + \frac{m}{q} \le 4\frac{m}{p}.$$

By Proposition 6.4, φ_1, ψ_1, and ξ are approximately unitarily equivalent to φ_1', ψ_1', and ξ', respectively, where $\varphi_1', \psi_1', \xi'\colon A \to B$ are $*$-homomorphisms of the form

$$\begin{aligned}\varphi_1'(f)(e^{2\pi i t}) &= u(t)\operatorname{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i F_1(t)}), \dots, f(e^{2\pi i F_L(t)})\big)\,u(t)^*,\\ \psi_1'(f)(e^{2\pi i t}) &= v(t)\operatorname{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i G_1(t)}), \dots, f(e^{2\pi i G_L(t)})\big)\,v(t)^*,\\ \xi'(f)(e^{2\pi i t}) &= w(t)\operatorname{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i H_1(t)}), \dots, f(e^{2\pi i H_L(t)})\big)\,w(t)^*,\end{aligned}$$

for integers $r_1, r_2, \dots, r_N$ with $0 \le r_i < \frac{n}{d_i}$, $i = 1, 2, \dots, N$, unitaries u, v, w in $C[0,1] \otimes M_m$ with $u(0) = v(0) = w(0)$, $u(1) = v(1) = w(1)$, and continuous functions $F_r, G_r, H_r\colon [0,1] \to \mathbb{R}$, $r = 1, 2, \dots, L$, such that for $t \in [0,1]$,

$$\begin{aligned}F_1(t) \le F_2(t) \le \cdots \le F_L(t) \le F_1(t) + 1,\\ G_1(t) \le G_2(t) \le \cdots \le G_L(t) \le G_1(t) + 1,\\ H_1(t) \le H_2(t) \le \cdots \le H_L(t) \le H_1(t) + 1,\end{aligned}$$

and such that for each $t \in [0,1]$,

$$\gamma(e^{2\pi i t}) = \sum_{r=1}^{L}\big(H_r(t) - G_r(t)\big). \tag{14}$$

Hence

$$\Big|\sum_{r=1}^{L}\big(H_r(t) - G_r(t)\big)\Big| < 4\frac{m}{p}. \tag{15}$$

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It follows from (13) that for each $t \in [0,1]$,

$$R_L\Big((e^{2\pi i F_1(t)}, \dots, e^{2\pi i F_L(t)}), (e^{2\pi i H_1(t)}, \dots, e^{2\pi i H_L(t)})\Big) \le \frac{3}{p}.$$

Let $t \in [0,1]$ and let I be a $2q$-arc. Then by (iii) and (iv)

$$\begin{aligned}
\#\{r : e^{2\pi i F_r(t)} \in I\} n &+ \sum_{\{i : x_i \in I\}} r_i d_i \\
&\le \operatorname{Tr}\Big(\varphi_1'(g_I^A \otimes 1)(e^{2\pi i t})\Big) \\
&< m\delta + \operatorname{Tr}\Big(\psi_1'(g_I^A \otimes 1)(e^{2\pi i t})\Big) \\
&\le m\delta + \#\left\{r : e^{2\pi i G_r(t)} \in I \pm \frac{1}{4q}\right\} n + \sum_{\{i : x_i \in \operatorname{supp} g_I^A\}} r_i d_i \\
&\le \#\left\{r : e^{2\pi i G_r(t)} \in I \pm \frac{1}{2q}\right\} n + \sum_{\{i : x_i \in \operatorname{supp} g_I^A\}} r_i d_i .
\end{aligned}$$

Hence

$$\#\{r : e^{2\pi i F_r(t)} \in I\} \le \#\left\{r : e^{2\pi i G_r(t)} \in I \pm \frac{1}{2q}\right\}.$$

It follows from Lemma 7.1 that for each $t \in [0,1]$,

$$R_L\Big((e^{2\pi i F_1(t)}, \dots, e^{2\pi i F_L(t)}), (e^{2\pi i G_1(t)}, \dots, e^{2\pi i G_L(t)})\Big) \le \frac{1}{2q} + \frac{1}{2q} = \frac{1}{q}.$$

We conclude that

$$R_L\Big((e^{2\pi i G_1(t)}, \dots, e^{2\pi i G_L(t)}), (e^{2\pi i H_1(t)}, \dots, e^{2\pi i H_L(t)})\Big) \le \frac{1}{q} + \frac{3}{p} \le \frac{4}{p}.$$

Since $f \mapsto \psi_1'(f)(y_j)$ and $f \mapsto \xi'(f)(y_j)$ are equivalent representations of A on M_m for $j = 1, 2, \dots, M$, it follows that

$$(e^{2\pi i G_1(t_j)}, \dots, e^{2\pi i G_L(t_j)}) = (e^{2\pi i H_1(t_j)}, \dots, e^{2\pi i H_L(t_j)})$$

as unordered L-tuples. Therefore, as $\gamma(y_j) = 0$, $j = 1, 2, \dots, M$, we see by Lemma 6.2 and (14) that

$$G_r(t_j) = H_r(t_j), \quad r = 1, 2, \dots, L,\ j = 1, 2, \dots, M.$$

As $v(0) = w(0)$, $v(1) = w(1)$, we may thus define a $*$-homomorphism $\mu\colon A \to B$ by

$$\mu(f)(e^{2\pi i t}) = v(t) \operatorname{diag}\Big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i H_1(t)}), \dots, f(e^{2\pi i H_L(t)})\Big) v(t)^*,$$

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for $f \in A$, $t \in [0,1]$. Since

$$\operatorname{Tr}\big(\mu(f)(z)\big) = \operatorname{Tr}\big(\xi'(f)(z)\big) = \operatorname{Tr}\big(\xi(f)(z)\big), \quad z \in \mathbb{T}, f \in A,$$

we get from [27, Theorem 1.4] that μ and ξ are approximately unitarily equivalent.

By (i) we have that for every l-arc J,

$$\#\{r : e^{2\pi i G_r(t)} \in J\} > 8\frac{m}{p}.$$

As $L\frac{4}{p} \leq 4\frac{m}{p}$, we conclude from (15) and Lemma 7.4 that

$$|G_r(t) - H_r(t)| \leq \frac{4}{p} + \frac{2}{l} \leq \frac{6}{l}.$$

Hence

$$\|\mu(f) - \psi_1'(f)\| < \frac{\epsilon}{6}, \quad f \in F.$$

Choose unitaries $U, V \in B$ such that

$$\|\xi(f) - U\mu(f)U^*\| < \frac{\epsilon}{6}, \quad f \in F,$$

$$\|\psi_1'(f) - V\psi_1(f)V^*\| < \frac{\epsilon}{6}, \quad f \in F.$$

Set $W = UV$. Then for $f \in F$,

$$\begin{aligned}
&\|\varphi(f) - W\psi(f)W^*\| \\
&\quad \leq \|\varphi(f) - \varphi_1(f)\| + \|\varphi_1(f) - \xi(f)\| + \|\xi(f) - U\mu(f)U^*\| \\
&\qquad + \|U\mu(f)U^* - U\psi_1'(f)U^*\| + \|U\psi_1'(f)U^* - UV\psi_1(f)V^*U^*\| \\
&\qquad + \|W\psi_1(f)W^* - W\psi(f)W^*\| \\
&\quad < \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \epsilon.
\end{aligned}$$ ■

Lemma 7.6 *Let $B = A(m, e_1, e_2, \ldots, e_M)$ be a building block and let $r \in B$ be a non-zero projection of rank $s \in \mathbb{Z}$. Let $C = rBr$ and let $u, v \in C$ be unitaries. Then*

$$D_C\big(q_C'(u), q_C'(v)\big) \leq 2\pi\frac{m}{s} D_B\Big(q_B'\big(u + (1-r)\big), q_B'\big(v + (1-r)\big)\Big).$$

Proof Let $\epsilon = D_B\Big(q_B'\big(u + (1-r)\big), q_B'\big(v + (1-r)\big)\Big)$. We may assume that $\epsilon < 1$. Let $b \in B$ be a self-adjoint element such that $uv^* + (1-r) = e^{2\pi i b}$ modulo $\overline{DU(B)}$ and $\|b\| \leq \epsilon$. Define $c \in C$ by $c(z) = \frac{1}{s}\operatorname{Tr}\big(b(z)\big)\, r$. Since $\widehat{b} = \widehat{c}$ we have that $e^{2\pi i b} = e^{2\pi i c}$ modulo $\overline{DU(B)}$. Thus

$$uv^* + (1-r) = e^{2\pi i c} = re^{2\pi i c}r + (1-r) \quad \text{modulo } \overline{DU(B)}.$$

C is a building block by Corollary 3.5, and therefore it follows from Proposition 5.3 that $uv^* = re^{2\pi i c}r$ modulo $\overline{DU(C)}$. Thus

$$D_C\big(q'_C(u), q'_C(v)\big) \le \|re^{2\pi i c}r - r\| \le \|e^{2\pi i c} - 1\| \le 2\pi\|c\| \le 2\pi\frac{m}{s}\epsilon. \qquad \blacksquare$$

Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_R$, where $A_i = A(n_i, d_1^i, d_2^i, \ldots, d_{N_i}^i)$ is a building block. For each $i = 1, 2, \ldots, R$, we define unitaries in A by

$$v_i^A = (1, \ldots, 1, v^{A_i}, 1, \ldots, 1).$$
$$w_{i,k}^A = (1, \ldots, 1, w_k^{A_i}, 1, \ldots, 1), \quad k = 1, 2, \ldots, N_i.$$

Set $U^A = \bigcup_{i=1}^R \{w_{i,k}^A : k = 1, 2, \ldots, N_i\}$. If p is a positive integer, we set

$$H(A, p) = \bigcup_{i=1}^R \iota_i\big(H(A_i, p)\big),$$
$$\widetilde{H}(A, p) = \bigcup_{i=1}^R \iota_i\big(\widetilde{H}(A_i, p)\big),$$

where $\iota_i \colon A_i \to A$ denotes the inclusion, $i = 1, 2, \ldots, R$.

Theorem 7.7 *Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_R$ be a finite direct sum of building blocks. Let $p_1, p_2, \ldots, p_R$ be the minimal non-zero central projections in A. Let $\epsilon > 0$ and let $F \subseteq A$ be a finite set. There exists a positive integer l such that if p and q are positive integers with $l \le p \le q$, if B is a finite direct sum of building blocks, if $\varphi, \psi \colon A \to B$ are unital $*$-homomorphisms, if $\delta > 0$, if*

(i) $\widehat{\psi}(\widehat{h}) > \frac{8}{p}$, $h \in H(A, l)$,
(ii) $\widehat{\psi}(\widehat{h}) > \frac{2}{q}$, $h \in H(A, p) \cup \{p_1, p_2, \ldots, p_R\}$,
(iii) $\|\widehat{\varphi}(\widehat{h}) - \widehat{\psi}(\widehat{h})\| < \delta$, $h \in \widetilde{H}(A, 2q)$,
(iv) $\widehat{\psi}(\widehat{h}) > \delta$, $h \in H(A, 4q)$,
(v) $D_B\Big(\varphi^\#\big(q'_A(v_i^A)\big), \psi^\#\big(q'_A(v_i^A)\big)\Big) < \frac{1}{4q^2}$, $i = 1, 2, \ldots, R$;

and if at least one of the two statements

(vi) $[\varphi] = [\psi]$ *in* $KK(A, B)$,
(vii) $\varphi_* = \psi_*$ *on* $K_0(A)$ *and* $\varphi^\#(x) = \psi^\#(x)$, $x \in U^A$,

are true; then there exists a unitary $W \in B$ such that

$$\|\varphi(f) - W\psi(f)W^*\| < \epsilon, \quad f \in F.$$

Proof For each $i = 1, 2, \ldots, R$, let $\iota_i \colon A_i \to A$ be the inclusion and let $\pi_i \colon A \to A_i$ be the projection. Choose by Theorem 7.5 a positive integer l_0^i with respect to the finite set $\pi_i(F) \subseteq A_i$ and $\epsilon > 0$. Set $l = \max_i l_0^i$.

Let integers $q \geq p \geq l$, a finite direct sum of building blocks B, and unital $*$-homomorphisms $\varphi, \psi \colon A \to B$ be given such that the above holds. Since (vi) implies (vii) by Proposition 5.6, we may assume that (vii) holds. It is easy to reduce to the case where $B = A(m, e_1, e_2, \ldots, e_M)$ is a single building block.

Since $\varphi_*[p_i] = \psi_*[p_i]$ in $K_0(B)$ for $i = 1, 2, \ldots, R$, there is by Lemma 3.4 a unitary $u \in B$ such that $u\varphi(p_i)u^* = \psi(p_i)$ for every $i = 1, 2, \ldots, R$. Hence we may assume that $\varphi(p_i) = \psi(p_i)$, $i = 1, 2, \ldots, R$. Set $q_i = \psi(p_i)$. It follows from (ii) that $q_i \neq 0$, $i = 1, 2, \ldots, R$. Let t_i be the (normalized) trace of q_i.

Let $\varphi_i, \psi_i \colon A_i \to q_iBq_i$ be the induced maps. Note that q_iBq_i is a building block by Corollary 3.5. Fix some $i = 1, 2, \ldots, R$.

Every tracial state on q_iBq_i is of the form $\frac{1}{t_i}\omega|_{q_iBq_i}$ for some $\omega \in T(B)$. Therefore φ_i and ψ_i satisfy (i)–(iv) of Theorem 7.5, with δ replaced by $\frac{\delta}{t_i}$. Note that $t_i > \frac{2}{q}$ by (ii). Since

$$D_B\Big(q_B'\big(\varphi_i(v^{A_i}) + (1 - q_i)\big), q_B'\big(\psi_i(v^{A_i}) + (1 - q_i)\big) \Big) < \frac{1}{4q^2}$$

by (vi), we have that

$$D_{q_iBq_i}\Big(\varphi_i^{\#}\big(q_{A_i}'(v^{A_i})\big), \psi_i^{\#}\big(q_{A_i}'(v^{A_i})\big) \Big) \leq 2\pi\frac{1}{t_i}\frac{1}{4q^2} < 2\pi\frac{q}{2}\frac{1}{4q^2} < \frac{1}{q}$$

by Lemma 7.6, which is (v) of Theorem 7.5 for φ_i and ψ_i. Similarly we get that $\varphi_i^{\#}(w_k^{A_i}) = \psi_i^{\#}(w_k^{A_i})$, $k = 1, 2, \ldots, N_i$, which is (vi) of Theorem 7.5. Hence there exists a unitary $W_i \in q_iBq_i$ such that

$$\|\varphi_i(f) - W_i\psi_i(f)W_i{}^*\| < \epsilon, \quad f \in \pi_i(F).$$

Set $W = \sum_{i=1}^R W_i$. Then $W \in B$ is a unitary and

$$\|\varphi(f) - W\psi(f)W^*\| < \epsilon, \quad f \in F. \qquad \blacksquare$$

8 Existence

The goal of this section is to prove an existence theorem that is the counterpart of the uniqueness theorem of the previous section.

Let A and B be building blocks and let $\varphi \colon A \to B$ be a $*$-homomorphism. We say that continuous functions $\lambda_1, \lambda_2, \ldots, \lambda_N \colon \mathbb{T} \to \mathbb{T}$ are eigenvalue functions for φ if $\lambda_1(z), \lambda_2(z), \ldots, \lambda_N(z)$ are eigenvalues for the matrix $\varphi(\iota \otimes 1)(z)$ (counting multiplicities) for every $z \in \mathbb{T}$, where $\iota \colon \mathbb{T} \to \mathbb{C}$ denotes the inclusion.

Theorem 8.1 *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block, let $\epsilon > 0$, and let C be a positive integer. There exists a positive integer K such that if*

(i) $B = A(m, e_1, e_2, \ldots, e_M)$ *is a building block with* $s(B) \geq K$*,*
(ii) $\kappa \in KK(A, B)_e$*,*
(iii) $\lambda_1, \lambda_2, \ldots, \lambda_C \colon \mathbb{T} \to \mathbb{T}$ *are continuous functions,*
(iv) $u \in B$ *is a unitary such that* $\kappa_*[v^A] = [u]$ *in* $K_1(B)$*;*

then there exists a unital $*$*-homomorphism* $\varphi\colon A \to B$ *such that* $\lambda_1, \lambda_2, \ldots, \lambda_C$ *are eigenvalue functions for* φ*, and such that*

$$[\varphi] = \kappa \quad \text{in } KK(A, B),$$

$$\varphi^{\#}\big(q_A'(v^A)\big) = q_B'(u), \quad \text{in } U(B)/\overline{DU(B)},$$

$$\Big\| \widehat{\varphi}(f) - \frac{1}{C}\sum_{k=1}^{C} f \circ \lambda_k \Big\| < \epsilon \|f\|, \quad f \in \operatorname{Aff} T(A),$$

when we identify Aff $T(A)$ *and* Aff $T(B)$ *with* $C_{\mathbb{R}}(\mathbb{T})$ *as order unit spaces.*

Proof We may assume that $\epsilon < 4$ and, by repeating the functions $\lambda_1, \lambda_2, \ldots, \lambda_C$, that $C > \frac{8}{\epsilon}$. Let K be a positive integer such that

$$K \geq \frac{4(N + C + 2)n}{\epsilon}.$$

Let B, $\lambda_1, \lambda_2, \ldots, \lambda_C$, κ, and u be as above. By Proposition 4.3 there are integers h_{ji}, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$, with $0 \leq h_{ji} < \frac{n}{d_i}$ for $i \neq N$, such that

$$\begin{pmatrix} \kappa^*([\Lambda_1^B]) \\ \kappa^*([\Lambda_2^B]) \\ \vdots \\ \kappa^*([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & h_{22} & \cdots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \cdots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}.$$

As in the proof of Theorem 4.7 we see that

$$\sum_{i=1}^{N} h_{ji} d_i = e_j, \tag{16}$$

since $\kappa_*\colon K_0(A) \to K_0(B)$ preserves the order unit, and $h_{jN} > \frac{n}{d_N}$, because $s(B) \geq Nn$. By Proposition 4.3 we have for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$,

$$\frac{m}{e_j} h_{ji} = l_{ji} \frac{n}{d_i} + s_i, \tag{17}$$

where l_{ji} and s_i are integers such that $0 \leq s_i < \frac{n}{d_i}$. Note that $l_{ji} \geq 0$. For $j = 1, 2, \ldots, M$, choose integers h_{jN}^o, $0 \leq h_{jN}^o < \frac{n}{d_N}$, and $r_j \geq 0$ such that

$$h_{jN} = r_j \frac{n}{d_N} + h_{jN}^o, \tag{18}$$

and note that

$$\frac{m}{e_j}h^o_{jN} = l^o_{jN}\frac{n}{d_N} + s_N \tag{19}$$

for some integers $l^o_{jN} \geq 0$, $j = 1, 2, \ldots, M$. Then

$$l_{jN} - l^o_{jN} = \frac{m}{e_j}r_j. \tag{20}$$

Let for each j

$$r_j = k_j(C+2) + u_j \tag{21}$$

for some integers $k_j \geq 0$ and $0 \leq u_j < C + 2$ and set

$$b = \min_{1\leq j\leq M} k_j \frac{m}{e_j}.$$

Note that for $j = 1, 2, \ldots, M$,

$$\begin{aligned} e_j = \sum_{i=1}^{N} h_{ji}d_i &< (N-1)n + r_j n + h^o_{jN}d_N < Nn + r_j n \\ &= (N + C + 2)n + \big(r_j - (C+2)\big)\, n \leq \frac{\epsilon}{4}e_j + \big(r_j - (C+2)\big)\, n. \end{aligned}$$

Hence

$$\Big(1 - \frac{\epsilon}{4}\Big)\, e_j < \big(r_j - (C+2)\big)\, n.$$

Therefore

$$nk_j(C+2)\frac{m}{e_j} = n(r_j - u_j)\frac{m}{e_j} > n\big(r_j - (C+2)\big)\,\frac{m}{e_j} > \Big(1 - \frac{\epsilon}{4}\Big)\, m. \tag{22}$$

Since by (16)

$$nk_j(C+2)\frac{m}{e_j} \leq nr_j\frac{m}{e_j} \leq h_{jN}d_N\frac{m}{e_j} \leq m,$$

we see that

$$nb\frac{8}{\epsilon} \leq nbC \leq nb(C+2) \leq m. \tag{23}$$

By this and (22),

$$m\Big(1 - \frac{\epsilon}{4}\Big) < nb(C+2) \leq nbC + \frac{\epsilon}{4}m.$$

Hence from (23) we conclude that

$$0 \leq 1 - \frac{nbC}{m} < \frac{\epsilon}{2}.$$

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Let $x_1, x_2, \ldots, x_N$ denote the exceptional points of A and let $y_1, y_2, \ldots, y_M$ be those of B. Set for $j = 1, 2, \ldots, M$,

$$a_j = \Big(\prod_{i=1}^{N-1} \operatorname{Det}\big(\Lambda_i(v^A)\big)^{h_{ji}}\Big) \operatorname{Det}\big(\Lambda_N(v^A)\big)^{h^o_{jN}},$$

then by (17) and (19)

$$a_j^{\frac{m}{e_j}} = \Big(\prod_{i=1}^{N-1} x_i^{\,l_{ji}}\Big) x_N^{\,l^o_{jN}} \prod_{i=1}^{N} \operatorname{Det}\big(\Lambda_i(v^A)\big)^{s_i}. \tag{24}$$

Set for $j = 1, 2, \ldots, M$,

$$c_j = \operatorname{Det}\big(\Lambda_j(u)\big)$$

and note that

$$c_j^{\frac{m}{e_j}} = \operatorname{Det}\big(u(y_j)\big). \tag{25}$$

By (22) we see that $k_j \neq 0$, $j = 1, 2, \ldots, M$, and hence there exists a continuous function $\lambda_{C+1} \colon \mathbb{T} \to \mathbb{T}$ such that

$$\big(\lambda_{C+1}(y_j)\big)^{-k_j} = a_j c_j^{-1} \prod_{k=1}^{C} \big(\lambda_k(y_j)\big)^{k_j}, \quad j = 1, 2, \ldots, M. \tag{26}$$

Let for $f \in A$ and $j = 1, 2, \ldots, M$, $D_j(f)$ be the $m \times m$ matrix

$$\operatorname{diag}\Big(\Lambda_1^{s_1}(f), \ldots, \Lambda_N^{s_N}(f), \underbrace{f(x_1), \ldots, f(x_1)}_{l_{j1}\text{ times}}, \ldots, \underbrace{f(x_{N-1}), \ldots, f(x_{N-1})}_{l_{j(N-1)}\text{ times}},$$
$$\underbrace{f(x_N), \ldots, f(x_N)}_{l^o_{jN}\text{ times}}, \underbrace{f\big(\lambda_1(y_j)\big), \ldots, f\big(\lambda_1(y_j)\big)}_{k_j \frac{m}{e_j} - b\text{ times}}, \ldots,$$
$$\underbrace{f\big(\lambda_{C+1}(y_j)\big), \ldots, f\big(\lambda_{C+1}(y_j)\big)}_{k_j \frac{m}{e_j} - b\text{ times}}, \underbrace{f(1), \ldots, f(1)}_{(k_j+u_j)\frac{m}{e_j} - b\text{ times}},$$
$$\underbrace{f\big(\lambda_1(y_j)\big), \ldots, f\big(\lambda_1(y_j)\big)}_{b\text{ times}}, \ldots, \underbrace{f\big(\lambda_{C+1}(y_j)\big), \ldots, f\big(\lambda_{C+1}(y_j)\big)}_{b\text{ times}},$$
$$\underbrace{f(1), \ldots, f(1)}_{b\text{ times}}\Big).$$

Since $D_j(f)$ is a block-diagonal matrix with $\frac{m}{e_j} h_{ji}$ blocks of the form $\Lambda_i(f)$, $i = 1, 2, \ldots, N-1$, $\frac{m}{e_j} h^o_{jN}$ blocks of the form $\Lambda_N(f)$, $k_j \frac{m}{e_j}$ blocks of the form $f\big(\lambda_k(y_j)\big)$,

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$k = 1, 2, \ldots, C+1$, and $(k_j + u_j)\frac{m}{e_j}$ blocks of the form $f(1)$, there exists a unitary $W_j \in M_m$ such that

$$W_j D_j(f) W_j^* \in M_{e_j} \subseteq M_m$$

for every $f \in A$. Set $L = \frac{1}{n}(m - \sum_{i=1}^N s_i d_i) - (C+2)b$. For each $j = 1, 2, \ldots, M$, we have by (16), (17), (20), (21) that

$$L = \sum_{i=1}^{N} l_{ji} - (C+2)b = \sum_{i=1}^{N-1} l_{ji} + l^o_{jN} + \frac{m}{e_j}\big(k_j(C+2) + u_j\big) - (C+2)b.$$

Choose for $k = 1, 2, \ldots, L$ continuous functions $\mu_k \colon \mathbb{T} \to \mathbb{T}$ such that for each $j = 1, 2, \ldots, M$,

$$\begin{aligned}
&\big(\mu_1(y_j), \mu_2(y_j), \ldots, \mu_L(y_j)\big)\\
&\quad = \big(\underbrace{x_1, \ldots, x_1}_{l_{j1}\text{ times}}, \ldots, \underbrace{x_{N-1}, \ldots, x_{N-1}}_{l_{j(N-1)}\text{ times}}, \underbrace{x_N, \ldots, x_N}_{l^o_{jN}\text{ times}},\\
&\qquad \underbrace{\lambda_1(y_j), \ldots, \lambda_1(y_j)}_{k_j\frac{m}{e_j} - b\text{ times}}, \ldots, \underbrace{\lambda_{C+1}(y_j), \ldots, \lambda_{C+1}(y_j)}_{k_j\frac{m}{e_j} - b\text{ times}}, \underbrace{1, 1, \ldots, 1}_{(k_j+u_j)\frac{m}{e_j} - b\text{ times}}\big)
\end{aligned}$$

as ordered tuples.

Choose a unitary $W \in C(\mathbb{T}) \otimes M_m$ such that $W(y_j) = W_j$ for $j = 1, 2, \ldots, M$. Define a continuous function $g \colon \mathbb{T} \to \mathbb{T}$ such that

$$g(z) \prod_{k=1}^{L} \mu_k(z) \prod_{k=1}^{C+1} \big(\lambda_k(z)\big)^b \prod_{i=1}^{N} \operatorname{Det}\big(\Lambda_i(v^A)\big)^{s_i} = \operatorname{Det}\big(u(z)\big), \quad z \in \mathbb{T}.$$

Then by (24), (25), and (26) we have that $g(y_j) = 1$ for $j = 1, 2, \ldots, M$. Define a unital $*$-homomorphism $\varphi \colon A \to B$ by

$$\begin{aligned}
\varphi(f)(z) = W(z) \operatorname{diag}\Big(&\Lambda_1^{s_1}(f), \ldots, \Lambda_N^{s_N}(f), f\big(\mu_1(z)\big), \ldots, f\big(\mu_L(z)\big),\\
&\underbrace{f\big(\lambda_1(z)\big), \ldots, f\big(\lambda_1(z)\big)}_{b\text{ times}}, \ldots,\\
&\underbrace{f\big(\lambda_{C+1}(z)\big), \ldots, f\big(\lambda_{C+1}(z)\big)}_{b\text{ times}},\\
&f\big(g(z)\big), \underbrace{f(1), \ldots, f(1)}_{b-1\text{ times}}\Big) W(z)^*.
\end{aligned}$$

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By the remarks following the definition of $D_j(f)$ we see that for $j = 1, 2, \dots, M$,

$$\begin{aligned}\varphi^*[\Lambda_j^B] &= \sum_{i=1}^{N-1} h_{ji}[\Lambda_i^A] + h_{jN}^o[\Lambda_N^A] + \bigl(k_j(C+1) + (k_j + u_j)\bigr)\frac{n}{d_N}[\Lambda_N^A]\\ &= \sum_{i=1}^{N-1} h_{ji}[\Lambda_i^A] + \Bigl(h_{jN}^o + r_j\frac{n}{d_N}\Bigr)[\Lambda_N^A] = \sum_{i=1}^{N} h_{ji}[\Lambda_i^A] = \kappa^*[\Lambda_j^B],\end{aligned}$$

and hence $\varphi^* = \kappa^*$ in $\operatorname{Hom}\bigl(K^0(B), K^0(A)\bigr)$. Furthermore, as $\operatorname{Det}\bigl(\nu^A(\cdot)\bigr)$ is the identity map on $\mathbb{T}$, we have that for $z \in \mathbb{T}$,

$$\operatorname{Det}\bigl(\varphi(\nu^A)(z)\bigr) = \prod_{i=1}^{N}\operatorname{Det}\bigl(\Lambda_i(\nu^A)\bigr)^{s_i}\prod_{k=1}^{L}\mu_k(z)\Bigl(\prod_{k=1}^{C+1}\lambda_k(z)^b\Bigr)g(z) = \operatorname{Det}\bigl(u(z)\bigr),$$

and by (26), for $j = 1, 2, \dots, M$,

$$\begin{aligned}\operatorname{Det}\bigl(\Lambda_j\circ\varphi(\nu^A)\bigr) &= \Bigl(\prod_{i=1}^{N-1}\operatorname{Det}\bigl(\Lambda_i(\nu^A)\bigr)^{h_{ji}}\Bigr)\operatorname{Det}\bigl(\Lambda_N(\nu^A)\bigr)^{h_{jN}^o}\prod_{k=1}^{C+1}\lambda_k(y_j)^{k_j}\\ &= \Bigl(\prod_{i=1}^{N-1}\operatorname{Det}\bigl(\Lambda_i(\nu^A)\bigr)^{h_{ji}}\Bigr)\operatorname{Det}\bigl(\Lambda_N(\nu^A)\bigr)^{h_{jN}^o}a_j^{-1}c_j = \operatorname{Det}\bigl(\Lambda_j(u)\bigr).\end{aligned}$$

Hence $q_B'\bigl(\varphi(\nu^A)\bigr) = q_B'(u)$ in $U(B)/\overline{DU(B)}$ by Proposition 5.3. It follows from Theorem 4.7 that $[\varphi] = \kappa$ in $KK(A, B)$.

Finally, for $\omega \in T(B)$ and $f \in \operatorname{Aff} T(A) \cong C_{\mathbb{R}}(\mathbb{T})$,

$$\begin{aligned}&\Bigl|\widehat{\varphi}(f)(\omega) - \frac{1}{C}\sum_{i=1}^{C} f\circ\lambda_k(\omega)\Bigr|\\ &\quad= \Bigl|\omega\bigl(\varphi(f\otimes 1)\bigr) - \frac{1}{C}\sum_{k=1}^{C}\omega\bigl((f\circ\lambda_k)\otimes 1\bigr)\Bigr|\\ &\quad\le \Bigl|\frac{1}{m}(m - Cbn)\Bigr|\,\|f\| + \Bigl\|\frac{1}{m}bn\sum_{k=1}^{C} f\circ\lambda_k - \frac{1}{C}\sum_{k=1}^{C} f\circ\lambda_k\Bigr\|\\ &\quad\le \Bigl|\frac{1}{m}(m - Cbn)\Bigr|\,\|f\| + \Bigl|\frac{1}{m}bn - \frac{1}{C}\Bigr|C\|f\| = 2\Bigl|1 - \frac{Cbn}{m}\Bigr|\,\|f\| < \epsilon\|f\|.\end{aligned}$$

Hence

$$\Bigl\|\widehat{\varphi}(f) - \frac{1}{C}\sum_{k=1}^{C} f\circ\lambda_k\Bigr\| < \epsilon\|f\|. \qquad \blacksquare$$

The following result is due to Li [17, Theorem 2.1]. It generalizes a theorem of Thomsen [26, Theorem 2.1] and it is the key stone in the proof of Theorem 8.3 below.

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Theorem 8.2 *Let X be a path-connected compact metric space, let $F \subseteq C_{\mathbb{R}}(X)$ be a finite subset and let $\epsilon > 0$. There exists a positive integer L such that for all integers $N \geq L$, for all compact metric spaces Y, and for all positive linear order unit preserving maps $\Theta\colon C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(Y)$, there exist continuous functions $\lambda_k\colon Y \to X$, $k = 1, 2, \dots, N$, such that*

$$\left\| \Theta(f) - \frac{1}{N} \sum_{k=1}^{N} f \circ \lambda_k \right\| < \epsilon, \quad f \in F.$$

Theorem 8.3 *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block, let $\epsilon > 0$, let $F \subseteq \operatorname{Aff} T(A)$ be a finite set, and let C be a non-negative integer. There exists a positive integer K such that if*

(i) $B = A(m, e_1, e_2, \dots, e_M)$ *is a building block with* $s(B) \geq K$,
(ii) $\kappa \in KK(A, B)_e$,
(iii) $\lambda_1, \lambda_2, \dots, \lambda_C\colon \mathbb{T} \to \mathbb{T}$ *are continuous functions,*
(iv) $\Theta\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ *is a positive linear order unit preserving map,*
(v) $u \in B$ *is a unitary such that* $\kappa_*[v^A] = [u]$ *in* $K_1(B)$;

then there exists a unital $*$-homomorphism $\varphi\colon A \to B$ such that $\lambda_1, \lambda_2, \dots, \lambda_C$ are eigenvalue functions for φ and such that

$$[\varphi] = \kappa \quad \textit{in } KK(A, B),$$

$$\varphi^{\#}\big(q'_A(v^A)\big) = q'_B(u) \quad \textit{in } U(B)/\overline{DU(B)}$$

$$\|\widehat{\varphi}(f) - \Theta(f)\| < \epsilon, \quad f \in F.$$

Proof We may assume that $\|f\| \leq 1$, $f \in F$. Choose by Theorem 8.2 an integer L with respect to $F \subseteq \operatorname{Aff} T(A) \cong C_{\mathbb{R}}(\mathbb{T})$ and $\frac{\epsilon}{3}$. We may assume that $L > C$ and that $1 - \frac{L-C}{C+L} < \frac{\epsilon}{3}$. Then choose by Theorem 8.1 an integer K with respect to $C + L$ and $\frac{\epsilon}{3}$.

Now let B, Θ, $\lambda_1, \lambda_2, \dots, \lambda_C$, κ and u be given as above. Choose continuous functions $\lambda_{C+1}, \lambda_{C+2}, \dots, \lambda_{C+L}\colon \mathbb{T} \to \mathbb{T}$ such that in $\operatorname{Aff} T(B) \cong C_{\mathbb{R}}(\mathbb{T})$,

$$\left\| \Theta(f) - \frac{1}{L} \sum_{k=C+1}^{C+L} f \circ \lambda_k \right\| < \frac{\epsilon}{3}, \quad f \in F.$$

By Theorem 8.1 there exists a unital $*$-homomorphism $\varphi\colon A \to B$ such that λ_1, $\lambda_2, \dots, \lambda_{C+L}$ are eigenvalue functions for φ and such that

$$[\varphi] = \kappa \quad \text{in } KK(A, B),$$

$$\varphi^{\#}\big(q'_A(v^A)\big) = q'_B(u) \quad \text{in } U(B)/\overline{DU(B)},$$

$$\left\| \widehat{\varphi}(f) - \frac{1}{C+L} \sum_{k=1}^{C+L} f \circ \lambda_k \right\| < \frac{\epsilon}{3}\|f\|, \quad f \in \operatorname{Aff} T(A).$$

Since for $f \in \operatorname{Aff} T(A)$,

$$\begin{aligned}\Big\| \frac{1}{C+L} \sum_{k=1}^{C+L} f \circ \lambda_k - \frac{1}{L} \sum_{k=C+1}^{C+L} f \circ \lambda_k \Big\| \\ \leq \Big\| \frac{1}{C+L} \sum_{k=C+1}^{C+L} f \circ \lambda_k - \frac{1}{L} \sum_{k=C+1}^{C+L} f \circ \lambda_k \Big\| + \Big\| \frac{1}{C+L} \sum_{k=1}^{C} f \circ \lambda_k \Big\| \\ \leq \Big| \frac{1}{C+L} - \frac{1}{L} \Big| L\|f\| + \frac{1}{C+L} C\|f\| = \Big(1 - \frac{L-C}{C+L}\Big) \|f\| < \frac{\epsilon}{3}\|f\|,\end{aligned}$$

we get that

$$\|\widehat{\varphi}(f) - \Theta(f)\| < \epsilon, \quad f \in F. \qquad \blacksquare$$

Lemma 8.4 *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block, let $p \in A$ be a non-zero projection, and let $u \in A$ be a unitary. Then there exists a unitary $w \in pAp$ such that*

$$q'_A(u) = q'_A\big(w + (1-p)\big) \quad \text{in } U(A)/\overline{DU(A)}.$$

Proof Note that pAp is a building block by Corollary 3.5. Hence by Lemma 5.4 there exists a unitary $w \in pAp$ such that

$$\operatorname{Det}\big(w(z)\big) = \operatorname{Det}\big(u(z)\big), \quad z \in \mathbb{T},$$
$$\operatorname{Det}\big(\Lambda_i(w)\big) = \operatorname{Det}\big(\Lambda_i(u)\big), \quad i = 1, 2, \dots, N.$$

Then $q'_A(u) = q'_A\big(w + (1-p)\big)$ in $U(A)/\overline{DU(A)}$ by Theorem 5.3. ■

Theorem 8.5 *Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_R$ be a finite direct sum of building blocks. Let $F \subseteq \operatorname{Aff} T(A)$ be a finite set and let $\epsilon > 0$. There exists a positive integer K such that if*

(i) $B = B_1 \oplus B_2 \oplus \cdots \oplus B_S$ is a finite direct sum of building blocks and κ is an element in $KK(A, B)_e$,
(ii) for every minimal non-zero central projection p in A we have that

$$s(B)\rho_B(\kappa_*[p]) \geq K \quad \text{in } \operatorname{Aff} T(B),$$

(iii) there exists a linear positive order unit preserving map $\Theta\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ such that the diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \operatorname{Aff} T(A) \\ {\scriptstyle\kappa_*}\downarrow & & \downarrow{\scriptstyle\Theta} \\ K_0(B) & \xrightarrow[\rho_B]{} & \operatorname{Aff} T(B) \end{array}$$

commutes,

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(iv) $u_1, u_2, \ldots, u_N \in B$ *are unitaries such that*

$$\kappa_*[v_i^A] = [u_i] \quad \textit{in } K_1(A), \quad i = 1, 2, \ldots, R;$$

then there exists a unital $*$-homomorphism $\varphi\colon A \to B$ such that $[\varphi] = \kappa$ in $KK(A, B)$, and such that

$$\|\widehat{\varphi}(f) - \Theta(f)\| < \epsilon, \quad f \in F,$$

$$\varphi^{\#}\big(q_A'(v_i^A)\big) = q_B'(u_i) \quad \textit{in } U(B)/\overline{DU(B)}, \quad i = 1, 2, \ldots, R.$$

Proof Let $\pi_i^A\colon A \to A_i$ be the projection and $\iota_i^A\colon A_i \to A$ be the inclusion, $i = 1, 2, \ldots, R$. Let $p_1, p_2, \ldots, p_R$ denote the minimal non-zero central projections in A. Choose by Theorem 8.3 an integer K_i with respect to $\widehat{\pi_i^A}(F) \subseteq \operatorname{Aff} T(A_i)$, $\epsilon > 0$ and $C = 0$. Set $K = \max_{1 \le i \le R} K_i$.

Let B, κ, Θ, and $u_1, u_2, \ldots, u_N$ be as above. We may assume that $S = 1$. To see this, assume that the case $S = 1$ has been settled. Let $\pi_l^B\colon B \to B_l$ be the projection and let $\iota_l^B\colon B_l \to B$ be the inclusion. As the diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\ \rho_A\ } & \operatorname{Aff} T(A) \\ {\scriptstyle \pi^B_{l\,*}\circ\kappa_*}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\pi_l^B}\circ\Theta} \\ K_0(B_l) & \xrightarrow[\ \rho_{B_l}\ ]{} & \operatorname{Aff} T(B_l) \end{array}$$

commutes for $l = 1, 2, \ldots, S$, and since $s(B_l)\rho_{B_l}(\pi^B_{l\,*}\circ\kappa_*[p_i]) \ge K$ for $i = 1, 2, \ldots, R$, $l = 1, 2, \ldots, S$, we get unital $*$-homomorphisms $\varphi_l\colon A \to B_l$ such that

$$[\varphi_l] = [\pi_l^B] \cdot \kappa \quad \text{in } KK(A, B_l),$$

$$\|\widehat{\varphi}_l(f) - \widehat{\pi_l^B} \circ \Theta(f)\| < \epsilon, \quad f \in F,$$

$$\varphi_l^{\#}\big(q_A'(v_i^A)\big) = q_{B_l}'\big(\pi_l^B(u_i)\big) \quad \text{in } U(B_l)/\overline{DU(B_l)}, \quad i = 1, 2, \ldots, R.$$

Define $\varphi\colon A \to B$ by $\varphi(a) = \big(\varphi_1(a), \varphi_2(a), \ldots, \varphi_S(a)\big)$. Then

$$[\varphi] = \Big[\sum_{l=1}^{S} \iota_l^B \circ \varphi_l\Big] = \sum_{l=1}^{S} [\iota_l^B] \cdot [\pi_l^B] \cdot \kappa = \kappa \quad \text{in } KK(A, B),$$

$$\|\widehat{\varphi}(f) - \Theta(f)\| = \max_l \|\widehat{\pi_l^B} \circ \widehat{\varphi}(f) - \widehat{\pi_l^B} \circ \Theta(f)\| < \epsilon, \quad f \in F,$$

$$\varphi^{\#}\big(q_A'(v_i^A)\big) = q_B'(u_i) \quad \text{in } U(B)/\overline{DU(B)}, \quad i = 1, 2, \ldots, R.$$

So assume $B = A(m, e_1, e_2, \ldots, e_M)$. Note that by assumption $\kappa_*[p_i] > 0$ in $K_0(B)$ for $i = 1, 2, \ldots, R$. Let $e = \gcd(e_1, e_2, \ldots, e_M)$. Choose by Corollary 3.6 orthogonal non-zero projections $q_i \in M_e \subseteq B$, for $i = 1, 2, \ldots, R$, with sum 1 such

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that $\kappa_*[p_i] = [q_i]$. Let $t_i > 0$ be the normalized trace of q_i. Note that we have a well-defined map $J_i\colon \operatorname{Aff} T(A_i) \to \operatorname{Aff} T(A)$ such that $J_i(\widehat{a}) = \widehat{\iota_i^A(a)}$ for every self-adjoint element $a \in A_i$. Define $\Theta_i\colon \operatorname{Aff} T(A_i) \to \operatorname{Aff} T(q_i B q_i)$ by

$$\Theta_i(f)\left(\frac{1}{t_i}\tau \circ \epsilon_i\right) = \frac{1}{t_i}\Theta\big(J_i(f)\big)(\tau), \quad \tau \in T(B),$$

where $\epsilon_i\colon q_i B q_i \to B$ denotes the inclusion.

Θ_i is a linear positive map, and it preserves the order unit since

$$\Theta_i(1)\left(\frac{1}{t_i}\tau \circ \epsilon_i\right) = \frac{1}{t_i}\Theta(\widehat{p_i})(\tau) = \frac{1}{t_i}\rho_B \circ \kappa_*[p_i](\tau) = 1.$$

By [23, Theorem 7.3] we get that $[\epsilon_i] \in KK(q_i B q_i, B)$ is a KK-equivalence. Note that

$$[\epsilon_i]^{-1} \cdot \kappa \cdot [\iota_i^A] \in KK(A_i, q_i B q_i)_e.$$

By Corollary 3.5 we have that $q_i B q_i \cong A(t_i m, t_i e_1, t_i e_2, \dots, t_i e_M)$. Using Lemma 8.4, choose a unitary $w_i \in q_i B q_i$ such that

$$q'_B\big(w_i + (1 - q_i)\big) = q'_B(u_i) \quad \text{in } U(B)/\overline{DU(B)}.$$

Since $t_i e_j \geq K$ for $j = 1, 2, \dots, M$, we get by Theorem 8.3 a unital $*$-homomorphism $\varphi_i\colon A_i \to q_i B q_i$ such that

$$[\varphi_i] = [\epsilon_i]^{-1} \cdot \kappa \cdot [\iota_i^A] \quad \text{in } KK(A_i, q_i B q_i),$$

$$\|\widehat{\varphi}_i(f) - \Theta_i(f)\| < \epsilon, \quad f \in \widehat{\pi_i^A}(F),$$

$$\varphi_i(v^{A_i}) = w_i \mod \overline{DU(q_i B q_i)}.$$

Now define $\varphi\colon A \to B$ by

$$\varphi(a) = \sum_{i=1}^{R} \epsilon_i \circ \varphi_i \circ \pi_i^A(a).$$

φ is a unital $*$-homomorphism and

$$[\varphi] = \sum_{i=1}^{R} [\epsilon_i] \cdot [\varphi_i] \cdot [\pi_i^A] = \sum_{i=1}^{R} \kappa \cdot [\iota_i^A] \cdot [\pi_i^A] = \kappa \quad \text{in } KK(A, B).$$

For $f \in \operatorname{Aff} T(A)$, $\tau \in T(B)$, we have that

$$\Theta(f)(\tau) = \sum_{i=1}^{R} \Theta\Big(J_i\big(\widehat{\pi_i^A}(f)\big)\Big)(\tau) = \sum_{i=1}^{R} t_i \Theta_i\big(\widehat{\pi_i^A}(f)\big)\left(\frac{1}{t_i}\tau \circ \epsilon_i\right),$$

and

$$\widehat{\varphi}(f)(\tau) = f(\tau \circ \varphi) = f\left(\sum_{i=1}^{R} t_i \frac{1}{t_i} \tau \circ \epsilon_i \circ \varphi_i \circ \pi_i^A\right) = \sum_{i=1}^{R} t_i \widehat{\varphi}_i\big(\widehat{\pi_i^A}(f)\big) \left(\frac{1}{t_i}\tau \circ \epsilon_i\right).$$

It follows that

$$\|\widehat{\varphi}(f) - \Theta(f)\| < \epsilon, \quad f \in F.$$

Finally, for $i = 1, 2, \ldots, R$,

$$\varphi(v_i^A) = \epsilon_i \circ \varphi_i(v^{A_i}) + (1 - q_i) = w_i + (1 - q_i) = u_i$$

modulo $\overline{DU(B)}$. ∎

9 Injective Connecting Maps

The purpose of this section is to show that a simple unital infinite dimensional inductive limit of a sequence of finite direct sums of building blocks can be realized as an inductive limit of a sequence of finite direct sums of building blocks with unital and injective connecting maps.

From now on, we will consider inductive limits in the category of order unit spaces and linear positive order unit preserving maps, as introduced by Thomsen [26]. It follows from [26, Lemma 3.3] that $\operatorname{Aff} T(\cdot)$ is a continuous functor from the category of separable unital C^*-algebras and unital $*$-homomorphisms to the category of order unit spaces. We will also need Elliott's approximative intertwining argument, see [9, Theorem 2.1] or [25].

Lemma 9.1 *Let A be a finite direct sum of building blocks, interval building blocks, and matrix algebras. Let $\epsilon > 0$ and let $F \subseteq A$ be a finite set. There exists a finite set of positive non-zero elements $H \subseteq A$ such that if B is a building block or an interval building block, and $\varphi\colon A \to B$ is a unital $*$-homomorphism with $\varphi(h) \neq 0$, $h \in H$, then there exists a unital injective $*$-homomorphism $\psi\colon A \to B$ such that $\|\varphi(f) - \psi(f)\| < \epsilon$, $f \in F$.*

Proof By Corollary 3.5 (and the corresponding result for interval building blocks) we may assume that A is a building block, an interval building block or a matrix algebra rather than a finite direct sum of such algebras. We will carry out the proof in the case that $A = A(n, d_1, d_2, \ldots, d_N)$ is a circle building block. The proof in the case that A is an interval building block is similar, and the matrix algebra case is trivial.

Choose $\delta > 0$ such that for $x, y \in \mathbb{T}$,

$$\rho(x, y) < 2\delta \Longrightarrow \|f(x) - f(y)\| < \epsilon, \quad f \in F.$$

Let $\mathbb{T} = \bigcup_{i=1}^{K} V_i$ where each V_i is an open arc-segment of length less than δ. Choose for each $i = 1, 2, \ldots, K$, a non-zero continuous function $\chi_i\colon \mathbb{T} \to [0, 1]$ with support in V_i such that χ_i is zero at every exceptional point of A. Set

$$H = \{\chi_1 \otimes 1, \chi_2 \otimes 1, \ldots, \chi_K \otimes 1\}.$$

Let $\varphi\colon A \to B$ be given such that $\varphi(h) \neq 0$, $h \in H$. By [27, Chapter 1] we may assume that

$$\varphi(f)(e^{2\pi it}) = u(t)\operatorname{diag}\Big(\Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f), f\big(\lambda_1(t)\big), \dots, f\big(\lambda_L(t)\big)\Big)u(t)^*$$

if $B = A(m, e_1, e_2, \dots, e_M)$ is a circle building block, and

$$\varphi(f)(t) = u(t)\operatorname{diag}\Big(\Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f), f\big(\lambda_1(t)\big), \dots, f\big(\lambda_L(t)\big)\Big)u(t)^*$$

if $B = I(m, e_1, e_2, \dots, e_M)$ is an interval building block. Here $u \in C[0,1] \otimes M_m$ is a unitary, $\lambda_1, \dots, \lambda_L\colon [0,1] \to \mathbb{T}$ are continuous functions, and $s_1, s_2, \dots, s_N$ are non-negative integers. Since $\varphi(h) \neq 0$, $h \in H$, it follows that the set $\bigcup_{k=1}^{L} \lambda_k([0,1])$ intersects non-trivially with every V_i.

If B is an interval building block, let $t_1, t_2, \dots, t_M \in [0,1]$ be the exceptional points of B. If B is a circle building block, let $t_1, t_2, \dots, t_M \in [0,1]$ be numbers such that $e^{2\pi it_j}$, $j = 1, 2, \dots, M$, are the exceptional points of B.

For each $k = 1, 2, \dots, L$, choose a continuous function $\mu_k\colon [0,1] \to \mathbb{T}$ such that $\rho\big(\mu_k(t), \lambda_k(t)\big) < 2\delta$, $t \in [0,1]$, such that $\mu_k(t) = \lambda_k(t)$ for $t \in \{t_1, t_2, \dots, t_M, 0, 1\}$, and such that $\bigcup_{i=1}^{k} \mu_k([0,1]) = \bigcup_{i=1}^{k} V_i = \mathbb{T}$. Define $\psi\colon A \to B$ by

$$\psi(f)(e^{2\pi it}) = u(t)\operatorname{diag}\Big(\Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f), f\big(\mu_1(t)\big), \dots, f\big(\mu_L(t)\big)\Big)u(t)^*$$

if B is a circle building block, and

$$\psi(f)(t) = u(t)\operatorname{diag}\Big(\Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f), f\big(\mu_1(t)\big), \dots, f\big(\mu_L(t)\big)\Big)u(t)^*$$

if B is an interval building block. Note that ψ is injective and $\|\varphi(f) - \psi(f)\| < \epsilon$, $f \in F$. ■

Lemma 9.2 *Let A be a unital C^*-algebra that is the inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks. Then A is the inductive limit of a similar sequence, with unital connecting maps.

Proof Note that we may assume that $\alpha_{n,\infty}(p) \neq 0$ for every positive integer n and every minimal non-zero central projection $p \in A_n$. By Lemma 2.2 it follows that $\alpha_{n,\infty}(q) \neq 0$ for every non-zero projection $q \in A_n$. Let $1_n \in A_n$ denote the unit. Since $\{\alpha_{n,\infty}(1_n)\}_{n=1}^{\infty}$ is an approximate unit for A there exists a positive integer N such that $\alpha_{k,\infty}(1_k) = 1$ for all $k \geq N$. Hence $\alpha_k(1_k) = 1_{k+1}$, $k \geq N$. ■

Lemma 9.3 *Let $X \subseteq \mathbb{T}$ be a closed set and let $G \subseteq X$ be a finite subset. Let $\delta > 0$ be given. There exist a closed subset $R \subseteq X$ with finitely many connected components such that $G \subseteq R$, together with a continuous surjective map $g\colon X \to R$ such that $g(z) = z$, $z \in G$, and $\rho\big(g(z), z\big) \leq \delta$, $z \in X$.*

Proof Let $G = \{e^{2\pi i t_j} : j = 1, 2, \dots, N\}$ where $0 \le t_1 < t_2 < \cdots < t_N < 1$. Set $t_{N+1} = t_1 + 1$. Set $I_j = \{e^{2\pi i t} : t \in [t_j, t_{j+1}]\}$. We may assume that $t_{j+1} - t_j < \delta$ unless the interior of I_j intersects non-trivially with X. On each I_j let either g be the identity map (if $I_j \subseteq X$) or a continuous map onto $\{e^{2\pi i t_j}, e^{2\pi i t_{j+1}}\}$ that is constant on the set of boundary points of I_j. Set $R = g(X)$. ∎

Lemma 9.4 *Let A be a quotient of a finite direct sum of building blocks. Let $F \subseteq A$ be a finite set and let $\epsilon > 0$. There exists a finite direct sum of building blocks, interval building blocks and matrix algebras B, and unital $*$-homomorphisms $\varphi\colon A \to B$ and $\psi\colon B \to A$ such that ψ is injective and $\|\psi \circ \varphi(f) - f\| < \epsilon$, $f \in F$.*

Proof We may assume that A is a quotient of a building block rather than of a finite direct sum of building blocks. Hence by Lemma 2.2

$$A = \{f \in C(X) \otimes M_n : f(x_i) \in M_{d_i}, i = 1, 2, \dots, N\}$$

where $X \subseteq \mathbb{T}$ is a closed subset and $x_1, x_2, \dots, x_N \in X$. Choose $\delta > 0$ such that

$$y, z \in X, \quad \rho(y, z) \le \delta \Longrightarrow \|f(y) - f(z)\| < \epsilon, \quad f \in F.$$

Choose by Lemma 9.3 a closed subset $R \subseteq X$ with finitely many connected components such that $x_1, x_2, \dots, x_N \in R$, and a continuous surjective map $g\colon X \to R$ such that $g(x_i) = x_i$, $i = 1, 2, \dots, N$, and such that $\rho\big(g(z), z\big) \le \delta$, $z \in X$. Let

$$B = \{f \in C(R) \otimes M_n : f(x_i) \in M_{d_i}, i = 1, 2, \dots, N\}.$$

Define $\psi\colon B \to A$ by $\psi(f) = f \circ g$ and let $\varphi\colon A \to B$ be restriction. Then $\|\psi \circ \varphi(f) - f\| < \epsilon$, $f \in F$. ∎

Proposition 9.5 *Let A be a unital simple inductive limit of a sequence of finite direct sums of building blocks. Then A is the inductive limit of a sequence of finite direct sums of building blocks, interval building blocks and matrix algebras, with unital and injective connecting maps.*

Proof By Lemma 9.2 we have that A is the inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

where each α_n is unital and injective and each A_n is a quotient of a finite direct sum of building blocks. We will construct a strictly increasing sequence of positive integers $\{n_k\}$, a sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

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of finite direct sums of building blocks, interval building blocks and matrix algebras with unital connecting maps, unital $*$-homomorphisms $\mu_k\colon A_{n_k} \to B_{k+1}$, and unital injective $*$-homomorphisms $\psi_k\colon B_k \to A_{n_k}$ such that the diagram

$$\begin{array}{ccccccc} A_{n_1} & \xrightarrow{\alpha_{n_1,n_2}} & A_{n_2} & \xrightarrow{\alpha_{n_2,n_3}} & A_{n_3} & \xrightarrow{\alpha_{n_3,n_4}} & \cdots \\ \uparrow{\scriptstyle \psi_1} & \searrow^{\mu_1} & \uparrow{\scriptstyle \psi_2} & \searrow^{\mu_2} & \uparrow{\scriptstyle \psi_3} & \searrow^{\mu_3} & \\ B_1 & \xrightarrow[\beta_1]{} & B_2 & \xrightarrow[\beta_2]{} & B_3 & \xrightarrow[\beta_3]{} & \cdots \end{array}$$

becomes an approximate intertwining. Furthermore β_k should be injective unless B_{k+1} is finite dimensional. This is sufficient since the proposition is trivial if A is an AF-algebra.

It is easy to construct B_1, n_1 and ψ_1. Assume that B_k, n_k and ψ_k have been constructed. Let $\epsilon > 0$ and finite sets $F \subseteq A_{n_k}$ and $G \subseteq B_k$ be given. Choose $H \subseteq B_k$ by Lemma 9.1 with respect to $\epsilon > 0$ and G. Since A is simple we may choose n_{k+1} such that $\widehat{\alpha_{n_k,n_{k+1}}}(\widehat{\psi_k(h)}) > 0$ for $h \in H$. Choose B_{k+1}, φ_{k+1} and ψ_{k+1} by Lemma 9.4 with respect to $\epsilon > 0$ and $\alpha_{n_k,n_{k+1}}(F)$. Set $\mu_k = \varphi_{k+1} \circ \alpha_{n_k,n_{k+1}}$. Then

$$\|\psi_{k+1} \circ \mu_k(x) - \alpha_{n_k,n_{k+1}}(x)\| < \epsilon, \quad x \in F.$$

Since $\widehat{\mu_k} \circ \widehat{\psi_k}(\widehat{h}) > 0$, $h \in H$, there exists by Lemma 9.1 a unital $*$-homomorphism $\beta_k\colon B_k \to B_{k+1}$ such that

$$\|\mu_k \circ \psi_k(x) - \beta_k(x)\| < \epsilon, \quad x \in G,$$

and such that β_k is injective if B_{k+1} is infinite dimensional. ■

Lemma 9.6 *Let A be a simple infinite dimensional inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks, interval building blocks and matrix algebras, with unital and injective connecting maps. Then $s(A_m) \to \infty$.

Proof The lemma is well-known if A is an AF-algebra. We may therefore assume that A_k is infinite dimensional for some k. Let L be a positive integer. Let $b_1, b_2, \dots, b_L \in A_k$ be positive non-zero mutually orthogonal elements. Since A is simple and the connecting maps are injective, there exists an integer $N \geq k$ such that

$$\widehat{\alpha_{k,N}}(\widehat{b_j}) > 0, \quad j = 1, 2, \dots, L.$$

Hence if $m \geq N$ and $\mu\colon A_m \to M_n$ is a unital $*$-homomorphism, we see that the elements $\mu \circ \alpha_{1,m}(b_j)$, $j = 1, 2, \dots, L$, are non-zero and mutually orthogonal. Thus $n \geq L$. ■

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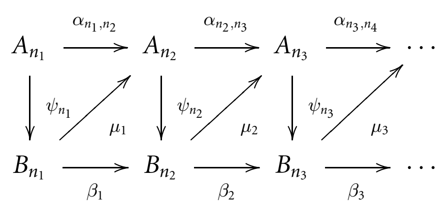
Proposition 9.7 *Let A be a simple unital infinite dimensional inductive limit of a sequence of finite direct sums of building blocks. Then A is the inductive limit of a sequence of finite direct sums of building blocks and interval building blocks with unital and injective connecting maps.*

Proof By Proposition 9.5 we have that A is the inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

where each α_k is unital and injective and each A_k is of the form $C_k \oplus F_k$ for a finite (possibly trivial) direct sum of building blocks C_k and a finite dimensional C^*-algebra F_k. Set $B_k = C_k \oplus \big(C(\mathbb{T}) \otimes F_k\big)$ and let $\psi_k \colon A_k \to B_k$ be the canonical $*$-homomorphism. It suffices to construct a strictly increasing sequence of positive integers $\{n_k\}$, unital $*$-homomorphisms $\mu_k \colon B_{n_k} \to A_{n_{k+1}}$, and unital injective $*$-homomorphisms $\beta_k \colon B_{n_k} \to B_{n_{k+1}}$ such that the diagram



becomes an approximate intertwining. This is done by induction. Set $n_1 = 1$.

Assume that n_k has been constructed. Let $\epsilon > 0$ and a finite set $G \subseteq B_{n_k}$ be given. It suffices to construct $n_{k+1} > n_k$, a unital $*$-homomorphism $\mu_k \colon B_{n_k} \to A_{n_{k+1}}$ such that $\mu_k \circ \psi_{n_k}$ and $\alpha_{n_k,n_{k+1}}$ are approximately unitarily equivalent, and a unital injective $*$-homomorphism $\beta_k \colon B_{n_k} \to B_{n_{k+1}}$ such that

$$\|\beta_k(x) - \psi_{n_{k+1}} \circ \mu_k(x)\| < \epsilon, \quad x \in G. \tag{27}$$

Let $F_{n_k} = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_N}$ and let $p_1, p_2, \ldots, p_N$ be the minimal non-zero central projections in $F_{n_k} \subseteq A_{n_k}$. Let $\pi_i \colon B_{n_k} \to C(\mathbb{T}) \otimes M_{m_i}$ be the projection, $i = 1, 2, \ldots, N$. Choose by Lemma 9.1 a finite set $H_i \subseteq C(\mathbb{T}) \otimes M_{m_i}$ of positive non-zero elements with respect to ϵ and $\pi_i(G)$. Let h_i be the cardinality of H_i.

Since A is simple there exists a $\delta > 0$ such that

$$\widehat{\alpha_{n_k,\infty}}(\widehat{p}_i) > \delta, \quad i = 1, 2, \ldots, N.$$

By Lemma 9.6 there exists an integer $n_{k+1} > n_k$ such that

$$\widehat{\alpha_{n_k,n_{k+1}}}(\widehat{p}_i) > \delta, \quad i = 1, 2, \ldots, N,$$
$$s(A_{n_{k+1}}) > \delta^{-1} \max_i (h_i m_i), \quad i = 1, 2, \ldots, N.$$

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Let $q_i = \alpha_{n_k,n_{k+1}}(p_i)$ and note that

$$s(q_i A_{n_{k+1}} q_i) > \delta s(A_{n_{k+1}}) > h_i m_i, \quad i = 1, 2, \dots, N.$$

Hence there exists a unital $*$-homomorphism $\lambda_i\colon C(\mathbb{T}) \otimes M_{m_i} \to q_i A_{n_{k+1}} q_i$ such that $\widehat{\lambda}_i(\widehat{h}) > 0$, $h \in H_i$. Let $\mu_k\colon B_{n_k} \to A_{n_{k+1}}$ be the $*$-homomorphism that agrees with $\alpha_{n_k,n_{k+1}}$ on C_{n_k} and with λ_i on $C(\mathbb{T}) \otimes M_{m_i}$. The $*$-homomorphism $x \mapsto \lambda_i(1 \otimes x)$ from M_{m_i} to $q_i A_{n_{k+1}} q_i$ is by [27, Chapter 1] approximately unitarily equivalent to the $*$-homomorphism induced by $\alpha_{n_k,n_{k+1}}$. Hence $\mu_k \circ \psi_{n_k}$ and $\alpha_{n_k,n_{k+1}}$ are approximately unitarily equivalent.

Let $e_i = \psi_{n_{k+1}}(q_i)$, $i = 1, 2, \dots, N$ and let $\xi_i\colon q_i A_{n_{k+1}} q_i \to e_i B_{n_{k+1}} e_i$ be the unital $*$-homomorphism induced by $\psi_{n_{k+1}}$. Since $\widehat{\xi}_i \circ \widehat{\lambda}_i(\widehat{h}) > 0$ there exists by Lemma 9.1 a unital injective $*$-homomorphism $\varphi_i\colon C(\mathbb{T}) \otimes M_{m_i} \to e_i B_{n_{k+1}} e_i$ such that

$$\|\varphi_i(x) - \xi_i \circ \lambda_i(x)\| < \epsilon, \quad x \in \pi_i(G), \quad i = 1, 2, \dots, N.$$

Let β_k be the $*$-homomorphism that agrees with $\psi_{n_{k+1}} \circ \mu_k$ on C_{n_k} and with φ_i on $C(\mathbb{T}) \otimes M_{m_i}$. Note that β_k is unital and injective and that (27) holds. ■

It remains to replace interval building blocks with building blocks. This turns out to be much more complicated than in [20, Lemma 1.5] or [27, Lemma 4.7], since our building blocks may be unital projectionless. We will use the following lemma, which resembles a uniqueness result for interval building blocks. The proof is inspired by Elliott's proof of the uniqueness lemma for interval algebras [8].

Lemma 9.8 *Let $A = I(n, d_1, d_2, \dots, d_N)$ be an interval building block. Let $F \subseteq A$ be a finite set and let $\epsilon > 0$ be given. There is a finite set $H \subseteq A$ of positive elements of norm 1 such that if $B = I(m, e_1, e_2, \dots, e_M)$ is an interval building block with exceptional points $y_1, y_2, \dots, y_M$, if $\varphi, \psi\colon A \to B$ are unital $*$-homomorphisms and if $\delta > 0$, such that*

(i) $\|\widehat{\varphi}(\widehat{h}) - \widehat{\psi}(\widehat{h})\| < \delta, \quad h \in H,$
(ii) $\widehat{\varphi}(\widehat{h}) > \delta$, $h \in H$,
(iii) $\widehat{\psi}(\widehat{h}) > \delta$, $h \in H$,
(iv) *$f \mapsto \varphi(f)(y_j)$ and $f \mapsto \psi(f)(y_j)$ are equivalent representations of A on M_m, $j = 1, 2, \dots, M$;*

then there is a unitary $W \in B$ such that

$$\|\varphi(f) - W\psi(f)W^*\| < \epsilon, \quad f \in F.$$

Proof We may assume that $\|f\| \le 1$ for $f \in F$. Let $x_1, x_2, \dots, x_N$ be the exceptional points of A. Choose a positive integer q such that

$$\frac{2}{q} < \min\{|x_i - x_j| : i \neq j\},$$

$$|x - y| \le \frac{3}{q} \Rightarrow \|f(x) - f(y)\| < \frac{\epsilon}{3}, \quad f \in F.$$

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For $r = 1, 2, \ldots, q$, define a continuous function $h_r \colon [0,1] \to [0,1]$ by

$$h_r(t) = \begin{cases} 0 & 0 \le t \le \frac{r-1}{q}, \\ qt - (r-1) & \frac{r-1}{q} \le t \le \frac{r}{q}, \\ 1 & \frac{r}{q} \le t \le 1. \end{cases}$$

Set

$$H = \{h_1 \otimes 1, h_2 \otimes 1, \ldots, h_q \otimes 1\} \cup \{(h_1 - h_2) \otimes 1, \ldots, (h_{q-1} - h_q) \otimes 1\}.$$

Let $\varphi, \psi \colon A \to B$ be unital $*$-homomorphisms that satisfy (i)–(iv). By [27, Chapter 1] we see that φ and ψ are approximately unitarily equivalent to $*$-homomorphisms of the form

$$\varphi'(f)(t) = u(t)\operatorname{diag}\Big(\Lambda_1^{r_1}(f), \ldots, \Lambda_N^{r_N}(f), f\big(\lambda_1(t)\big), \ldots, f\big(\lambda_L(t)\big)\Big)u(t)^*$$

$$\psi'(f)(t) = v(t)\operatorname{diag}\Big(\Lambda_1^{s_1}(f), \ldots, \Lambda_N^{s_N}(f), f\big(\mu_1(t)\big), \ldots, f\big(\mu_K(t)\big)\Big)v(t)^*$$

for continuous functions $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_L$, $\mu_1 \le \mu_2 \le \cdots \le \mu_K \colon [0,1] \to [0,1]$, integers r_i and s_i with $0 \le r_i < \frac{n}{d_i}$, $0 \le s_i < \frac{n}{d_i}$ for $i = 1, 2, \ldots, N$, and unitaries $u, v \in C[0,1] \otimes M_m$. By (iv) we have that $f \mapsto \varphi'(f)(y_j)$ and $f \mapsto \psi'(f)(y_j)$ are equivalent representations of A on M_m, $j = 1, 2, \ldots, M$. It follows that $r_i = s_i$, $i = 1, 2, \ldots, N$, that $K = L$, and that

$$\big(\lambda_1(y_j), \lambda_2(y_j), \ldots, \lambda_L(y_j)\big) = \big(\mu_1(y_j), \mu_2(y_j), \ldots, \mu_L(y_j)\big)$$

as unordered L-tuples, $j = 1, 2, \ldots, M$. Hence

$$\lambda_k(y_j) = \mu_k(y_j), \quad j = 1, 2, \ldots, M, k = 1, 2, \ldots, L. \tag{28}$$

For every $t \in [0,1]$, $r = 2, 3, \ldots, q$, we have that

$$\begin{aligned} \#\Big\{k : \lambda_k(t) \ge \frac{r}{q}\Big\} n + \sum_{i : x_i \ge \frac{r}{q}} r_i d_i &\le \operatorname{Tr}\big(\varphi'(h_r \otimes 1)(t)\big) \\ &< m\delta + \operatorname{Tr}\big(\psi'(h_r \otimes 1)(t)\big) \\ &< \operatorname{Tr}\big(\psi'(h_{r-1} \otimes 1)(t)\big) \\ &\le \#\Big\{k : \mu_k(t) \ge \frac{r-2}{q}\Big\} n + \sum_{i : x_i \ge \frac{r-2}{q}} r_i d_i. \end{aligned}$$

As $[\frac{r-2}{q}, \frac{r}{q}]$ at most contains one of the exceptional points of A, we see that

$$\#\Big\{k : \lambda_k(t) \ge \frac{r}{q}\Big\} n < \#\Big\{k : \mu_k(t) \ge \frac{r-2}{q}\Big\} n + n.$$

Thus
$$\#\left\{k : \lambda_k(t) \geq \frac{r}{q}\right\} \leq \#\left\{k : \mu_k(t) \geq \frac{r-2}{q}\right\}.$$
It follows that $\lambda_k(t) \leq \mu_k(t) + \frac{3}{q}$. Symmetry allows us to conclude that for all $t \in [0,1]$,
$$|\lambda_k(t) - \mu_k(t)| \leq \frac{3}{q}, \quad k = 1, 2, \dots, L.$$
By (28) we can define a $*$-homomorphism $\beta\colon A \to B$ by
$$\beta(f)(t) = \nu(t)\operatorname{diag}\Big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f\big(\lambda_1(t)\big), \dots, f\big(\lambda_L(t)\big)\Big)\nu(t)^*$$
Note that
$$\|\psi'(f) - \beta(f)\| < \frac{\epsilon}{3}, \quad f \in F.$$
Since $\operatorname{Tr}\big(\beta(f)(t)\big) = \operatorname{Tr}\big(\varphi(f)(t)\big)$, $f \in A$, $t \in [0,1]$, it follows that β and φ are approximately unitarily equivalent by [27, Corollary 1.5]. Hence there exists a unitary $U \in B$ such that
$$\|\varphi(f) - U\beta(f)U^*\| < \frac{\epsilon}{3}, \quad f \in F.$$
Choose a unitary $V \in B$ such that
$$\|\psi'(f) - V\psi(f)V^*\| < \frac{\epsilon}{3}, \quad f \in F.$$
Set $W = UV$. Then for $f \in F$,
$$\begin{aligned}\|\varphi(f) - W\psi(f)W^*\| &\leq \|\varphi(f) - U\beta(f)U^*\| + \|U\beta(f)U^* - U\psi'(f)U^*\| \\ &\quad + \|U\psi'(f)U^* - UV\psi(f)V^*U^*\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$
■

Define a continuous function $\kappa\colon \mathbb{T} \to [0,1]$ by
$$\kappa(e^{2\pi it}) = \begin{cases} 2t & t \in [0, \frac{1}{2}], \\ 2 - 2t & t \in [\frac{1}{2}, 1]. \end{cases}$$
Define continuous functions $\iota_1, \iota_2\colon [0,1] \to \mathbb{T}$ by $\iota_1(t) = e^{\pi it}$, $\iota_2(t) = e^{-\pi it}$. Note that $\kappa \circ \iota_1 = \kappa \circ \iota_2 = \mathrm{id}_{[0,1]}$.

Let $A = I(n, d_1, d_2, \dots, d_N)$ be an interval building block with exceptional points $t_1, t_2, \dots, t_N$. Define a circle building block by
$$A^{\mathbb{T}} = \big\{f \in C(\mathbb{T}) \otimes M_n : f\big(\iota_1(t_i)\big), f\big(\iota_2(t_i)\big) \in M_{d_i}, i = 1, 2, \dots, N\big\}.$$
Define unital $*$-homomorphisms $\xi_A\colon A \to A^{\mathbb{T}}$ by $\xi_A(f) = f \circ \kappa$, $f \in A$, and $j_A^1, j_A^2\colon A^{\mathbb{T}} \to A$ by $j_A^1(g) = g \circ \iota_1$, $j_A^2(g) = g \circ \iota_2$, $g \in A^{\mathbb{T}}$. Then $j_A^1 \circ \xi_A = j_A^2 \circ \xi_A = \mathrm{id}_A$.

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Let A be a finite direct sum of building blocks and interval building blocks. It follows from the above that there exists a finite direct sum of building blocks $A^{\mathbb{T}}$ together with unital $*$-homomorphisms $\xi_A\colon A \to A^{\mathbb{T}}$ and $j_A^1, j_A^2\colon A^{\mathbb{T}} \to A$ such that $j_A^1 \circ \xi_A = j_A^2 \circ \xi_A = \mathrm{id}_A$ and

$$j_A^1(f) = j_A^2(f) = 0 \Longrightarrow f = 0, \quad f \in A^{\mathbb{T}}. \tag{29}$$

Theorem 9.9 *Let A be a simple unital infinite dimensional inductive limit of a sequence of finite direct sums of circle building blocks. Then A is the inductive limit of a sequence of finite direct sums of circle building blocks with unital and injective connecting maps.*

Proof By Proposition 9.7 we see that A is the inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

where each A_n is a finite direct sum of circle and interval building blocks and each α_n is a unital and injective $*$-homomorphism.

By passing to a subsequence, if necessary, we may assume that either, every A_n is a circle or an interval building block or, every A_n is a finite direct sum of at least two circle or interval building blocks.

Let us first assume that the latter is the case.

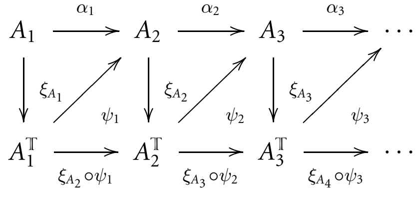
Let $A_n = A_1^n \oplus A_2^n \oplus \cdots \oplus A_{N_n}^n$ where each A_i^n is a circle or an interval building block. For each n let $\pi_i^n\colon A_n \to A_i^n$ denote the coordinate projections, $i = 1, 2, \ldots, N_n$. First we claim that we may assume that all the maps $\pi_i^{n+1} \circ \alpha_n$ are injective.

By Elliott's approximate intertwining argument it suffices to show that given a finite set $G \subseteq A_n$ and $\epsilon > 0$ there exists an integer $m > n$ and a unital $*$-homomorphism $\psi\colon A_n \to A_m$ such that $\|\alpha_{n,m}(g) - \psi(g)\| < \epsilon$, $g \in G$, and such that $\pi_i^m \circ \psi$ is injective, $i = 1, 2, \ldots, N_m$. Choose by Lemma 9.1 a finite set $H \subseteq A_n$ of positive non-zero elements with respect to G and ϵ. As A is simple and the connecting maps are injective, we have that $\widehat{\alpha_{n,\infty}}(\widehat{h}) > 0$, $h \in H$. Thus there exists an integer $m > n$ such that $\widehat{\alpha_{n,m}}(\widehat{h}) > 0$, $h \in H$. Hence $\pi_i^m \circ \alpha_{n,m}(h) \neq 0$, $i = 1, 2, \ldots, N_m$, and the claim follows by N_m applications of Lemma 9.1.

Define a unital $*$-homomorphism $\psi_n\colon A_n^{\mathbb{T}} \to A_{n+1}$ by

$$\psi_n(x) = \left(\pi_1^{n+1} \circ \alpha_n \circ j_{A_n}^1(x), \pi_2^{n+1} \circ \alpha_n \circ j_{A_n}^2(x), \ldots, \pi_{N_{n+1}}^{n+1} \circ \alpha_n \circ j_{A_n}^2(x)\right).$$

Since the maps $\pi_i^{n+1} \circ \alpha_n$ are injective, $i = 1, 2, \ldots, N_{n+1}$, and as $N_{n+1} \geq 2$, it follows from (29) that ψ_n is injective. The theorem therefore follows in this case from the commutativity of the diagram



It remains to prove the theorem in the first case. By passing to a subsequence we may assume that each A_n is an interval building block. Let $\epsilon > 0$, let k be a positive integer, and let $F \subseteq A_k$ be finite. Again by Elliott's approximative intertwining argument, it suffices to show that there exists an integer $l > k$ and a unital and injective $*$-homomorphism $\psi\colon A_k^{\mathbb{T}} \to A_l$ such that

$$\|\alpha_{k,l}(x) - \psi \circ \xi_{A_k}(x)\| < \epsilon, \quad x \in F.$$

Choose by Lemma 9.8 a finite set $H \subseteq A_k$ of positive elements of norm 1 with respect to F and ϵ. Since A is simple and the connecting maps are injective there exists a $\delta > 0$ such that $\widehat{\alpha_{k,\infty}}(\widehat{h}) > 2\delta$, $h \in H$. Let $A_k = I(n, d_1, d_2, \dots, d_N)$. By Lemma 9.6 there exists an integer $l > k$ such that $s(A_l) > \frac{2n}{\delta}$ and such that

$$\widehat{\alpha_{k,l}}(\widehat{h}) > 2\delta, \quad h \in H.$$

Let $A_l = I(m, e_1, e_2, \dots, e_M)$. By [27, Chapter 1] $\alpha_{k,l} \circ j^1_{A_k}\colon A_k^{\mathbb{T}} \to A_l$ is approximately unitarily equivalent to a $*$-homomorphism $\beta\colon A_k^{\mathbb{T}} \to A_l$ of the form

$$\beta(f)(t) = u(t)\,\mathrm{diag}\Big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f\big(\mu_1(t)\big), \dots, f\big(\mu_L(t)\big)\Big)\,u(t)^*, \qquad t \in [0,1],$$

where $u \in C[0,1] \otimes M_m$ is a unitary and $\mu_1, \mu_2, \dots, \mu_L\colon [0,1] \to \mathbb{T}$ are continuous functions. Choose a continuous function $\mu_1'\colon [0,1] \to \mathbb{T}$ such that $\mu_1' = \mu_1$ at the exceptional points of A_l and such that μ_1' is surjective. Define $\varphi\colon A_k^{\mathbb{T}} \to A_l$ by

$$\varphi(f)(t) = u(t)\,\mathrm{diag}\Big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f\big(\mu_1'(t)\big), f\big(\mu_2(t)\big), \dots, f\big(\mu_L(t)\big)\Big)\,u(t)^*.$$

Note that φ is injective, and that for $h \in H$,

$$\|\widehat{\varphi} \circ \widehat{\xi_{A_k}}(\widehat{h}) - \widehat{\alpha_{k,l}}(\widehat{h})\| = \big\|\widehat{\varphi}\big(\widehat{\xi_{A_k}}(\widehat{h})\big) - \widehat{\alpha_{k,l}} \circ \widehat{j^1_{A_k}}\big(\widehat{\xi_{A_k}}(\widehat{h})\big)\big\| \le \|\widehat{\varphi} - \widehat{\beta}\| \le \frac{2n}{m} < \delta.$$

Finally, as $\Lambda_j \circ \varphi = \Lambda_j \circ \beta$, $j = 1, 2, \dots, M$, we see by Lemma 9.8 that there exists a unitary $W \in A_l$ such that

$$\|W\varphi \circ \xi_{A_k}(f)W^* - \alpha_{k,l}(f)\| < \epsilon, \quad f \in F.$$

Set $\psi(x) = W\varphi(x)W^*$, $x \in A_k^{\mathbb{T}}$. ■

10 Construction of a Certain Map

In [22] Rørdam defined the bifunctor KL to be a certain quotient of KK. Some of our main results are more elegantly formulated in terms of KL than KK, and we will

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therefore from now on use KL instead of KK. Recall from [22] that the Kasparov product yields a product $KL(B,C) \times KL(A,B) \to KL(A,C)$. Furthermore, if $K_*(A)$ is finitely generated then $KL(A,\cdot) \cong KK(A,\cdot)$, and this functor is continuous by [23, Theorem 1.14] and [23, Theorem 7.13]. Finally, approximately unitarily equivalent $*$-homomorphisms define the same element of KL [22, Proposition 5.4]. It should be noted that KL is related to homomorphisms of K-theory with coefficients, [6].

Let A and B be unital C^*-algebras. Let $KL(A,B)_e$ be the set of elements $\kappa \in KK(A,B)$ for which $\kappa_*\colon K_0(A) \to K_0(B)$ preserves the order unit. Let $KL(A,B)_T$ be those elements $\kappa \in KL(A,B)_e$ for which there exists an affine continuous map $\varphi_T\colon T(B) \to T(A)$ such that

$$r_B(\omega)\big(\kappa_*(x)\big) = r_A\big(\varphi_T(\omega)\big)(x), \quad x \in K_0(A), \omega \in T(B).$$

Lemma 10.1 *Let C be a finite direct sum of building blocks, let $\epsilon > 0$, and let $F \subseteq \operatorname{Aff} T(C)$ be a finite set. Let B be the inductive limit of a sequence of finite direct sums of building blocks*

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

with unital connecting maps. Let $J\colon \operatorname{Aff} T(C) \to \operatorname{Aff} T(B)$ be a linear positive order unit preserving map and let $\kappa \in KL(C,B)_e$. There exists a positive integer n, a linear positive order unit preserving map $M\colon \operatorname{Aff} T(C) \to \operatorname{Aff} T(B_n)$, and an element $\omega \in KK(C,B_n)_e$ such that

$$\|J(f) - \widehat{\beta_{n,\infty}} \circ M(f)\| < \epsilon, \quad f \in F,$$
$$\kappa = [\beta_{n,\infty}] \cdot \omega \quad \text{in } KL(C,B).$$

Proof We may assume that $\|f\| \le 1$, $f \in F$. Decompose $C = C_1 \oplus C_2 \oplus \cdots \oplus C_N$ as a finite direct sum of building blocks and let $\pi_i\colon C \to C_i$ denote the projection, $i = 1,2,\dots,N$.

For every $i = 1,2,\dots,N$, identify $\operatorname{Aff} T(C_i)$ and $C_{\mathbb{R}}(\mathbb{T})$. Choose open sets V_1, $V_2,\dots,V_{k_i} \subseteq \mathbb{T}$ such that $\bigcup_{j=1}^{k_i} V_j = \mathbb{T}$ and such that

$$x,y \in V_j \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{2}, \quad f \in \widehat{\pi}_i(F).$$

Let $\{h_j : j = 1,2,\dots,k_i\}$ be a continuous partition of unity in $C_{\mathbb{R}}(\mathbb{T})$ subordinate to the cover $\{V_j : j = 1,2,\dots,k_i\}$ and let $x_j \in V_j$ be an arbitrary point, $j = 1,2,\dots,k_i$. Define linear positive order unit preserving maps $T_i\colon \operatorname{Aff} T(C_i) \to \mathbb{R}^{k_i}$ and $S_i\colon \mathbb{R}^{k_i} \to \operatorname{Aff} T(C_i)$ by

$$T_i(f) = \big(f(x_1), f(x_2), \dots, f(x_{k_i})\big),$$
$$S_i(t_1,t_2,\dots,t_{k_i}) = \sum_{j=1}^{k_i} t_j h_j.$$

Note that

$$\|S_i \circ T_i(f) - f\| < \frac{\epsilon}{2}, \quad f \in \widehat{\pi}_i(F).$$

Hence there exist linear positive order unit preserving maps

$$T\colon \operatorname{Aff} T(C) \longrightarrow \mathbb{R}^k,$$
$$S\colon \mathbb{R}^k \longrightarrow \operatorname{Aff} T(C),$$

where $k = \sum_{i=1}^{N} k_i$, such that

$$\|S \circ T(f) - f\| < \frac{\epsilon}{2}, \quad f \in F.$$

Let $\{e_j : j = 1, 2, \dots, k\}$ be the standard basis in $\mathbb{R}^k$. As $\{J \circ S(e_j) : j = 1, 2, \dots, k\}$ are positive elements with sum 1 in $\operatorname{Aff} T(B)$, there exist a positive integer l and positive elements $x_1, x_2, \dots, x_k \in \operatorname{Aff} T(B_l)$ such that $\sum_{j=1}^{k} x_j = 1$ and

$$\|\widehat{\beta_{l,\infty}}(x_j) - J \circ S(e_j)\| < \frac{\epsilon}{2k}, \quad j = 1, 2, \dots, k.$$

Define linear positive order unit preserving maps $V\colon \mathbb{R}^k \to \operatorname{Aff} T(B_l)$ by

$$V\Big(\sum_{j=1}^{k} t_j e_j\Big) = \sum_{j=1}^{k} t_j x_j,$$

and $W\colon \operatorname{Aff} T(C) \to \operatorname{Aff} T(B_l)$ by $W = V \circ T$. Since

$$\|\widehat{\beta_{l,\infty}} \circ V - J \circ S\| < \frac{\epsilon}{2}$$

we see that

$$\|\widehat{\beta_{l,\infty}} \circ W(f) - J(f)\| < \epsilon, \quad f \in F.$$

By continuity of $KL(C, \cdot)$ there exist an integer m and an element $\nu \in KL(C, B_m)$ such that $[\beta_{m,\infty}] \cdot \nu = \kappa$. As

$$\beta_{m,\infty *} \circ \nu_*[1] = \kappa_*[1] = [1] = \beta_{m,\infty *}[1] \quad \text{in } K_0(B)$$

we see that there exists an integer $n \geq m, l$ such that $[\beta_{m,n}] \cdot \nu \in KL(C, B_n)_e$. Set $\omega = [\beta_{m,n}] \cdot \nu$ and $M = \widehat{\beta_{l,n}} \circ W$. ■

Proposition 10.2 *Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks. Let B be the inductive limit of a sequence*

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

of finite direct sums of building blocks with unital connecting maps. Assume that there exist a $\kappa \in KL(A,B)_e$ and an affine continuous map $\varphi_T\colon T(B) \to T(A)$ such that

$$r_B(\omega)\big(\kappa_*(x)\big) = r_A\big(\varphi_T(\omega)\big)(x), \quad x \in K_0(A), \omega \in T(B).$$

Let C be a finite direct sum of building blocks and let $\psi\colon C \to A$ be a unital $*$-homomorphism. Let $\epsilon > 0$ and let $F \subseteq \operatorname{Aff} T(C)$ be a finite subset. There exist a positive integer m and a linear positive order unit preserving map $M\colon \operatorname{Aff} T(C) \to \operatorname{Aff} T(B_m)$ such that

$$\|\widehat{\beta_{m,\infty}} \circ M(f) - \varphi_{T_*} \circ \widehat{\psi}(f)\| < \epsilon, \quad f \in F,$$

and an element $\omega \in KL(C, B_m)_e$ such that

$$[\beta_{m,\infty}] \cdot \omega = \kappa \cdot [\psi] \quad \text{in } KL(C,B),$$

$$M \circ \rho_C = \rho_{B_m} \circ \omega_* \quad \text{on } K_0(C).$$

Proof We may assume that $\|f\| \le 1$, $f \in F$. Decompose $C = C_1 \oplus C_2 \oplus \cdots \oplus C_N$ as a finite direct sum of building blocks. Let $r_1, r_2, \ldots, r_N \in C$ be projections such that $[r_1], [r_2], \ldots, [r_N]$ generate $K_0(C)$. By factoring ψ through the C^*-algebra obtained from C by erasing those direct summands C_i for which $\psi(r_i) = 0$, we may assume that $\psi(r_i) \ne 0$, $i = 1, 2, \ldots, N$. There exist positive integers $d_1, d_2, \ldots, d_N$ such that

$$\sum_{i=1}^{N} d_i[r_i] = [1] \quad \text{in } K_0(C).$$

Since A is simple there exists a $\delta_0 > 0$ such that

$$\widehat{\psi}(\widehat{r_i}) > \delta_0, \quad i = 1, 2, \ldots, N.$$

Choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta(1 + \sum_{i=1}^{N} d_i) < \epsilon$.

By Lemma 10.1 there exist a positive integer l and a linear positive order unit preserving map $V\colon \operatorname{Aff} T(C) \to \operatorname{Aff} T(B_l)$ such that

$$\|\widehat{\beta_{l,\infty}} \circ V(f) - \varphi_{T*} \circ \widehat{\psi}(f)\| < \delta, \quad f \in F \cup \{\widehat{r_1}, \widehat{r_2}, \ldots, \widehat{r_N}\},$$

and an element $\nu \in KK(C, B_l)_e$ such that

$$[\beta_{l,\infty}] \cdot \nu = \kappa \cdot [\psi] \quad \text{in } KL(C,B).$$

Since by assumption $\rho_B \circ \kappa_* = \varphi_{T*} \circ \rho_A$ on $K_0(A)$ we see that for $i = 1, 2, \ldots, N$,

$$\widehat{\beta_{l,\infty}} \circ \rho_{B_l} \circ \nu_*[r_i] = \rho_B \circ {\beta_{l,\infty}}_* \circ \nu_*[r_i] = \varphi_{T*} \circ \rho_A \circ \psi_*[r_i] = \varphi_{T*} \circ \widehat{\psi}(\widehat{r_i}) > \delta_0.$$

Hence

$$\|\widehat{\beta_{l,\infty}} \circ \rho_{B_l} \circ \nu_*[r_i] - \widehat{\beta_{l,\infty}} \circ V(\widehat{r_i})\| < \delta, \quad i = 1, 2, \ldots, N.$$

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Choose $m > l$ such that for $i = 1, 2, \ldots, N$,

$$\widehat{\beta_{l,m}} \circ \rho_{B_l} \circ \nu_*[r_i] > \delta_0,$$
$$\|\widehat{\beta_{l,m}} \circ \rho_{B_l} \circ \nu_*[r_i] - \widehat{\beta_{l,m}} \circ V(\widehat{r_i})\| < \delta.$$

Define $W\colon \operatorname{Aff} T(C) \to \operatorname{Aff} T(B_m)$ by $W = \widehat{\beta_{l,m}} \circ V$. Define $\omega \in KK(C, B_m)_e$ by $\omega = [\beta_{l,m}] \cdot \nu$.

Decompose $B_m = B_1^m \oplus B_2^m \oplus \cdots \oplus B_L^m$ as a finite direct sum of building blocks and let $\pi_j\colon B_m \to B_j^m$ be the projection, $j = 1, 2, \ldots, L$. Identify $\operatorname{Aff} T(B_m)$ with $\bigoplus_{j=1}^L C_{\mathbb{R}}(\mathbb{T})$. Fix some $j = 1, 2, \ldots, L$. Set $W_j = \widehat{\pi}_j \circ W$. $W_j(\widehat{r_i})$ is a strictly positive function in $C_{\mathbb{R}}(\mathbb{T})$, since $\delta < \delta_0$. Thus for each $i = 1, 2, \ldots, N$, we can define $M_j\colon \operatorname{Aff} T(A_n) \cong \bigoplus_{i=1}^N \operatorname{Aff} T(C_i) \to C_{\mathbb{R}}(\mathbb{T})$ by

$$M_j(f_1, f_2, \ldots, f_N) = \sum_{i=1}^N W_j(0, \ldots, 0, f_i, 0, \ldots, 0) \frac{1}{W_j(\widehat{r_i})} \widehat{\pi}_j(\rho_{B_m} \circ \omega_*[r_i]).$$

M_j is positive and linear, and it preserves the order unit since

$$M_j(1) = \sum_{i=1}^N W_j(d_i\widehat{r_i}) \frac{1}{W_j(\widehat{r_i})} \widehat{\pi}_j(\rho_{B_m} \circ \omega_*[r_i]) = \sum_{i=1}^N \widehat{\pi}_j\big(\rho_{B_m} \circ \omega_*(d_i[r_i])\big) = 1.$$

Let now $g \in C_{\mathbb{R}}(\mathbb{T}) \cong \operatorname{Aff} T(C_i)$, $\|g\| \le 1$, for $i = 1, 2, \ldots, N$. Since

$$-d_i\widehat{r_i} \le (0, \ldots, 0, g, 0, \ldots, 0) \le d_i\widehat{r_i}$$

in $\operatorname{Aff} T(C)$ we have that

$$\begin{aligned}
&\|M_j(0, \ldots, g, \ldots, 0) - W_j(0, \ldots, g, \ldots, 0)\| \\
&\quad = \left\| W_j(0, \ldots, g, \ldots, 0) \frac{1}{W_j(\widehat{r_i})} \big(\widehat{\pi}_j(\rho_{B_m} \circ \omega_*[r_i]) - W_j(\widehat{r_i})\big) \right\| \\
&\quad \le d_i \|\widehat{\pi}_j(\rho_{B_m} \circ \omega_*[r_i]) - W_j(\widehat{r_i})\| < \delta d_i.
\end{aligned}$$

Hence if $f \in \operatorname{Aff} T(C)$, $\|f\| \le 1$, then

$$\|M_j(f) - W_j(f)\| < \sum_{i=1}^N \delta d_i.$$

Define $M\colon \operatorname{Aff} T(C) \to \operatorname{Aff} T(B_m)$ by

$$M(f) = \big(M_1(f), M_2(f), \ldots, M_L(f)\big).$$

Then

$$\|M(f) - W(f)\| < \sum_{i=1}^N \delta d_i, \quad f \in \operatorname{Aff} T(C), \|f\| \le 1,$$

and hence

$$\|\widehat{\beta_{m,\infty}} \circ M(f) - \varphi_{T*} \circ \widehat{\psi}(f)\| < \delta + \sum_{i=1}^{N} \delta d_i < \epsilon, \quad f \in F.$$

Finally, $M(\widehat{r}_i) = \rho_{B_m} \circ \omega_*[r_i]$, $i = 1, 2, \ldots, N$. It follows that $M \circ \rho_C = \rho_{B_m} \circ \omega_*$ on $K_0(C)$. ■

Lemma 10.3 *Let A be a unital simple inductive limit of a sequence of finite direct sums of building blocks with $K_0(A)$ non-cyclic. Then $\operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)}$ is torsion free.*

Proof The image of the canonical map $K_0(A) \to \operatorname{Aff} SK_0(A)$ is dense by [1, Proposition 3.1], since $K_0(A)$ is a simple countable dimension group. By definition ρ_A is the composition of this map with the linear bounded map $\operatorname{Aff} SK_0(A) \to \operatorname{Aff} T(A)$ induced by r_A. It follows that $\rho_A\big(K_0(A)\big)$ is dense in some subspace of $\operatorname{Aff} T(A)$. ■

Lemma 10.4 *Let A be an inductive limit of a sequence of finite direct sums of building blocks*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

with unital connecting maps. Assume that ρ_A is injective and that $\rho_A\big(K_0(A)\big)$ is a discrete subgroup of $\operatorname{Aff} T(A)$. Let n be a positive integer and let x, y be elements of the torsion subgroup of $U(A_n)/\overline{DU(A_n)}$ such that $\alpha^{\#}_{n,\infty}(x) = \alpha^{\#}_{n,\infty}(y)$. There exists an integer $k \geq n$ such that $\alpha^{\#}_{n,k}(x) = \alpha^{\#}_{n,k}(y)$.

Proof Since $\alpha_{n,\infty_*}\big(\pi_{A_n}(x)\big) = \alpha_{n,\infty_*}\big(\pi_{A_n}(y)\big)$ in $K_1(A)$ there is an integer $l \geq n$ such that $\alpha_{n,l_*}\big(\pi_{A_n}(x)\big) = \alpha_{n,l_*}\big(\pi_{A_n}(y)\big)$. By Proposition 5.2 we see that

$$\alpha^{\#}_{n,l}(x - y) = \lambda_{A_l}\bigg(q_{A_l}\Big(\frac{1}{m}\rho_{A_l}(z)\Big)\bigg)$$

for some positive integer m and an element $z \in K_0(A_l)$. Since $\rho_A\big(K_0(A)\big)$ is discrete and since $\lambda_A\bigg(q_A\Big(\frac{1}{m}\rho_A\big(\alpha_{l,\infty_*}(z)\big)\Big)\bigg) = 0$ we see that $\frac{1}{m}\rho_A\big(\alpha_{l,\infty_*}(z)\big) = \rho_A\big(\alpha_{j,\infty_*}(w)\big)$ for some positive integer j and an element $w \in K_0(A_j)$. Since ρ_A is injective we may choose an integer $k \geq l, j$ such that $\alpha_{l,k_*}(z) = \alpha_{j,k_*}(mw)$ in $K_0(A)$. Note that $\alpha^{\#}_{n,k}(x - y) = \lambda_{A_k}\bigg(q_{A_k}\Big(\frac{1}{m}\rho_{A_k}\big(\alpha_{k,l_*}(z)\big)\Big)\bigg) = 0$. ■

Proposition 10.5 *Let A be a unital C^*-algebra and let B be a unital inductive limit of a sequence of finite direct sums of building blocks such that the torsion subgroup of $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ is totally disconnected. Let $\varphi, \psi \colon A \to B$ be unital $*$-homomorphisms that are homotopic and let $x \in U(A)/\overline{DU(A)}$ be an element of finite order. Then $\varphi^{\#}(x) = \psi^{\#}(x)$.*

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Proof Let $u \in A$ be a unitary such that $x = q'_A(u)$. Let $(\varphi_t)_{t\in[0,1]}$ be a homotopy connecting φ to ψ. We may assume that $\|\varphi_t(u) - \varphi_0(u)\| < 1$ for $t \in [0,1]$. Thus

$$\varphi_t(u)\varphi_0(u)^* = e^{2\pi i b_t}$$

where $t \mapsto b_t$ is a continuous path of self-adjoint elements in B. Since $\lambda_B\big(q_B(\widehat{b_t})\big) = q'_B(e^{2\pi i b_t})$ we see that $q_B(\widehat{b_t})$ has finite order in $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$. Thus $t \mapsto q_B(\widehat{b_t})$ is a continuous path in a totally disconnected subset of a metric space. It follows that it is constant and hence $q_B(\widehat{b_t}) = 0$ for every $t \in [0,1]$. We conclude that $\varphi_0^\#\big(q'_A(u)\big) = \varphi_1^\#\big(q'_A(u)\big)$. ■

We leave it as an open question whether the torsion subgroup of the group $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ always is totally disconnected.

Proposition 10.6 *Let A be a finite direct sum of building blocks and let B be a unital inductive limit of a sequence of finite direct sums of building blocks such that the torsion subgroup of $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ is totally disconnected. Let $\varphi, \psi\colon A \to B$ be unital $*$-homomorphisms such that $[\varphi] = [\psi]$ in $KL(A,B)$. Let x be an element of the torsion subgroup of $U(A)/\overline{DU(A)}$. Then $\varphi^\#(x) = \psi^\#(x)$.*

Proof By [18, Corollary 15.1.3] and Theorem 2.4 there exist a positive integer m and $*$-homomorphisms $\lambda, \mu\colon A \to B_m$ such that φ is homotopic to $\beta_{m,\infty} \circ \lambda$ and ψ is homotopic to $\beta_{m,\infty} \circ \mu$. By increasing m we may assume that λ and μ are unital. There exists an integer $k \geq m$ such that $[\beta_{m,k}] \cdot [\lambda] = [\beta_{m,k}] \cdot [\mu]$ in $KL(A, B_k)$. Thus $\beta^\#_{m,k} \circ \lambda^\#(x) = \beta^\#_{m,k} \circ \mu^\#(x)$ by Proposition 5.6. Hence $\varphi^\#(x) = \beta^\#_{m,\infty} \circ \lambda^\#(x) = \beta^\#_{m,\infty} \circ \mu^\#(x) = \psi^\#(x)$ by Proposition 10.5. ■

Lemma 10.7 *Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks such that $K_0(A)$ is non-cyclic, and let B be a unital inductive limit of a sequence of finite direct sums of building blocks. If there exists an element $\kappa \in KL(A,B)_T$ then $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ is torsion free.*

Proof By Lemma 9.2 we may assume that A is the inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks with unital connecting maps. Similarly B is the inductive limit of a sequence of finite direct sums of building blocks

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

with unital connecting maps. Let $\epsilon > 0$. There exists a positive integer n such that for every $t \in \mathbb{R}$ we have that $d'_{A_n}\big(q_{A_n}(\widehat{t1}), 0\big) < \epsilon$. To see this choose a positive integer k such that $\frac{1}{k} < \epsilon$. Since $\operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)}$ is torsion free by Lemma 10.3, we may

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choose n such that $d'_{A_n}\big(q_{A_n}(\frac{j}{k}\widehat{1}),0\big) < \frac{\epsilon}{2}$, $j = 1,2,\ldots,k-1$. Let $t \in \mathbb{R}$. We may assume that $0 < t < 1$. Choose $j = 0,1,2,\ldots,k$ such that $|t - \frac{j}{k}| \le \frac{1}{2k} < \frac{\epsilon}{2}$. Then $d'_{A_n}\big(q_{A_n}(t\widehat{1}),0\big) < \epsilon$.

By Proposition 10.2 we get a positive integer l and a contractive group homomorphism $S\colon \operatorname{Aff} T(A_n)/\overline{\rho_{A_n}\big(K_0(A_n)\big)} \to \operatorname{Aff} T(B_l)/\overline{\rho_{B_l}\big(K_0(B_l)\big)}$ such that $S\big(q_{A_n}(r\widehat{1})\big) = q_{B_l}(r\widehat{1})$ for every $r \in \mathbb{R}$. Let $x \in \operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ be an element of order m. There is an integer $k \ge l$ such that $d'_B\left(x, q_B\left(\frac{1}{m}\rho_B\big(\beta_{k,\infty*}(y)\big)\right)\right) < \epsilon$ for some element $y \in K_0(B_k)$. We claim that $d'_{B_k}\left(q_{B_k}\big(\frac{1}{m}\rho_{B_k}(y)\big),0\right) < \epsilon$. To this end we may assume that B_k is a building block. Then $\rho_{B_k}(y) = w\widehat{1}$ for some $w \in \mathbb{Q}$. Hence

$$\begin{aligned} d'_{B_k}\left(q_{B_k}\left(\frac{1}{m}\rho_{B_k}(y)\right),0\right) &= d'_{B_k}\left(\widetilde{\beta_{l,k}} \circ S\left(q_{A_n}\left(\frac{w}{m}\widehat{1}\right)\right),0\right) \\ &\le d'_{A_n}\left(q_{A_n}\left(\frac{w}{m}\widehat{1}\right),0\right) < \epsilon. \end{aligned}$$

Thus $d'_B(x,0) < 2\epsilon$. Since $\epsilon > 0$ was arbitrary we conclude that $x = 0$. ■

Lemma 10.8 *Let A be a simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks with unital connecting maps. Let y be an element in $U(A)/\overline{DU(A)}$ of order $k < \infty$. Then there exist a positive integer m and an element $w \in U(A_m)/\overline{DU(A_m)}$ of order k such that $\alpha^{\#}_{m,\infty}(w) = y$.

Proof By continuity of K_1 there exist a positive integer l and an element z in $U(A_l)/\overline{DU(A_l)}$ such that $\alpha_{l,\infty*}\big(\pi_{A_l}(z)\big) = \pi_A(y)$ in $K_1(A)$. Since the short exact sequence of Proposition 5.2 splits we may assume that $kz = 0$. Note that $\pi_A\big(\alpha^{\#}_{l,\infty}(z)\big) = \pi_A(y)$ and hence

$$y = \alpha^{\#}_{l,\infty}(z) + \lambda_A\big(q_A(f)\big) \quad \text{in } U(A)/\overline{DU(A)}$$

for some $f \in \operatorname{Aff} T(A)$ with $kq_A(f) = 0$ in the group $\operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)}$. If $K_0(A)$ is non-cyclic then we see that $q_A(f) = 0$ by Lemma 10.3. Thus we may assume that $K_0(A) \cong \mathbb{Z}$ such that $\rho_A\big(K_0(A)\big)$ is a discrete subgroup of $\operatorname{Aff} T(A)$. It follows that $f = \frac{1}{k}\rho_A(x)$ for some $x \in K_0(A)$. By continuity of K_0 we have that $x = \alpha_{m,\infty*}(h)$ for some integer $m \ge l$ and some $h \in K_0(A_m)$. Define $w \in U(A_m)/\overline{DU(A_m)}$ by

$$w = \alpha^{\#}_{l,m}(z) + \lambda_{A_m}\left(q_{A_m}\left(\frac{1}{k}\rho_{A_m}(h)\right)\right).$$

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Then

$$\alpha^{\#}_{m,\infty}(w) = \alpha^{\#}_{l,\infty}(z) + \lambda_A\Big(q_A\Big(\frac{1}{k}\rho_A\big(\alpha_{m,\infty_*}(h)\big)\Big)\Big) = \alpha^{\#}_{l,\infty}(z) + \lambda_A\big(q_A(f)\big) = y.$$

Since y has order k and $kw = 0$ it follows that w has order k as well. ■

Theorem 10.9 *Let A be a unital simple inductive limit of a sequence of finite direct sums of building blocks and let B be an inductive limit of a similar sequence*

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

with unital connecting maps such that $s(B_k) \to \infty$ and such that the torsion subgroup of $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ is totally disconnected. Let $\kappa \in KL(A,B)_T$. Let C be a finite direct sum of building blocks and let $\varphi\colon C \to A$ be a unital $*$-homomorphism. Then there is a unital $*$-homomorphism $\psi\colon C \to B$ such that $[\psi] = \kappa \cdot [\varphi]$ in $KL(C,B)$.

Moreover, if C_1 is another finite direct sum of building blocks, if $\varphi_1\colon C_1 \to A$ and $\psi_1\colon C_1 \to B$ are unital $*$-homomorphisms such that $[\psi_1] = \kappa \cdot [\varphi_1]$ in $KL(A,B)$, and if $x \in U(C)/\overline{DU(C)}$ and $x_1 \in U(C_1)/\overline{DU(C_1)}$ are elements of finite order such that $\varphi^{\#}(x) = \varphi_1^{\#}(x_1)$, then $\psi^{\#}(x) = \psi_1^{\#}(x_1)$.

Proof Let a finite direct sum of building blocks C and a unital $*$-homomorphism $\varphi\colon C \to A$ be given. Let $\varphi_T\colon T(B) \to T(A)$ be a continuous affine map such that

$$r_B(\omega)\big(\kappa_*(x)\big) = r_A\big(\varphi_T(\omega)\big)(x), \quad x \in K_0(A), \omega \in T(B).$$

Let $p_1, p_2, \dots, p_N$ be the minimal non-zero central projections in C. As in the proof of Proposition 10.2 we see that we may assume that $\varphi(p_i) \neq 0$, $i = 1, 2, \dots, N$. Choose $\delta > 0$ such that $\widehat{\varphi}(\widehat{p_i}) > 2\delta$. Choose an integer K by Theorem 8.5 with respect to $F = \varnothing$ and $\epsilon = 1$. By Proposition 10.2 there exist a positive integer m and a linear positive order unit preserving map $M\colon \operatorname{Aff} T(C) \to \operatorname{Aff} T(B_m)$ such that

$$\|\widehat{\beta_{m,\infty}} \circ M(\widehat{p_i}) - \varphi_{T_*} \circ \widehat{\varphi}(\widehat{p_i})\| < \delta, \quad i = 1, 2, \dots, N,$$

and an element $\omega \in KL(C, B_m)_e$ such that

$$[\beta_{m,\infty}] \cdot \omega = \kappa \cdot [\varphi] \quad \text{in } KL(C,B),$$
$$M \circ \rho_C = \rho_{B_m} \circ \omega_* \quad \text{on } K_0(C).$$

Hence $\widehat{\beta_{m,\infty}} \circ M(\widehat{p_i}) > \delta$, $i = 1, 2, \dots, N$. Choose $k \geq m$ such that $s(B_k) \geq K\delta^{-1}$ and such that $\widehat{\beta_{m,k}} \circ M(\widehat{p_i}) > \delta$, $i = 1, 2, \dots, N$. Then $\rho_{B_k}(\beta_{m,k_*} \circ \omega_*[p_i]) > \delta$ and hence $s(B_k)\rho_{B_k}(\beta_{m,k_*} \circ \omega_*[p_i]) \geq K$. Furthermore $\widehat{\beta_{m,k}} \circ M \circ \rho_C = \rho_{B_k} \circ \beta_{m,k_*} \circ \omega_*$. It follows from Theorem 8.5 that there exists a unital $*$-homomorphism $\mu\colon C \to B_k$ such that $[\mu] = [\beta_{m,k}] \cdot \omega$. Set $\psi = \beta_{k,\infty} \circ \mu$. This proves the first part of the theorem.

To prove the second part of the theorem, let us first note that

$$\pi_B\big(\psi^\#(x)\big) = \psi_*\big(\pi_C(x)\big) = \kappa_* \circ \varphi_*\big(\pi_C(x)\big) = \psi_{1*}\big(\pi_{C_1}(x_1)\big) = \pi_B\big(\psi_1^\#(x_1)\big).$$

Hence if $K_0(A)$ is non-cyclic then $\psi^\#(x) = \psi_1^\#(x_1)$ by Lemma 10.7.

We may therefore assume that $K_0(A)$ is cyclic. By Lemma 9.2 we see that A is the inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

where each A_n is a finite direct sum of building blocks and each α_n is unital. By [18, Corollary 15.1.3] there exist a positive integer n and $*$-homomorphisms $\lambda\colon C \to A_n$ and $\lambda_1\colon C_1 \to A_n$ such that φ is homotopic to $\alpha_{n,\infty} \circ \lambda$ and φ_1 is homotopic to $\alpha_{n,\infty} \circ \lambda_1$. Note that λ and λ_1 are unital. Since

$$\alpha_{n,\infty}^\# \circ \lambda^\#(x) = \alpha_{n,\infty}^\# \circ \lambda_1^\#(x_1)$$

by Proposition 10.5, there exists by Lemma 10.4 a positive integer k such that

$$\alpha_{n,k}^\# \circ \lambda^\#(x) = \alpha_{n,k}^\# \circ \lambda_1^\#(x_1).$$

By the first part of the theorem there is a unital $*$-homomorphism $\gamma\colon A_k \to B$ such that $[\gamma] = \kappa \cdot [\alpha_{k,\infty}]$. Note that

$$[\gamma] \cdot [\alpha_{n,k}] \cdot [\lambda] = \kappa \cdot [\alpha_{n,\infty}] \cdot [\lambda] = \kappa \cdot [\varphi] = [\psi] \quad \text{in } KL(C, B)$$

$$[\gamma] \cdot [\alpha_{n,k}] \cdot [\lambda_1] = \kappa \cdot [\alpha_{n,\infty}] \cdot [\lambda_1] = \kappa \cdot [\varphi_1] = [\psi_1] \quad \text{in } KL(C_1, B).$$

Hence

$$\psi^\#(x) = \gamma^\# \circ \alpha_{n,k}^\# \circ \lambda^\#(x) = \gamma^\# \circ \alpha_{n,k}^\# \circ \lambda_1^\#(x_1) = \psi_1^\#(x_1)$$

by Proposition 10.6. ■

Let A, B and κ be as above. Let y be an element in $U(A)/\overline{DU(A)}$ of finite order. By Lemma 10.8 there is a finite direct sum of building blocks C, an element of finite order x in $U(C)/\overline{DU(C)}$, and a unital $*$-homomorphism $\varphi\colon C \to A$ such that $\varphi^\#(x) = y$. By the first part of the theorem above there exists a unital $*$-homomorphism $\psi\colon C \to B$ such that $[\psi] = \kappa \cdot [\varphi]$. Set $s_\kappa(y) = \psi^\#(x)$. By the second part $s_\kappa(y)$ is independent of the choice of φ, ψ and x. Thus we have a well-defined map

$$s_\kappa\colon \operatorname{Tor}\big(U(A)/\overline{DU(A)}\big) \longrightarrow \operatorname{Tor}\big(U(B)/\overline{DU(B)}\big).$$

It follows easily from Lemma 10.8 that s_κ is a group homomorphism. Note that if $\mu\colon A \to B$ is a unital $*$-homomorphism then $s_{[\mu]}(y) = \mu^\#(y)$ for every y in the torsion subgroup of $U(A)/\overline{DU(A)}$. Finally, we note that s_κ exists for trivial reasons if $K_0(A)$ is non-cyclic (since $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ is torsion free in this case, see Lemma 10.7). It is possible (as in [27]) to prove our classification theorem in the case of non-cyclic K_0-group without using the map s_κ, but we have chosen to construct it in general in order to obtain a unified proof of the classification theorem in the cases K_0 cyclic and K_0 non-cyclic.

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Lemma 10.10 *Let A be a unital simple infinite dimensional inductive limit of a sequence of finite direct sums of building blocks and let B be an inductive limit of a similar sequence*

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

with unital connecting maps such that $s(B_k) \to \infty$ and such that the torsion subgroup of $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ is totally disconnected. Let $\kappa \in KL(A,B)_e$ and let $\varphi_T\colon T(B) \to T(A)$ be a continuous affine map such that

$$r_B(\omega)\big(\kappa_*(x)\big) = r_A\big(\varphi_T(\omega)\big)(x), \quad x \in K_0(A), \omega \in T(B).$$

There exists a group homomorphism $\Phi\colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ such that $\Phi(y) = s_\kappa(y)$ for y in the torsion subgroup of $U(A)/\overline{DU(A)}$ and such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \widetilde{\varphi_T}} & & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \kappa_*} & & \\
0 & \longrightarrow & \operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B) & \longrightarrow & 0
\end{array}$$

commutes.

Proof It will be convenient to set $G_1 = \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)}$, $G_2 = K_1(A)$, and $H_1 = \operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$, $H_2 = K_1(B)$. Note that $U(A)/\overline{DU(A)} \cong G_1 \oplus G_2$ and $U(B)/\overline{DU(B)} \cong H_1 \oplus H_2$ by Proposition 5.2. Hence s_κ can be identified with a matrix of the form

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{ij}\colon \operatorname{Tor}(G_j) \to \operatorname{Tor}(H_i)$ is a group homomorphism, $i, j = 1, 2$.

Let $z \in \operatorname{Tor}(G_1)$. If $K_0(A)$ is cyclic then $z = q_A\big(\frac{1}{m}\rho_A(h)\big)$ for some positive integer m and $h \in K_0(A)$. Choose a finite direct sum of building blocks C and a unital $*$-homomorphism $\varphi\colon C \to A$ such that $\varphi_*(g) = h$ for some $g \in K_0(C)$. Choose a unital $*$-homomorphism $\psi\colon C \to B$ such that $[\psi] = \kappa \cdot [\varphi]$. Since $\varphi_{T*} \circ \rho_A = \rho_B \circ \kappa_*$ we see that

$$\begin{aligned}
s_\kappa\big(\lambda_A(z)\big) &= s_\kappa\left(\lambda_A\left(q_A\Big(\frac{1}{m}\rho_A\big(\varphi_*(g)\big)\Big)\right)\right) \\
&= s_\kappa\left(\varphi^\#\left(\lambda_C\left(q_C\Big(\frac{1}{m}\rho_C(g)\Big)\right)\right)\right) = \psi^\#\left(\lambda_C\left(q_C\Big(\frac{1}{m}\rho_C(g)\Big)\right)\right) \\
&= \lambda_B\left(q_B\Big(\frac{1}{m}\rho_B\big(\psi_*(g)\big)\Big)\right) = \lambda_B\left(q_B\left(\frac{1}{m}\varphi_{T*}\Big(\rho_A\big(\varphi_*(g)\big)\Big)\right)\right) \\
&= \lambda_B\big(\widetilde{\varphi_T}(z)\big).
\end{aligned}$$

Hence $f_{11}(z) = \widetilde{\varphi_T}(z)$ and $f_{21}(z) = 0$. By Lemma 10.7 this conclusion also holds if $K_0(A)$ is non-cyclic. Let $w \in \mathrm{Tor}(G_2)$. Choose an element $y \in U(A)/\overline{DU(A)}$ of finite order such that $\pi_A(y) = w$. Choose a finite direct sum of building blocks C and a unital $*$-homomorphism $\varphi\colon C \to A$ such that $\varphi^\#(x) = y$. Choose a unital $*$-homomorphism $\psi\colon C \to B$ such that $[\psi] = \kappa \cdot [\varphi]$ in $KL(C, B)$. Since

$$\pi_B\big(s_\kappa(y)\big) = \pi_B\big(\psi^\#(x)\big) = \psi_*\big(\pi_C(x)\big) = \kappa_* \circ \pi_A\big(\varphi^\#(x)\big) = \kappa_* \circ \pi_A(y)$$

we see that $f_{22}(w) = \kappa_*(w)$. Finally, since H_1 is a divisible group there exists by [11, Theorem 21.1] a group homomorphism $\lambda\colon G_2 \to H_1$ such that $\lambda(w) = f_{12}(w)$ for every $w \in G_2$ of finite order. Set

$$\Phi = \begin{pmatrix} \widetilde{\varphi_T} & \lambda \\ 0 & \kappa_* \end{pmatrix}.$$

It is easy to see that the diagram commutes. ∎

11 Main Results

Consider the category of abelian groups, equipped with a complete and translation invariant metric, and contractive group homomorphisms. Inductive limits can be constructed in this category in a way similar to the way that they are constructed in the category of C^*-algebras. Indeed, let

$$G_1 \xrightarrow{\mu_1} G_2 \xrightarrow{\mu_2} G_3 \xrightarrow{\mu_3} \cdots$$

be an inductive system. Let ρ_k denote the metric on G_k. Let H be the inductive limit in the category of groups. Define a pseudo-metric d on H by

$$d\big(\mu_{n,\infty}(x), \mu_{m,\infty}(y)\big) = \lim_{k\to\infty} \rho_k\big(\mu_{n,k}(x), \mu_{m,k}(y)\big).$$

Form the quotient of H by the subgroup $\{x \in H : d(x, 0) = 0\}$ and complete with respect to the induced metric to obtain the inductive limit.

It is an elementary exercise to prove that $U(\cdot)/\overline{DU(\cdot)}$ is a continuous functor from the category of unital C^*-algebras and unital $*$-homomorphisms, to the category of abelian groups equipped with a complete translation invariant metric, and contractive group homomorphisms.

Proposition 11.1 *Let A be a simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks with unital and injective connecting maps. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

with unital connecting maps such that $s(B_k) \to \infty$ and such that the torsion subgroup of $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ *is totally disconnected. Let* $\varphi_T \colon T(B) \to T(A)$ *be an affine continuous map, let* $\kappa \in KL(A,B)_e$ *be an element such that*

$$r_B(\omega)\big(\kappa_*(x)\big) = r_A\big(\varphi_T(\omega)\big)(x), \quad x \in K_0(A), \omega \in T(B),$$

and let $\Phi \colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ *be a homomorphism such that the diagram*

$$\begin{array}{ccccc} \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\ \Big\downarrow{\scriptstyle \widetilde{\varphi_T}} & & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \kappa_*} \\ \operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B) \end{array}$$

commutes. Assume finally that

$$s_\kappa(y) = \Phi(y), \quad y \in \operatorname{Tor}\big(U(A)/\overline{DU(A)}\big).$$

Let n be a positive integer and let $F_1 \subseteq \operatorname{Aff} T(A_n)$ and $F_2 \subseteq U(A_n)/\overline{DU(A_n)}$ be finite sets. There exist a positive integer m and a unital $*$-homomorphism $\psi \colon A_n \to B_m$ such that

$$[\beta_{m,\infty}] \cdot [\psi] = \kappa \cdot [\alpha_{n,\infty}] \quad \text{in } KL(A_n, B),$$

$$\|\widehat{\beta_{m,\infty}} \circ \widehat{\psi}(f) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)\| < \epsilon, \quad f \in F_1,$$

$$D_B\big(\beta^\#_{m,\infty} \circ \psi^\#(x), \Phi \circ \alpha^\#_{n,\infty}(x)\big) < \epsilon, \quad x \in F_2.$$

Proof Let $A_n = C_1 \oplus \cdots \oplus C_R$ where each C_i is a building block. By Proposition 5.2 and Proposition 3.2 there are for each $x \in U(A_n)/\overline{DU(A_n)}$ an element a_x in $\operatorname{Aff} T(A_n)/\overline{\rho_{A_n}\big(K_0(A_n)\big)}$, integers $k^1_x, k^2_x, \dots, k^R_x$, and an element y_x in the torsion subgroup of $U(A_n)/\overline{DU(A_n)}$ such that

$$x = \lambda_{A_n}(a_x) + \sum_{i=1}^{R} k^i_x q'_{A_n}(v^{A_n}_i) + y_x \quad \text{in } U(A_n)/\overline{DU(A_n)}.$$

Choose $b_x \in \operatorname{Aff} T(A_n)$ such that $q_{A_n}(b_x) = a_x$. Set $F_1' = F_1 \cup \{b_x : x \in F_2\}$. Choose $0 < \delta < \frac{1}{2}$ such that $\delta < \epsilon$ and such that

$$|e^{2\pi i\delta} - 1| + \delta \sum_{i=1}^{R} k^x_i < \epsilon, \quad x \in F_2.$$

Let $p_1, p_2, \dots, p_R$ denote the minimal non-zero central projections in A_n. Since A is simple and the connecting maps are injective, there exists a $\gamma > 0$ such that $\widehat{\alpha_{n,\infty}}(\widehat{p}_i) > \gamma$, $i = 1, 2, \dots, R$. By Proposition 10.2 there exist a positive integer

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l, a linear positive order unit preserving map $M\colon \operatorname{Aff} T(A_n) \to \operatorname{Aff} T(B_l)$, and an element $\omega \in KL(A_n, B_l)_e$ such that

$$[\beta_{l,\infty}] \cdot \omega = \kappa \cdot [\alpha_{n,\infty}] \quad \text{in } KL(A_n, B),$$
$$\|\widehat{\beta_{l,\infty}} \circ M(f) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)\| < \frac{\delta}{2}, \quad f \in F_1',$$
$$M \circ \rho_{A_n} = \rho_{B_l} \circ \omega_* \quad \text{on } K_0(A_n).$$

Choose an integer K by Theorem 8.5 with respect to $F_1' \subseteq \operatorname{Aff} T(A_n)$ and $\frac{\delta}{2}$. Choose a positive integer k and unitaries $u_1, u_2, \ldots, u_R \in B_k$ such that

$$D_B\Big(\beta^{\#}_{k,\infty}\big(q'_{B_k}(u_i)\big), \Phi \circ \alpha^{\#}_{n,\infty}\big(q'_{A_n}(v_i^{A_n})\big)\Big) < \delta, \quad i = 1, 2, \ldots, R.$$

Note that $\kappa_* \circ \alpha_{n,\infty\,*}[v_i^{A_n}] = \beta_{k,\infty\,*}[u_i]$ in $K_1(B)$. Hence

$$\beta_{l,\infty\,*} \circ \omega_*[v_i^{A_n}] = \beta_{k,\infty\,*}[u_i], \quad i = 1, 2, \ldots, R.$$

Since $\rho_B \circ \kappa_* = \varphi_{T*} \circ \rho_A$ we see that for $i = 1, 2, \ldots, R$,

$$\begin{aligned}\widehat{\beta_{l,\infty}}\big(\rho_{B_l}(\omega_*[p_i])\big) &= \rho_B(\beta_{l,\infty\,*} \circ \omega_*[p_i]) = \varphi_{T*} \circ \rho_A \circ \alpha_{n,\infty\,*}[p_i] \\ &= \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(\widehat{p}_i) > \gamma.\end{aligned}$$

Hence there exists an integer $m \geq k, l$ such that $s(B_m) \geq K\gamma^{-1}$ and such that

$$\widehat{\beta_{l,m}}\big(\rho_{B_l}(\omega_*[p_i])\big) > \gamma, \quad i = 1, 2, \ldots, R,$$
$$\beta_{l,m\,*} \circ \omega_*[v_i^{A_n}] = \beta_{k,m\,*}[u_i] \quad \text{in } K_1(B_m), \quad i = 1, 2, \ldots, R.$$

It follows that $s(B_m)\rho_{B_m}(\beta_{l,m\,*} \circ \omega_*[p_i]) \geq K$ and that

$$\widehat{\beta_{l,m}} \circ M \circ \rho_{A_n} = \widehat{\beta_{l,m}} \circ \rho_{B_l} \circ \omega_* = \rho_{B_m} \circ \beta_{l,m\,*} \circ \omega_* \quad \text{on } K_0(A_n).$$

Therefore by Theorem 8.5 there exists a unital $*$-homomorphism $\psi\colon A_n \to B_m$ such that

$$[\psi] = [\beta_{l,m}] \cdot \omega \quad \text{in } KL(A_n, B_m),$$
$$\psi^{\#}\big(q'_{A_n}(v_i^{A_n})\big) = q'_{B_m}\big(\beta_{k,m}(u_i)\big) \quad \text{in } U(B_m)/\overline{DU(B_m)}, \quad i = 1, 2, \ldots, R,$$
$$\|\widehat{\psi}(f) - \widehat{\beta_{l,m}} \circ M(f)\| < \frac{\delta}{2}, \quad f \in F_1'.$$

It follows that

$$[\beta_{m,\infty}] \cdot [\psi] = \kappa \cdot [\alpha_{n,\infty}] \quad \text{in } KL(A_n, B), \tag{30}$$

$$\|\widehat{\beta_{m,\infty}} \circ \widehat{\psi}(f) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)\| < \delta, \quad f \in F_1', \tag{31}$$

$$D_B\Big(\beta^{\#}_{m,\infty} \circ \psi^{\#}\big(q'_{A_n}(v_i^{A_n})\big), \Phi \circ \alpha^{\#}_{n,\infty}\big(q'_{A_n}(v_i^{A_n})\big)\Big) < \delta, \quad i = 1, 2, \ldots, R. \tag{32}$$

Note that for $x \in F_2$,

$$d'_B\big(\widetilde{\beta_{m,\infty}} \circ \widetilde{\psi}(a_x), \widetilde{\varphi_T} \circ \widetilde{\alpha_{n,\infty}}(a_x)\big) = d'_B\Big(q_B\big(\widehat{\beta_{m,\infty}} \circ \widehat{\psi}(b_x)\big), q_B\big(\varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(b_x)\big)\Big)$$
$$\leq \|\widehat{\beta_{m,\infty}} \circ \widehat{\psi}(b_x) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(b_x)\| < \delta < \frac{1}{2}.$$

Hence
$$d_B\big(\widetilde{\beta_{m,\infty}} \circ \widetilde{\psi}(a_x), \widetilde{\varphi_T} \circ \widetilde{\alpha_{n,\infty}}(a_x)\big) < |e^{2\pi i\delta} - 1|, \quad x \in F_2.$$

By Proposition 5.2, λ_B is an isometry when $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ is equipped with the metric d_B. It follows that

$$D_B\big(\lambda_B \circ \widetilde{\beta_{m,\infty}} \circ \widetilde{\psi}(a_x), \lambda_B \circ \widetilde{\varphi_T} \circ \widetilde{\alpha_{n,\infty}}(a_x)\big) < |e^{2\pi i\delta} - 1|, \quad x \in F_2.$$

Thus

$$D_B\big(\beta^{\#}_{m,\infty} \circ \psi^{\#} \circ \lambda_{A_n}(a_x), \Phi \circ \alpha^{\#}_{n,\infty} \circ \lambda_{A_n}(a_x)\big) < |e^{2\pi i\delta} - 1|, \quad x \in F_2.$$

Since s_κ and Φ agree on the torsion subgroup of $U(A)/\overline{DU(A)}$, we see by (30) and the definition of s_κ that

$$\beta^{\#}_{m,\infty} \circ \psi^{\#}(y_x) = \Phi \circ \alpha^{\#}_{n,\infty}(y_x).$$

Hence for $x \in F_2$,

$$\begin{aligned} D_B\big(\beta^{\#}_{m,\infty} &\circ \psi^{\#}(x), \Phi \circ \alpha^{\#}_{n,\infty}(x)\big) \\ &\leq D_B\Big(\beta^{\#}_{m,\infty} \circ \psi^{\#}\big(\lambda_{A_n}(a_x)\big), \Phi \circ \alpha^{\#}_{n,\infty}\big(\lambda_{A_n}(a_x)\big)\Big) \\ &\qquad + \sum_{i=1}^{R} k_x^i D_B\Big(\beta^{\#}_{m,\infty} \circ \psi^{\#}\big(q'_{A_n}(v_i^{A_n})\big), \Phi \circ \alpha^{\#}_{n,\infty}\big(q'_{A_n}(v_i^{A_n})\big)\Big) \\ &< |e^{2\pi i\delta} - 1| + \sum_{i=1}^{R} k_x^i \delta < \epsilon. \end{aligned}$$

■

Theorem 11.2 *Let A be a unital simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

with unital connecting maps such that $s(B_k) \to \infty$ and such that the torsion subgroup of $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ is totally disconnected. Let $\varphi_T \colon T(B) \to T(A)$ be an affine continuous map, let $\kappa \in KL(A,B)_e$ be an element such that

$$r_B(\omega)\big(\kappa_*(x)\big) = r_A\big(\varphi_T(\omega)\big)(x), \quad x \in K_0(A), \omega \in T(B),$$

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and let $\Phi\colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be a homomorphism such that the diagram

$$\begin{array}{ccccc} \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\ \widetilde{\varphi_T}\big\downarrow & & \Phi\big\downarrow & & \big\downarrow\kappa_* \\ \operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B) \end{array}$$

commutes. Assume finally that

$$s_\kappa(y) = \Phi(y), \quad y \in \operatorname{Tor}\big(U(A)/\overline{DU(A)}\big).$$

There exists a unital $*$-homomorphism $\psi\colon A \to B$ such that $\psi^* = \varphi_T$ on $T(B)$, such that $\psi^\# = \Phi$ on $U(A)/\overline{DU(A)}$, and such that $[\psi] = \kappa$ in $KL(A, B)$.

Proof We may assume that A is infinite dimensional. Hence by Theorem 9.9 we may assume that each α_n is unital and injective. Let $A_n = A_1^n \oplus A_2^n \oplus \cdots \oplus A_{R_n}^n$ where each A_i^n is a building block and let P_n be the set of minimal non-zero central projections in A_n. For each positive integer n, choose a finite set $G_n \subseteq A_n$ such that G_n generates A_n as a C^*-algebra and such that $\alpha_n(G_n) \subseteq G_{n+1}$. Choose by uniqueness, Theorem 7.7, a positive integer l_n with respect to $G_n \subseteq A_n$ and 2^{-n}. Since A is simple and the connecting maps are injective there exists a positive integer p_n such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{8}{p_n}, \quad h \in H(A_n, l_n).$$

Next, there exists a positive integer q_n such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{2}{q_n}, \quad h \in H(A_n, p_n) \cup P_n.$$

Finally choose $\delta_n > 0$ such that $\delta_n < \frac{1}{4q_n^2}$ and such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \delta_n, \quad h \in H(A_n, 4q_n).$$

Choose for each n finite sets $F_n \subseteq \operatorname{Aff} T(A_n)$ such that $\widetilde{H}(A_n, 2q_n) \subseteq F_n$, such that $\widehat{\alpha_n}(F_n) \subseteq F_{n+1}$, and such that $\bigcup_{n=1}^{\infty} \widehat{\alpha_{n,\infty}}(F_n)$ is dense in $\operatorname{Aff} T(A)$.

Next, choose finite sets $V_n \subseteq U(A_n)/\overline{DU(A_n)}$ such that $q'_{A_n}(v_i^{A_n}) \in V_n$ for $i = 1, 2, \ldots, R_n$, such that $\alpha_n^\#(V_n) \subseteq V_{n+1}$, and such that $\bigcup_{n=1}^{\infty} \alpha_{n,\infty}^\#(V_n)$ is dense in $U(A)/\overline{DU(A)}$.

We will construct by induction strictly increasing sequences $\{n_k\}$ and $\{m_k\}$ and unital $*$-homomorphisms $\psi_k\colon A_{n_k} \to B_{m_k}$ such that

(i) $\|\beta_{m_{k-1},m_k} \circ \psi_{k-1}(x) - \psi_k \circ \alpha_{n_{k-1},n_k}(x)\| < 2^{-n_{k-1}}$, $x \in G_{n_{k-1}}$, $k \geq 2$,

(ii) $\|\widehat{\beta_{m_k,\infty}} \circ \widehat{\psi_k}(f) - \varphi_{T*} \circ \widehat{\alpha_{n_k,\infty}}(f)\| < \min\{2^{-n_k}, \frac{\delta_{n_k}}{2}\}$, $f \in F_{n_k}$,

(iii) $D_B\big(\beta_{m_k,\infty}^\# \circ \psi_k^\#(x), \Phi \circ \alpha_{n_k,\infty}^\#(x)\big) < \min\{2^{-n_k}, \frac{\delta_{n_k}}{2}\}$, $x \in V_{n_k}$,

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(iv) $[\beta_{m_k,\infty}]\cdot[\psi_k] = \kappa\cdot[\alpha_{n_k,\infty}]$ in $KL(A_{n_k}, B)$.

The integers n_k, m_k, and the $*$-homomorphism ψ_k are constructed in step k. The case $k = 1$ follows immediately from Proposition 11.1.

Assume that n_k, m_k, and ψ_k have been constructed such that (i)–(iv) hold. Choose $n_{k+1} > n_k$ such that

$$\widehat{\alpha_{n_k,n_{k+1}}}(\widehat{h}) > \frac{8}{p_{n_k}}, \quad h \in H(A_{n_k}, l_{n_k}),$$

$$\widehat{\alpha_{n_k,n_{k+1}}}(\widehat{h}) > \frac{2}{q_{n_k}}, \quad h \in H(A_{n_k}, p_{n_k}) \cup P_n,$$

$$\widehat{\alpha_{n_k,n_{k+1}}}(\widehat{h}) > \delta_{n_k}, \quad h \in H(A_{n_k}, 4q_{n_k}).$$

Choose by Proposition 11.1 a positive integer l and a unital $*$-homomorphism $\lambda\colon A_{n_{k+1}} \to B_l$ such that

$$\|\widehat{\beta_{l,\infty}} \circ \widehat{\lambda}(f) - \varphi_{T*} \circ \widehat{\alpha_{n_{k+1},\infty}}(f)\| < \min\left\{2^{-n_{k+1}}, \frac{\delta_{n_k}}{2}, \frac{\delta_{n_{k+1}}}{2}\right\}, \quad f \in F_{n_{k+1}},$$

$$D_B\big(\beta^{\#}_{l,\infty} \circ \lambda^{\#}(x), \Phi \circ \alpha^{\#}_{n_{k+1},\infty}(x)\big) < \min\left\{2^{-n_{k+1}}, \frac{\delta_{n_k}}{2}, \frac{\delta_{n_{k+1}}}{2}\right\}, \quad x \in V_{n_{k+1}},$$

$$[\beta_{l,\infty}]\cdot[\lambda] = \kappa\cdot[\alpha_{n_{k+1},\infty}] \quad \text{in } KL(A_{n_{k+1}}, B).$$

It follows that

$$\|\widehat{\beta_{l,\infty}} \circ \widehat{\lambda} \circ \widehat{\alpha_{n_k,n_{k+1}}}(f) - \widehat{\beta_{m_k,\infty}} \circ \widehat{\psi_k}(f)\| < \delta_{n_k}, \quad f \in F_{n_k},$$

$$D_B\big(\beta^{\#}_{l,\infty} \circ \lambda^{\#} \circ \alpha^{\#}_{n_k,n_{k+1}}(x), \beta^{\#}_{m_k,\infty} \circ \psi^{\#}_k(x)\big) < \delta_{n_k} < \frac{1}{4q_{n_k}}, \quad x \in V_{n_k},$$

$$[\beta_{m_k,\infty}]\cdot[\psi_k] = [\beta_{l,\infty}]\cdot[\lambda]\cdot[\alpha_{n_k,n_{k+1}}] \quad \text{in } KL(A_k, B).$$

Hence there exists an integer $m_{k+1} \geq l$ such that

$$\|\widehat{\beta_{l,m_{k+1}}} \circ \widehat{\lambda} \circ \widehat{\alpha_{n_k,n_{k+1}}}(f) - \widehat{\beta_{m_k,m_{k+1}}} \circ \widehat{\psi_k}(f)\| < \delta_{n_k}, \quad f \in F_{n_k},$$

$$D_B\big(\beta^{\#}_{l,m_{k+1}} \circ \lambda^{\#} \circ \alpha^{\#}_{n_k,n_{k+1}}(x), \beta^{\#}_{m_k,m_{k+1}} \circ \psi^{\#}_k(x)\big) < \frac{1}{4q_{n_k}}, \quad x \in V_{n_k},$$

$$[\beta_{l,m_{k+1}}]\cdot[\lambda]\cdot[\alpha_{n_k,n_{k+1}}] = [\beta_{m_k,m_{k+1}}]\cdot[\psi_k] \quad \text{in } KL(A_k, B_{m_{k+1}}).$$

By uniqueness, Theorem 7.7, there exists a unitary $W \in B_{m_{k+1}}$ such that

$$\|\beta_{m_k,m_{k+1}} \circ \psi_k(x) - W\beta_{l,m_{k+1}} \circ \lambda \circ \alpha_{n_k,n_{k+1}}(x)W^*\| < 2^{-n_k}, \quad x \in G_{n_k}.$$

Set $\psi_{k+1}(x) = W\beta_{l,m_{k+1}} \circ \lambda(x)W^*$, $x \in A_{n_{k+1}}$. It is easily seen that (i)–(iv) are satisfied with $k+1$ in place of k. This completes the induction step.

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By Elliott's approximate intertwining argument, see *e.g.* [25, Lemma 1], there exists a $*$-homomorphism $\psi\colon A \to B$ such that

$$\psi\big(\alpha_{n,\infty}(x)\big) = \lim_{k\to\infty} \beta_{m_k,\infty} \circ \psi_k \circ \alpha_{n,n_k}(x), \quad x \in A_n.$$

Clearly, ψ is unital. Let $f \in F_n$, $\omega \in T(B)$. The sequence $\omega \circ \beta_{m_k,\infty} \circ \psi_k \circ \alpha_{n,n_k}$ converges to $\omega \circ \psi \circ \alpha_{n,\infty}$ in $T(A_n)$ as $k \to \infty$. Hence it follows that

$$\widehat{\beta_{m_k,\infty}} \circ \widehat{\psi_k} \circ \widehat{\alpha_{n,n_k}}(f)(\omega) \to \widehat{\psi} \circ \widehat{\alpha_{n,\infty}}(f)(\omega) \quad \text{as } k \to \infty.$$

On the other hand, from (ii) it follows that

$$\widehat{\beta_{m_k,\infty}} \circ \widehat{\psi_k} \circ \widehat{\alpha_{n,n_k}}(f)(\omega) \to \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)(\omega) \quad \text{as } k \to \infty.$$

Hence $\widehat{\psi} = \varphi_{T*}$ on $\operatorname{Aff} T(A)$ and thus $\psi^* = \varphi_T$ on $T(B)$. If $y, z \in U(A)/\overline{DU(A)}$ then $D_B\big(\Phi(y), \Phi(y)\big) \le D_A(y,z)$. This is clear in the case that $\pi_A(y) \neq \pi_A(z)$ since then $D_A(y,z) = 2$, and otherwise it follows since λ_A and λ_B are isometries and φ_{T*} is contractive (with respect to d_A and d_B). Thus Φ is continuous and by arguments similar to those applied above we see that $\psi^\# = \Phi$.

Let finally n be a positive integer. Since A_n is semiprojective there exists by [18, Theorem 15.1.1] a positive integer $l \ge n$ such that $\psi \circ \alpha_{n,\infty}$ is homotopic to $\beta_{m_l,\infty} \circ \psi_l \circ \alpha_{n,n_l}$. Hence

$$[\psi] \cdot [\alpha_{n,\infty}] = [\beta_{m_l,\infty}] \cdot [\psi_l] \cdot [\alpha_{n,n_l}] = \kappa \cdot [\alpha_{n_l,\infty}] \cdot [\alpha_{n,n_l}] = \kappa \cdot [\alpha_{n,\infty}]$$

in $KL(A_n, B)$. It follows from [22, Lemma 5.8] that $[\psi] = \kappa$ in $KL(A,B)$. ■

The following corollary generalizes a theorem of Thomsen [27, Theorem A].

Corollary 11.3 *Let A be a unital simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks such that $K_0(A)$ is non-cyclic. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

with unital connecting maps such that $s(B_k) \to \infty$. Let $\varphi_T\colon T(B) \to T(A)$ be an affine continuous map, let $\kappa \in KL(A,B)_e$ be an element such that

$$r_B(\omega)\big(\kappa_*(x)\big) = r_A\big(\varphi_T(\omega)\big)(x), \quad x \in K_0(A), \omega \in T(B),$$

and let $\Phi\colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ *be a homomorphism such that the diagram*

$$\begin{array}{ccccc} \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\ {\scriptstyle \widetilde{\varphi_T}}\big\downarrow & & {\scriptstyle \Phi}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_*} \\ \operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B) \end{array}$$

commutes. There exists a unital $*$*-homomorphism* $\psi\colon A \to B$ *such that* $\psi^* = \varphi_T$ *on* $T(B)$*, such that* $\psi^\# = \Phi$ *on* $U(A)/\overline{DU(A)}$*, and such that* $[\psi] = \kappa$ *in* $KL(A, B)$*.*

Proof By Lemma 10.7 we have that $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ is torsion free such that s_κ is defined. It follows by Proposition 5.2 that $s_\kappa(y) = \Phi(y)$ for y in the torsion subgroup of $U(A)/\overline{DU(A)}$. Apply Theorem 11.2. ∎

Corollary 11.4 *Let A be a unital inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

with unital connecting maps such that $s(B_k) \to \infty$ *and such that the torsion subgroup of* $\operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)}$ *is totally disconnected. Let* $\varphi_T\colon T(B) \to T(A)$ *be an affine continuous map, let* $\varphi_0\colon K_0(A) \to K_0(B)$ *be an order unit preserving group homomorphism such that*

$$r_B(\omega)\big(\varphi_0(x)\big) = r_A\big(\varphi_T(\omega)\big)(x), \quad x \in K_0(A), \omega \in T(B),$$

and let $\varphi_1\colon K_1(A) \to K_1(B)$ *be a group homomorphism. There exists a unital* $*$*-homomorphism* $\psi\colon A \to B$ *such that* $\psi^* = \varphi_T$ *on* $T(B)$*, such that* $\psi_* = \varphi_0$ *on* $K_0(A)$*, and such that* $\psi_* = \varphi_1$ *on* $K_1(A)$*.*

Proof Choose an element $\kappa \in KL(A, B)$ such that $\kappa_* = \varphi_0$ on $K_0(A)$ and such that $\kappa_* = \varphi_1$ on $K_1(A)$. By Lemma 10.10 there exists a group homomorphism $\Phi\colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ such that s_κ and Φ agree on the torsion subgroup of $U(A)/\overline{DU(A)}$ and such that the diagram

$$\begin{array}{ccccc} \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\ {\scriptstyle \widetilde{\varphi_T}}\big\downarrow & & {\scriptstyle \Phi}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_*} \\ \operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B) \end{array}$$

commutes. The conclusion follows from Theorem 11.2. ∎

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Theorem 11.5 *Let A and B be unital inductive limits of sequences of finite direct sums of building blocks, with A simple. Let $\varphi, \psi \colon A \to B$ be unital $*$-homomorphisms such that $\varphi^* = \psi^*$ on $T(B)$, $\varphi^\# = \psi^\#$ on $U(A)/\overline{DU(A)}$, and $[\varphi] = [\psi]$ in $KL(A, B)$. Then φ and ψ are approximately unitarily equivalent.*

Proof We may assume that A is infinite dimensional, and hence by Theorem 9.9 we see that A is the inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks with unital and injective connecting maps. By Lemma 9.2 we have that B is the inductive limit of a sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

of finite direct sums of building blocks with unital connecting maps. Let $A_n = A_1^n \oplus A_2^n \oplus \cdots \oplus A_{R_n}^n$ where each A_i^n is a building block. Let P_n be the set of minimal non-zero central projections in A_n.

Let $F \subseteq A$ be a finite set and let $\epsilon > 0$. It suffices to see that there exists a unitary $U \in B$ such that

$$\|\varphi(x) - U\psi(x)U^*\| < \epsilon, \quad x \in F.$$

We may assume that $F \subseteq \alpha_{n,\infty}(G)$ for a positive integer n and a finite set $G \subseteq A_n$.

Choose by uniqueness, Theorem 7.7, a positive integer l with respect to G and $\frac{\epsilon}{3}$. Since A is simple and the connecting maps are injective there exists an integer $p \geq l$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{9}{p}, \quad h \in H(A_n, l).$$

Next choose $q \geq p$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{3}{q}, \quad h \in H(A_n, p) \cup P_n.$$

Finally, choose $\delta > 0$ such that $3\delta < \epsilon$, $2\delta < \frac{1}{4q^2}$, and such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > 3\delta, \quad h \in H(A_n, 4q).$$

Since A_n by Theorem 2.4 is semiprojective there exist by [18, Corollary 15.1.3] a positive integer r and $*$-homomorphisms $\varphi_1, \psi_1 \colon A_n \to B_r$ such that $\beta_{r,\infty} \circ \varphi_1$ is homotopic to $\varphi \circ \alpha_{n,\infty}$ and $\beta_{r,\infty} \circ \psi_1$ is homotopic to $\psi \circ \alpha_{n,\infty}$, and such that if

$$x \in G \cup H(A_n, l) \cup H(A_n, p) \cup H(A_n, 4q) \cup \widetilde{H}(A_n, 2q) \cup P_n \cup \{v_1^{A_n}, v_2^{A_n}, \ldots, v_{R_n}^{A_n}\}$$

then

$$\|\beta_{r,\infty} \circ \varphi_1(x) - \varphi \circ \alpha_{n,\infty}(x)\| < \delta,$$
$$\|\beta_{r,\infty} \circ \psi_1(x) - \psi \circ \alpha_{n,\infty}(x)\| < \delta.$$

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By increasing r we may assume that φ_1 and ψ_1 are unital. Note that

$$D_B\Big(\beta^{\#}_{r,\infty}\circ\varphi^{\#}_1\big(q'_{A_n}(v^{A_n}_i)\big),\beta^{\#}_{r,\infty}\circ\psi^{\#}_1\big(q'_{A_n}(v^{A_n}_i)\big)\Big)<2\delta<\frac{1}{4q^2},\quad i=1,2,\dots,R_n,$$

$$\|\widehat{\beta_{r,\infty}}\circ\widehat{\varphi_1}(\widehat{h})-\widehat{\beta_{r,\infty}}\circ\widehat{\psi_1}(\widehat{h})\|<2\delta,\quad h\in\widetilde{H}(A_n,2q),$$

and

$$\widehat{\beta_{r,\infty}}\circ\widehat{\psi_1}(\widehat{h})>2\delta,\quad h\in H(A_n,4q),\tag{33}$$

$$\widehat{\beta_{r,\infty}}\circ\widehat{\psi_1}(\widehat{h})>\frac{8}{p},\quad h\in H(A_n,l),\tag{34}$$

$$\widehat{\beta_{r,\infty}}\circ\widehat{\psi_1}(\widehat{h})>\frac{2}{q},\quad h\in H(A_n,p)\cup P_n,\tag{35}$$

$$[\beta_{r,\infty}]\cdot[\psi_1]=[\beta_{r,\infty}]\cdot[\varphi_1]\quad\text{in } KL(A_n,B).\tag{36}$$

Choose an integer $m\geq r$ such that

$$\|\widehat{\beta_{r,m}}\circ\widehat{\varphi_1}(\widehat{h})-\widehat{\beta_{r,m}}\circ\widehat{\psi_1}(\widehat{h})\|<2\delta,\quad h\in\widetilde{H}(A_n,2q),\tag{37}$$

$$\widehat{\beta_{r,m}}\circ\widehat{\psi_1}(\widehat{h})>2\delta,\quad h\in H(A_n,4q),\tag{38}$$

$$\widehat{\beta_{r,m}}\circ\widehat{\psi_1}(\widehat{h})>\frac{8}{p},\quad h\in H(A_n,l),\tag{39}$$

$$\widehat{\beta_{r,m}}\circ\widehat{\psi_1}(\widehat{h})>\frac{2}{q},\quad h\in H(A_n,p)\cup P_n,\tag{40}$$

$$D_{B_m}\Big(\beta^{\#}_{r,m}\circ\varphi^{\#}_1\big(q'_{A_n}(v^{A_n}_i)\big),\beta^{\#}_{r,m}\circ\psi^{\#}_1\big(q'_{A_n}(v^{A_n}_i)\big)\Big)<\frac{1}{4q^2},\quad i=1,\dots,R_n,\tag{41}$$

$$[\beta_{r,m}]\cdot[\psi_1]=[\beta_{r,m}]\cdot[\varphi_1]\quad\text{in } KL(A_n,B_m).\tag{42}$$

By Theorem 7.7 there exists a unitary $W\in B_m$ such that

$$\|\beta_{r,m}\circ\varphi_1(x)-W\beta_{r,m}\circ\psi_1(x)W^*\|<\frac{\epsilon}{3},\quad x\in G.\tag{43}$$

If we put $U=\beta_{m,\infty}(W)$ we have that

$$\begin{aligned}\|\varphi\circ\alpha_{n,\infty}(x)-U\psi\circ\alpha_{n,\infty}(x)U^*\|&\leq\|\varphi\circ\alpha_{n,\infty}(x)-\beta_{r,\infty}\circ\varphi_1(x)\|\\&\quad+\|\beta_{r,\infty}\circ\varphi_1(x)-U\beta_{r,\infty}\circ\psi_1(x)U^*\|\\&\quad+\|\beta_{r,\infty}\circ\psi_1(x)-\psi\circ\alpha_{n,\infty}(x)\|\\&<\delta+\frac{\epsilon}{3}+\delta<\epsilon,\quad x\in G.\end{aligned}$$

■

In view of Theorem 7.5 one might think that equality in KL in the above theorem could be replaced by equality in K_0. This is however impossible in general, see [6, pp. 375–376] or [27, Theorem 8.4]. But in some cases, *e.g.* when $K_0(B)$ is cyclic, the KL-condition can be relaxed:

Theorem 11.6 *Assume furthermore that ρ_B is injective and $\rho_B\big(K_0(B)\big)$ is a discrete subgroup of $\operatorname{Aff} T(B)$. If $\varphi_* = \psi_*$ on $K_0(A)$, $\varphi^* = \psi^*$ on $T(B)$ and $\varphi^\# = \psi^\#$ on $U(A)/\overline{DU(A)}$, then φ and ψ are approximately unitarily equivalent.*

Proof As above, but with the following changes. Instead of (36) we get by Proposition 10.5 that

$$\beta^\#_{r,\infty} \circ \psi^\#_1(x) = \beta^\#_{r,\infty} \circ \varphi^\#_1(x), \quad x \in U^{A_n},$$
$$\beta_{r,\infty} \circ \psi_{1*} = \beta_{r,\infty} \circ \varphi_{1*} \quad \text{on } K_0(A_n).$$

By Lemma 10.4 we may now replace (42) by

$$\beta^\#_{r,m} \circ \psi^\#_1(x) = \beta^\#_{r,m} \circ \varphi^\#_1(x), \quad x \in U^{A_n},$$
$$\beta_{r,m} \circ \psi_{1*} = \beta_{r,m} \circ \varphi_{1*} \quad \text{on } K_0(A_n).$$

Finally, (43) follows again by Theorem 7.7. ■

Theorem 11.7 *Let A and B be simple unital infinite dimensional inductive limits of sequences of finite direct sum of building blocks. Let $\varphi_0 \colon K_0(A) \to K_0(B)$ be an isomorphism of groups with order units, let $\varphi_1 \colon K_1(A) \to K_1(B)$ be an isomorphism of groups, and let $\varphi_T \colon T(B) \to T(A)$ be an affine homeomorphism such that*

$$r_B(\omega)\big(\varphi_0(x)\big) = r_A\big(\varphi_T(\omega)\big)(x), \quad x \in K_0(A), \omega \in T(B).$$

There exists a $*$-isomorphism $\varphi \colon A \to B$ such that $\varphi_* = \varphi_0$ on $K_0(A)$, such that $\varphi_* = \varphi_1$ on $K_1(A)$, and such that $\varphi_T = \varphi^*$ on $T(B)$.

Proof By Theorem 9.9 we may assume that A is the inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks with unital and injective connecting maps. Similarly we may assume that B is the inductive limit of a sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

of finite direct sums of building blocks with unital and injective connecting maps. By Lemma 9.6 we have that $s(A_n) \to \infty$ and $s(B_n) \to \infty$ as $n \to \infty$.

By [23, Theorem 7.3] there exists an invertible element $\kappa \in KL(A,B)$ such that $\kappa_* = \varphi_0$ on $K_0(A)$ and $\kappa_* = \varphi_1$ on $K_1(A)$. By Lemma 10.10 there exists a group isomorphism $\Phi \colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Aff} T(A)/\overline{\rho_A\big(K_0(A)\big)} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \widetilde{\varphi_T}} & & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \kappa_*} & & \\
0 & \longrightarrow & \operatorname{Aff} T(B)/\overline{\rho_B\big(K_0(B)\big)} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B) & \longrightarrow & 0
\end{array}$$

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commutes and such that $s_\kappa(y) = \Phi(y)$ for y in the torsion subgroup of $U(A)/\overline{DU(A)}$.

By Theorem 11.2 there exists a unital $*$-homomorphism $\lambda\colon A \to B$ such that $\lambda^* = \varphi_T$ on $T(B)$, such that $\lambda^\# = \Phi$ on $U(A)/\overline{DU(A)}$, and such that $[\lambda] = \kappa$ in $KL(A,B)$. Note that $\kappa^{-1} \in KL(B,A)_T$. It is easy to see that s_κ is a bijection with inverse $s_{\kappa^{-1}}$. Hence $s_{\kappa^{-1}} = \Phi^{-1}$ on $\mathrm{Tor}\big(U(B)/\overline{DU(B)}\big)$. Thus there exists a unital $*$-homomorphism $\psi\colon B \to A$ such that $\psi^* = \varphi_T{}^{-1}$ on $T(A)$, such that $\psi^\# = \Phi^{-1}$ on $U(B)/\overline{DU(B)}$, and such that $[\psi] = \kappa^{-1}$ in $KL(B,A)$. By Theorem 11.5 the $*$-homomorphisms $\psi \circ \lambda$ and id_A are approximately unitarily equivalent. Similarly $\lambda \circ \psi$ and id_B are approximately unitarily equivalent. Hence by [21, Proposition A] λ is approximately unitarily equivalent to an isomorphism $\varphi\colon A \to B$. ■

12 Range of the Invariant

The purpose of this section is to determine the range of the Elliott invariant, *i.e.*, to answer the question which quadruples $\big(K_0(A), K_1(A), T(A), r_A\big)$ occur as the Elliott invariant for simple unital infinite dimensional C^*-algebras that are inductive limits of sequences of finite direct sums of building blocks. Villadsen [28] has answered this question in the case where A is an inductive limit of a sequence of finite direct sums of circle algebras. Using this result Thomsen has been able to determine the range of the Elliott invariant for those C^*-algebras that are inductive limits of finite direct sums of building blocks of the form $A(n, d, d, \ldots, d)$, see below.

We start out by examining the restrictions on $\big(K_0(A), K_1(A), T(A), r_A\big)$. Let A be a simple unital infinite dimensional inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks. We may by Theorem 9.9 assume that each α_n is unital and injective. By Corollary 3.6 each $K_0(A_k)$ is isomorphic (as an ordered group with order unit) to the K_0-group of a finite dimensional C^*-algebra. Thus $K_0(A)$ must be a countable dimension group. This group has to be simple as A is simple.

If $K_0(A) \cong \mathbb{Z}$ then by passing to a subsequence, if necessary, we may assume that A is the inductive limit of a sequence of building blocks, rather than finite direct sums of such algebras. By Lemma 3.9 it follows that $K_1(A)$ is an inductive limit of groups of the form $\mathbb{Z} \oplus H$, where H is any finite abelian group.

If $K_0(A)$ is not cyclic our only immediate conclusion is that $K_1(A)$ is a countable abelian group.

$T(A)$ must be a metrizable Choquet simplex. If B is a building block then obviously $r_B\colon T(B) \to SK_0(B)$ maps extreme points to extreme points. By [28, Corollary 1.6] and [28, Corollary 1.7] the same must be the case for r_A. Finally, r_A is surjective by either [3, Theorem 3.3] and [12], or [13, Corollary 9.18] (or, more elementary, because each $r_{A_k}\colon T(A_k) \to SK_0(A_k)$ is surjective). It follows from Theorem 12.1 and Corollary 12.5 that these are the only restrictions.

As mentioned above, Thomsen has calculated the range of the invariant for a subclass of the class we are considering. By [27, Theorem 9.2] we have the following:

Theorem 12.1 *Let G be a countable simple dimension group with order unit, H a countable abelian group, Δ a compact metrizable Choquet simplex, and $\lambda\colon \Delta \to SG$ an affine continuous extreme point preserving surjection. There exists a simple unital infinite dimensional inductive limit of a sequence of finite direct sums of building blocks A together with an isomorphism $\varphi_0\colon K_0(A) \to G$ of ordered groups with order unit, an isomorphism $\varphi_1\colon K_1(A) \to H$, and an affine homeomorphism $\varphi_T\colon \Delta \to T(A)$ such that*

$$r_A\big(\varphi_T(\omega)\big)(x) = \lambda(\omega)\big(\varphi_0(x)\big), \quad \omega \in \Delta, x \in K_0(A)$$

if and only if G is non-cyclic.

A can be realized as an inductive limit of a sequence of finite direct sums of circle algebras and interval building blocks of the form $I(n, d, d)$.

A different proof of this theorem could be based on Theorem 8.3 and [28, Theorem 4.2]. Combining the above theorem with Theorem 11.7 we get the following:

Theorem 12.2 *Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks such that $K_0(A)$ is non-cyclic. Then A is the inductive limit of a sequence of finite direct sums of circle algebras and interval building blocks of the form $I(n, d, d)$.*

We are left with the case of cyclic K_0-group. Note that the equation

$$r_A\big(\varphi_T(\omega)\big)(x) = \lambda(\omega)\big(\varphi_0(x)\big), \quad \omega \in \Delta, x \in K_0(A)$$

is trivial when A is a unital C^*-algebra with $K_0(A) \cong \mathbb{Z}$.

Lemma 12.3 *Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks. Then $\big(K_0(A), K_0(A)^+, [1]\big) \cong (\mathbb{Z}, \mathbb{Z}^+, 1)$ if and only if A is unital projectionless.*

Proof This follows easily from Theorem 9.9 and Lemma 3.8. ■

Theorem 12.4 *Let Δ be a metrizable Choquet simplex, and let H be the inductive limit of a sequence*

$$\mathbb{Z} \oplus H_1 \xrightarrow{h_1} \mathbb{Z} \oplus H_2 \xrightarrow{h_2} \mathbb{Z} \oplus H_3 \xrightarrow{h_3} \cdots$$

where each H_k is a finite abelian group. There exists an infinite dimensional simple unital projectionless C^*-algebra A that is an inductive limit of a sequence of building blocks, with $K_1(A) \cong H$ and such that $T(A)$ is affinely homeomorphic to Δ.

Proof By [26, Lemma 3.8] $\operatorname{Aff}\Delta$ is isomorphic to an inductive limit in the category of order unit spaces of a sequence

$$C_{\mathbb{R}}[0,1] \longrightarrow C_{\mathbb{R}}[0,1] \longrightarrow C_{\mathbb{R}}[0,1] \longrightarrow \cdots .$$

It is easy to see that this implies that $\operatorname{Aff}\Delta$ is isomorphic to an inductive limit of a sequence of the form

$$C_{\mathbb{R}}(\mathbb{T}) \xrightarrow{\Theta_1} C_{\mathbb{R}}(\mathbb{T}) \xrightarrow{\Theta_2} C_{\mathbb{R}}(\mathbb{T}) \xrightarrow{\Theta_3} \cdots .$$

Choose a dense sequence $\{x_k\}_{k=1}^{\infty}$ in $C_{\mathbb{R}}(\mathbb{T})$ and a dense sequence $\{z_k\}_{k=1}^{\infty}$ in $\mathbb{T}$.

For every positive integer k we will construct a unital projectionless building block A_k such that $K_1(A_k) \cong \mathbb{Z} \oplus H_k$, and a unital and injective $*$-homomorphism $\alpha_k \colon A_k \to A_{k+1}$ such that the (constant) functions $z \mapsto z_1, z \mapsto z_2, \ldots, z \mapsto z_k$ are eigenvalue functions for α_k, such that $\alpha_{k*} = h_k$ on $K_1(A_k)$ (under the identification $K_1(A_k) \cong \mathbb{Z} \oplus H_k$) and such that

$$\|\widehat{\alpha_k}(f) - \Theta_k(f)\| < 2^{-k}, \quad f \in F_k,$$

under the identification $\operatorname{Aff} T(A_k) \cong C_{\mathbb{R}}(\mathbb{T})$, where

$$F_k = \{x_1, x_2, \ldots, x_k\} \bigcup_{j=1}^{k-1} \Theta_{j,k}(\{x_1, x_2, \ldots, x_k\}) \bigcup_{j=1}^{k-1} \widehat{\alpha_{j,k}}(\{x_1, x_2, \ldots, x_k\}).$$

First choose by Lemma 3.9 a unital projectionless building block A_1 such that $K_1(A_1) \cong \mathbb{Z} \oplus H_1$.

Assume that A_k has been constructed. We will construct A_{k+1} and α_k. Choose K by Theorem 8.3 with respect to $F_k \subseteq \operatorname{Aff} T(A_k)$, $\epsilon = 2^{-k}$ and the integer $k+1$. By Lemma 3.9 there exists a unital projectionless building block A_{k+1} such that $s(A_{k+1}) \geq K$ and $K_1(A_{k+1}) \cong \mathbb{Z} \oplus H_{k+1}$. By Theorem 8.3 there exists a unital $*$-homomorphism $\alpha_k \colon A_k \to A_{k+1}$ such that the identity function on $\mathbb{T}$ and each of the functions $z \mapsto z_1, z \mapsto z_2, \ldots, z \mapsto z_k$ are among the eigenvalue functions for α_k and such that

$$\|\widehat{\alpha_k}(f) - \Theta_k(f)\| < 2^{-k}, \quad f \in F_k,$$
$$\alpha_{k*} = h_k \quad \text{on } K_1(A_k).$$

This completes the construction.

Set $A = \varinjlim(A_k, \alpha_k)$. A is infinite dimensional since the connecting maps are injective, and it is unital projectionless since the connecting maps are unital. By [26, Lemma 3.4] $\operatorname{Aff} T(A) \cong \varinjlim(C_{\mathbb{R}}[0,1], \Theta_k) \cong \operatorname{Aff}\Delta$, and hence $T(A)$ and Δ are affinely homeomorphic. Clearly $K_1(A) \cong H$.

Let $I \subseteq A$ be a closed two-sided ideal in A, $I \neq \{0\}$. By (the proof of) [5, Lemma 3.1],

$$I = \overline{\bigcup_{n=1}^{\infty} \alpha_{n,\infty}\big(\alpha_{n,\infty}{}^{-1}(I)\big)}.$$

Choose a positive integer n such that $\alpha_{n,\infty}{}^{-1}(I) \neq \{0\}$. Choose $f \in \alpha_{n,\infty}{}^{-1}(I)$ such that $f \neq 0$. Choose $k > n$ such that $f(z_k) \neq 0$. Then $\alpha_{n,l}(f)(z) \neq 0$ for every $z \in \mathbb{T}$ and $l > k$. Hence by Lemma 2.2 we see that $\alpha_{l,\infty}{}^{-1}(I) = A_l$ for every $l > k$. It follows that $I = A$. Thus A is simple. ∎

In the above theorem, let $H = 0$ and Δ be a one-point set. Then we obtain by Lemma 12.3 and Theorem 11.7 the C^*-algebra $\mathcal{Z}$ constructed by Jiang and Su [16].

Corollary 12.5 *Let d be a positive integer, let Δ be a metrizable Choquet simplex and let H be a countable abelian group. There exists an infinite dimensional simple unital inductive limit of a sequence of finite direct sums of building blocks A such that $\big(K_0(A), K_0(A)^+, [1]\big) \cong (\mathbb{Z}, \mathbb{Z}^+, d)$, $T(A) \cong \Delta$ and $K_1(A) \cong H$ if and only if H is the inductive limit of a sequence*

$$\mathbb{Z} \oplus H_1 \longrightarrow \mathbb{Z} \oplus H_2 \longrightarrow \mathbb{Z} \oplus H_3 \longrightarrow \cdots$$

where each H_k is a finite abelian group.

The C^*-algebra A is isomorphic to $M_d(B)$ where B is a simple unital projectionless C^*-algebra that is an inductive limit of a sequence of building blocks.

Proof Combine Theorem 12.4, Lemma 12.3 and Theorem 11.7. ■

Theorem 12.1 and Corollary 12.5 together determine the range of the Elliott invariant for the class of C^*-algebras for which our classification theorem applies. Let us conclude this paper by comparing our classification theorem with the classification theorems of Thomsen [27] and Jiang and Su [16].

It follows from [27, Theorem 9.2] that a C^*-algebra in our class is contained in Thomsen's class if and only if K_0 is non-cyclic. By calculating the range of the invariant for the C^*-algebras contained in Jiang's and Su's class, one can show that a C^*-algebra in our class with K_0 non-cyclic is contained in Jiang's and Su's class if and only if K_1 is a torsion group. A C^*-algebra in our class with cyclic K_0-group is contained in Jiang's and Su's class if and only if the K_1-group is an inductive limit of a sequence of finite cyclic groups, see [16, Theorem 4.5]. Thus our classification theorem can be applied to C^*-algebras that cannot be realized as inductive limits of finite direct sums of the building blocks considered in [27], or in [16], namely those that have cyclic K_0-group and a K_1-group that is not an inductive limit of a sequence of finite cyclic groups.

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