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Collineation groups which are sharply transitive on an oval

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Let G be a group of collineations in a projective plane Π of order n. Suppose that one of the point orbits of G is an oval $\underline{\underline{C}}$ of Π, and that G acts regularly on this orbit. We prove that G fixes a non-incident point-line pair if either n is even, or n is odd and G is abelian, or $n \neq 11, 23, 59$ is odd and $\underline{\underline{C}}$ is a pseudo-conic. It is then easy to deduce information about the lengths of the other orbits of G, and about the structure of G as an abstract group.

1. Introduction

General results on the relations between the (point and line) orbits of a collineation group in a finite projective plane have been obtained by, for example, Dembowski [*6*], Foulser and Sandler [*8*], and Piper [*16*]. These results depend on the fact that the orbits form a tactical decomposition of the plane. Parker [*15*], Hughes [*12*], and Dembowski [*6*] proved independently that the number of point orbits is equal to the number of line orbits.

Let Π be a finite projective plane of order n. An *oval* of Π is a set of $n + 1$ points in Π no three of which are collinear. The elementary properties of ovals are described in Qvist [*17*] and Dembowski [*5*]. If G is a group of collineations of Π and one of the point orbits of G is an oval $\underline{\underline{C}}$ of Π, then also one of the line orbits of G consists of the $n + 1$ tangents of $\underline{\underline{C}}$, and each of the remaining point (line) orbits either consists entirely of exterior points (chords) or consists entirely of interior points (non-secants).

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By making further assumptions about the way G acts on the oval $\underline{\underline{C}}$, about the geometric structure of $\underline{\underline{C}}$, or about the structure of G as an abstract group, we might hope to obtain more detailed descriptions of the remaining orbits of G.

Only the identity collineation fixes every point of the oval $\underline{\underline{C}}$, that is the collineation group G (with orbit $\underline{\underline{C}}$) acts faithfully on $\underline{\underline{C}}$; we say that G acts *regularly* or *sharply transitively* on $\underline{\underline{C}}$ if it is transitive on the points of $\underline{\underline{C}}$ and no non-trivial collineation in G fixes a point of $\underline{\underline{C}}$. If G acts regularly on $\underline{\underline{C}}$ then $|G| = n + 1$, and we shall then call $(G, \underline{\underline{C}})$ a *sharply transitive oval*. Singer's Theorem [*18*] guarantees the existence of sharply transitive ovals in every finite desarguesian plane, the ovals being conics and the groups cyclic.

An oval is a *pseudo-conic* in the sense of Ostrom [*14*] if it is the set of all absolute points of a polarity of Π.

The results proved in this paper will imply the following:

THEOREM. *If* $(G, \underline{\underline{C}})$ *is a sharply transitive oval in a finite projective plane of order* n, *then* G *fixes a non-incident point - line pair provided that either*

(i) n *is even, or*

(ii) n *is odd and* G *is abelian, or*

(iii) $n \neq 11, 23, 59$ *is odd and* $\underline{\underline{C}}$ *is a pseudo-conic.*

2. Assumed results

We shall assume the following theorems from the theory of finite projective planes and the theory of finite groups. Dembowski [5] or Hughes and Piper [*13*] is suggested as a general reference.

RESULT 1 (Baer [*2*]). If θ is an involutory collineation of a finite projective plane of order n, then either θ is an elation and n is even, or θ is an homology and n is odd, or the fixed points and lines of θ form a subplane of order $\sqrt{n}$.

RESULT 2 (Baer [*1*]). Every polarity of a finite projective plane has absolute points.

RESULT 3 (Parker [*15*], Hughes [*12*], Dembowski [*6*]). The number of

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point orbits, of any collineation group of a finite projective plane, is equal to the number of its line orbits.

RESULT 4 (Hering [11], Dembowski [5], p. 179). Let Γ be an abelian 2-group of collineations of a projective plane of order $n \equiv 3 \pmod 4$.

(a) If $|\Gamma| > 2$ and Γ is elementary abelian, then $|\Gamma| = 4$ and the fixed points and lines of Γ are the vertices and sides of a triangle.

(b) If Γ is not elementary then Γ fixes exactly one point and exactly one line, and the point does not lie on the line.

RESULT 5 (Hering [11]). If Γ is a 2-group of collineations of a projective plane of order $n \equiv 3 \pmod 4$ then Γ is cyclic, dihedral, semi-dihedral, or a generalized quaternion group.

RESULT 6 (Piper [16]). Let Γ be an abelian collineation group of order N in a projective plane of order n , and suppose Γ has exactly one point orbit of length N . Then either the fixed substructure of Γ is a line and at least three points on the line; or it is a point and at least three lines through the point; or $N = n^2 + n + 1, n^2, n^2 - 1,$ $n^2 - \sqrt{n}, n(n-1), (n-1)^2$, or $(n-\sqrt{n}+1)^2$; or $N = 9$ and $n = 4$.

RESULT 7 (see Hall [10]). Let G be a finite group. If G is soluble then G has an elementary abelian characteristic subgroup. If G is a p-group, for some prime p , then G has a non-trivial centre.

RESULT 8 (see Wielandt [19]). Suppose G is a permutation group on a finite set S , and $P \in S$. Then

$$|G| = |G_P| \cdot |P^G| ,$$

where G_P is the stabilizer in G of P and P^G is the orbit of G containing P . Also, G permutes the orbits of any normal subgroup H of G ; in particular, G permutes the fixed points of H . Finally, if G is abelian and transitive on S then G is sharply transitive on S .

RESULT 9 (see Wielandt [19]). If G is a permutation group on a finite set S , and if $\chi(g)$ denotes the number of elements of S fixed by $g \in G$, then the number t of orbits of G is given by:

$$\sum_{g \in G} \chi(g) = t|G| .$$

RESULT 10 (Frobenius' Theorem, see Hall [10], p. 292). The kernel of any Frobenius group G is a normal subgroup of G.

RESULT 11 (Feit and Thompson [7]). Every group of odd order is soluble.

RESULT 12 (Burnside [4]). If a finite group G has cyclic Sylow 2-subgroups then G has a normal 2-complement.

RESULT 13 (Brauer and Suzuki [3]). If a finite group G has generalized quaternion Sylow 2-subgroups then $G/O(G)$ has a non-trivial centre, where $O(G)$ denotes the largest normal subgroup of odd order in G.

RESULT 14 (see Gorenstein [9], pp. 260-265). Let G be a finite simple group whose Sylow 2-subgroups are either dihedral or semi-dihedral. Then G has only one conjugacy class of involutions.

3. Sharply transitive ovals of even order

If $(G, \underline{\underline{C}})$ is a sharply transitive oval in a projective plane Π of even order n, then G certainly fixes the knot F (point of concurrency of the $n + 1$ tangents to $\underline{\underline{C}}$), since every collineation which maps $\underline{\underline{C}}$ to itself fixes F. Also, no non-trivial element of G fixes a point $X \neq F$, since every line through F is tangent to $\underline{\underline{C}}$ and G acts regularly on the tangents of $\underline{\underline{C}}$. Thus every point orbit of G, apart from $\{F\}$, has length $n + 1$; and G has exactly $n + 1$ point orbits. It follows (Result 3) that G has exactly $n + 1$ line orbits.

Since any line orbit of length less than $n + 1$ has length at most $\frac{1}{3}(n+1)$, simple counting shows that G must have n line orbits of length $n + 1$ and one fixed line. We have proved:

THEOREM 1. *If* $(G, \underline{\underline{C}})$ *is a sharply transitive oval in a projective plane of even order* n, *then* G *fixes exactly one point and one line, the point does not lie on the line, and all other orbits of* G *have length* $n + 1$.

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4. Abelian sharply transitive ovals of odd order

In this section we prove:

THEOREM 2. *Let* $(G, \underline{C})$ *be a sharply transitive oval in a projective plane* Π *of odd order* n *, and suppose that* G *is abelian. Then the involutions of* G *are homologies, and* G *fixes the centre and axis of every involutory homology in* G *.*

Proof. Choose an involution θ in G . If θ is an homology then, since $\langle\theta\rangle \triangleleft G$, the whole group G must fix the centre and axis of θ .

Suppose now that θ is not an homology. Then the fixed points and lines of θ form a subplane Λ of order $\sqrt{n}$ (Result 1). Choose any point Q of $\underline{C}$, let $R = Q^{\theta}$, let q, r be the tangents at Q, R respectively, and let $P = q \cap r$. Then $G_P = \langle\theta\rangle$ and so $|P^G| = \frac{1}{2}(n+1)$, which means that G induces on Λ an abelian collineation group H of order $\frac{1}{2}(n+1)$.

The group H has at least one point orbit of length $h = |H| = \frac{1}{2}(n+1)$ and, since Λ has order $\sqrt{n}$, at most two such point orbits. If there is exactly one, Result 6 implies that the fixed substructure of H consists either of a line and at least three points on the line, or of a point and at least three lines through the point. (The other alternatives, namely various relations between h and $\sqrt{n}$, are easily seen to be impossible.)

If the fixed substructure is a line and at least three points on it, then the line is a non-secant of $\underline{C}$ and the fixed points are interior to $\underline{C}$. The fixed points determine at least three distinct chord orbits of length $\frac{1}{2}(n+1)$ for G , and these orbits determine at least three distinct involutions in G . By Result 4 these involutions generate a group of order 4 whose fixed points are the vertices of a triangle. The alternative (dual) case similarly gives rise to a contradiction.

We assume therefore that H has two point orbits of length h , and let $m = \sqrt{n}$, so that $h = \frac{1}{2}(m^2+1)$ and m is the order of Λ . Piper ([16], p. 331) remarks that simple calculations show that in such a case there is a subplane of Λ whose points form a third point orbit for H , and that H has only three point orbits. In our situation, this third orbit must have length m $\left(= m^2+m+1-2h\right)$, which is impossible since $|H| = \frac{1}{2}\left(m^2+1\right)$.

To establish Piper's assertion, let X^H be a point orbit of length less than h . Then $|H_X| \neq 1$ and H_X fixes every point of X^H (Result 8). Unless X^H is a single point, a set of collinear points, or a triangle, the fixed points of H_X form an invariant subplane Λ_0 of Λ (with respect to H). The first three possibilities are easily ruled out using the fact that H has two point orbits of length h . Now any line of Λ_0 contains at least one point from the union of the two h-orbits, and the lines of Λ_0 through the points of a given h-orbit all belong to the same line orbit. So Λ_0 contains at most two line orbits of H ; in fact Λ_0 can contain only one line orbit, since the orbits have odd length (dividing $\frac{1}{2}\left(m^2+1\right)$) and the number of lines in Λ_0 is odd. Thus Λ_0 contains only one point orbit (Result 3); indeed, every invariant proper subplane contains only one point orbit. It follows that every point Y in $\Lambda\backslash\Lambda_0$ which lies on a line l of Λ_0 belongs to an h-orbit $\left(H_Y\right.$ fixes l and so fixes every line of Λ_0 $\left.\right)$. The possibility that the set of such points exhausts the two h-orbits is easily excluded by counting. Thus if k is the order of Λ_0 then

$$\left(k^2+k+1\right)(m-k) = |H| = \frac{1}{2}\left(m^2+1\right) ,$$

and $|H_X| = m - k$. Now suppose $\phi \in H_X$ and $\phi \neq 1$; then, since each invariant proper subplane (for H) contains only one point orbit of H , ϕ fixes no point of $\Lambda\backslash\Lambda_0$. So H_X acts semi-regularly on the points of $\Lambda\backslash\Lambda_0$, and therefore every invariant proper subplane other than Λ_0 contains at least $m - k$ points. But

$$(m-k) + k^2 + k + 1 > m \left(= m^2+m+1-2h\right) ,$$

that is H leaves only one proper subplane invariant. So H has exactly three point orbits.

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This completes the proof of Theorem 2. We note that the intersections with a fixed line, of the chords of $\underline{\underline{C}}$ passing through a fixed point not on that line, form a point orbit of length $\frac{1}{2}(n+1)$ for G, and that the remaining points on these chords split into $\frac{1}{2}(n-1)$ orbits of length $n + 1$, plus the fixed point. A dual assertion can of course be made about line orbits.

5. Sharply transitive pseudo-conics of odd order

Let $(G, \underline{\underline{C}})$ be a sharply transitive pseudo-conic (in a projective plane Π of odd order n), with associated polarity α. Then every collineation ϕ in G commutes with α and so α induces a polarity on the incidence structure formed by the fixed points and lines of ϕ. If $\phi \neq 1$ this structure cannot be a subplane of Π, since ϕ fixes no point of $\underline{\underline{C}}$ and every polarity of a finite projective plane has absolute points (Result 2). Thus the involutions of G are homologies.

Consider any ψ in G which has prime order p and more than one fixed point, say ψ fixes (at least) the points A and B. Now AB cannot be an absolute line, and so $C = A^{\alpha} \cap B^{\alpha}$ is not on AB. But ψ fixes C and therefore, since the fixed points and lines form a closed substructure which is not a subplane, all further fixed points of ψ lie on one only of the lines AB, BC, CA, say on AB. By considering the action of ψ on the points of BC, we deduce that $p = 2$. It follows that if a non-trivial collineation in G fixes more than one point, then its order is a power of 2.

Now every collineation of prime order in G fixes at least one point, since $\left(|G|, n^2+n+1\right) = 1$. So every collineation in G whose order is not a power of 2 fixes exactly one point.

If χ in G has order 4 and χ fixes more than one point, consider the involution χ^2. The centre A and axis a of the homology χ^2 are fixed by χ, and all further fixed points of χ lie on a. Suppose χ fixes a point B on a, and consider the orbits of the group $\langle\chi\rangle$ acting on the points of AB: these are $\{A\}$, $\{B\}$ and further orbits all of length 4, so that $n - 1 \equiv 0 \pmod 4$, contradicting $n + 1 \equiv 0 \pmod 4$.

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We have proved:

LEMMA. *If* $(G, \underline{\underline{C}})$ *is a sharply transitive pseudo-conic in a projective plane of odd order, then the involutions of* G *are homologies, and every other non-trivial collineation in* G *fixes exactly one point.*

This lemma will be very useful in the proof of our main result:

THEOREM 3. *Suppose* $(G, \underline{\underline{C}})$ *is a sharply transitive pseudo-conic in a projective plane* Π *of odd order* $n \neq 3, 11, 23, 59$. *Then* G *fixes exactly one point and exactly one line, and the point does not lie on the line.*

Proof. We note first that it suffices to prove that G fixes exactly one point, since G then fixes the polar line of this point, and no other line; and the fixed point does not lie on its polar line since G acts regularly on $\underline{\underline{C}}$.

Let K be a non-trivial subnormal subgroup of G such that K is simple. K always exists, and $K = G$ if G is simple. The involutions in K are homologies (Lemma), and they form at most one conjugacy class of K (Results 5, 7, 12, 13, 14). Furthermore, no two involutory homologies in G have the same centre (or the same axis) since the action of such an homology on the oval $\underline{\underline{C}}$ is fully determined by the chords through its centre: it interchanges the two points of $\underline{\underline{C}}$ on each such chord. Thus the centres of the involutory homologies in K form a point orbit of K whose length equals the number of involutions in K .

If K has odd order, then K fixes exactly one point (Results 7, 11, Lemma), and this point is the unique fixed point of G . We assume therefore that K has even order.

Any S_2-subgroup (Sylow 2-subgroup) S of K has a non-trivial centre $Z(S)$. Let α be an involutory homology in $Z(S)$, let A be the centre and a the axis of α . Then $K_A = K_a = C_K(\alpha)$, the centralizer in K of α ; also $K_A \supseteq S$, and we have

$$k = 2^m rc ,$$

if $|K| = k$, $|S| = 2^m$, $|K_A| = 2^m r$ and $c = |A^K|$ is the number of

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involutions in K .

Let ϕ in K have odd prime order p , and fixed point F . Any S_p-subgroup P of K which contains ϕ fixes F , since $(n^2+n+1, p) = 1$. If $|K_F|$ were odd, then K would act as a Frobenius group on the points of F^K , that is K would have a proper non-trivial normal subgroup (Result 10), contradicting the simplicity of K . So $|K_F|$ is even, that is F is either the centre or lies on the axis of some involutory homology in K . If F is a centre then $F \in A^K$; while if F is not a centre then ϕ does not commute with any of the involutory homologies whose axis contains F , and so F lies on at least two axes. In the latter case, the S_2-subgroups of K_F each contain exactly one involution: otherwise the axes of two commuting involutions would both pass through F , which is impossible unless F is the centre of the product of these two involutions. It follows that these S_2-subgroups of K_F have order 2 , since if some ψ of order 4 in K fixed F then F would be the unique fixed point of an S_2-subgroup of K containing ψ , that is F would be the centre of an involutory homology.

If a point X lies on the axes of two involutions β and γ in K then $\langle \beta, \gamma \rangle$ fixes X and so either $\langle \beta, \gamma \rangle$ is a 2-group and X is fixed by an involution which commutes with both β and γ , that is X is a centre, or $\langle \beta, \gamma \rangle$ is not a 2-group and X is fixed by some collineation of odd order in K .

We have established that, for any point Y fixed by an involution in K , either $|K_Y| = 2$ or $|K_Y| = 2^m r$ or $|K_Y| = 2s_i$ for some odd $s_i > 1$ coprime to r . Furthermore,

$$k = 2^m r s_1 \ldots s_t ,$$

where $s_1, \ldots, s_t$ are the distinct numbers s_i so arising; and $s_1, \ldots, s_t$ are mutually coprime.

ASSUMPTION 1. Let us assume that K contains an element of odd order which fixes no centre, that is $t \geq 1$.

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Denote by c_0 the number of centres on the axis a, and let F_i be a point on a such that $|K_{F_i}| = 2s_i$. By Result 12, K_{F_i} has a normal 2-complement N. Since no involution in K_{F_i} commutes with an element of odd order in K_{F_i}, N acts semiregularly on the set of all axes through F_i. It follows that there are exactly s_i axes through F_i. But K is transitive on the set of all axes (of involutions in K) and on the points of F_i^K, so we may use simple counting to deduce that the number of points of F_i^K on a is exactly $2^{m-1}r$, for each $i = 1, 2, \ldots, t$.

To calculate the number b of orbits of K, considered as a permutation group on the $\frac{1}{2}n(n-1)$ interior points of $\underline{\underline{C}}$, we apply Result 9, obtaining

$$\tfrac{1}{2}n(n-1) + \tfrac{1}{2}(n+3)c + k - c - 1 = bk .$$

Writing $n + 1 = hk$, we have

$$b = \tfrac{1}{2}h(n+c) - h + 1 .$$

We return to the consideration of the points on an axis a. The interior points on a consist of: c_0 centres, $2^{m-1}rt$ points belonging to orbits F_i^K, and $\frac{1}{2}(n+1) - c_0 - 2^{m-1}rt$ points belonging to orbits of length $\frac{1}{2}k$. The third set determines $(\frac{1}{2}k)^{-1}c\left[\frac{1}{2}(n+1)-c_0-2^{m-1}rt\right]$ orbits, so if $c_0 \neq 0$ there are this number plus $t + 1$ orbits consisting of interior points which lie on at least one axis. Since $k|n+1$ and $2^{m-1}rc = \frac{1}{2}k$, k must divide $2cc_0$. But $c_0 + 1$ is the number of involutions in K_A, since c_0 is the number of axes through A and A is a centre. Also $|K_A| = 2^m r$, so that $c_0 + 1 \leq 2^m r - 1$, and $cc_0 < 2^m rc = k$. Thus either $k = 2cc_0$ or $c_0 = 0$.

ASSUMPTION 2. Assume that $k = 2cc_0$.

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Combining this with results obtained above, we deduce that there are exactly $\frac{1}{2}h(n-c) - h + 1$ orbits of interior points which lie on no axis, and therefore exactly $[\frac{1}{2}h(n-c)-h+1]k$ such points. We have now counted all the interior points: c centres, $\sum_{i=1}^{t} k(2s_i)^{-1}$ points which lie on at least two axes (but are not centres), $\left[\frac{1}{2}(n+1)-c_0-2^{m-1}rt\right]c$ points which lie on exactly one axis, and $[\frac{1}{2}h(n-c)-h+1]k$ points which lie on no axis. Thus

$$\frac{1}{2}n(n-1) = c + \sum_{i=1}^{t} k(2s_i)^{-1} + \left[\frac{1}{2}(n+1)-c_0-2^{m-1}rt\right]c + [\frac{1}{2}h(n-c)-h+1]k ,$$

from which we deduce the equation

$$(*) \qquad 1 = c + \sum_{i=1}^{t} k(2s_i)^{-1} + \frac{1}{2}k(1-t) ,$$

and thence (since each $s_i \geq 3$) the inequality $c - 1 \geq \frac{1}{2}k\left(\frac{2t}{3} - 1\right)$.

If $t \geq 3$ then $c \geq \frac{1}{2}k + 1$ and so (since $c|k$) $c = k$, that is every element of K is an involution, which is impossible. If $t = 1$ then, by (*), $1 = c + k(2s_1)^{-1}$ which is impossible since $c > 1$, $k > 0$ and $s_1 > 0$. So $t = 2$ and $c - 1 \geq \frac{1}{6}k$, that is $c = \frac{1}{5}k, \frac{1}{4}k, \frac{1}{3}k$ or $\frac{1}{2}k$. Now $c \neq \frac{1}{5}k$ or $\frac{1}{3}k$ since k is even and c is odd; and $c \neq \frac{1}{2}k$ since if $c = \frac{1}{2}k$ then K has a normal 2-complement, contrary to the simplicity of K.

Thus $t = 2$ and $c = \frac{1}{4}k$, that is $k = 4s_1s_2$ and so, by (*),

$$1 = 2s_1 + 2s_2 - s_1s_2 ,$$

from which we deduce that $\{s_1, s_2\} = \{3, 5\}$, $k = 60$, $c = 15$ and $c_0 = 2$. Each of the 15 involutions in K commutes with exactly 2 of the remaining 14, and the 15 centres of these involutory homologies can be partitioned into 5 disjoint sets $\underline{\underline{C}}_i$ of 3 non-collinear points which are the centres of the involutory homologies in an elementary abelian S_2-subgroup (of order 4) of K. The 15 centres form a unique point

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orbit $\underline{\underline{O}}$ of length 15 for K , the remaining point orbits having length 6, 10, 30 or 60 . It follows since K is subnormal in G , that $\underline{\underline{O}}^{G} = \underline{\underline{O}}$ and that G permutes the 5 sets $\underline{\underline{C}}_i$. Since $\underline{\underline{O}}^G = \underline{\underline{O}}$ and K (being simple) is generated by its 15 involutions, $K \trianglelefteq G$.

The representation of the simple group K as a permutation group on $W = \{\underline{\underline{C}}_1, \ldots, \underline{\underline{C}}_5\}$ is faithful. Let H be the kernel of the representation of G on W . Since $H \cap K = 1$, $H \triangleleft G$ and $K \trianglelefteq G$, every element of H commutes with each of the 15 involutory homologies in K . It follows readily that $H = 1$, so that G is isomorphic to a subgroup of S_5 . If $G \cong S_5$ then the normalizer in G of any Sylow 5-subgroup P of G contains an element ϕ of order 4 ; and ϕ must fix the point X fixed by P . But $|X^G| = 6$ and so ϕ must fix at least two points of X^G , contradicting our Lemma. Thus $G \ncong S_5$ and therefore $|G| = 60$, contradicting $n \neq 59$. We have shown that in all cases equation (*) leads to a contradiction.

Suppose that Assumption 2 is false. Then $c_0 = 0$ and so each S_2-subgroup S of K contains only one involution (otherwise, consider a pair of commuting involutions in S : the centre of one lies on the axis of the other). If S is cyclic then K has a normal 2-complement and so, since K is simple of even order, $|K| = 2$. If S is generalized quaternion then $K/O(K)$ has a non-trivial centre (Result 13), which is impossible. There is no other possibility (Result 5), so $|K| = 2$.

Now suppose that Assumption 1 is false. Then $k = 2^m r$, $c = 1$ and K contains exactly one involution. But K is simple, so $|K| = 2$.

Since either Assumption 1 or Assumption 2 is false, $|K| = 2$.

If K is a proper subnormal subgroup of some subnormal subgroup L of G which fixes more than one point, then L consists of involutions and the identity, that is L is elementary abelian. Since $K < L$, $|L| = 4$ (Result 4), and so L fixes exactly three points, the remaining point orbits of L having length 2 or 4 . Now G does not fix all three fixed points of L since $n \neq 3$, so either G fixes exactly one

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point or else G has exactly one point orbit of length 3. The latter case is impossible since the representation of G as a permutation group on this orbit would have kernel L of order 4, and the induced group would be isomorphic to a subgroup of S_3, contradicting $n \neq 11$ or 23.

If there is no such L then the centre of the involutory homology in K is the unique fixed point of G. This completes the proof of Theorem 3.

COROLLARY. *Suppose* $(G, \underline{\underline{C}})$ *satisfies the hypotheses of Theorem* 3. *Then either*

- *(i)* G *contains only one involution,* G *has two point (line) orbits of length* $\frac{1}{2}(n+1)$, *and* $n - 1$ *of length* $n + 1$; *or*
- *(ii)* $n \equiv 1 \pmod 4$, G *contains* $\frac{1}{2}(n+1)$ *conjugate involutions, and* G *has* $n + 1$ *point (line) orbits of length* $\frac{1}{2}(n+1)$ *and* $\frac{1}{2}(n-1)$ *of length* $n + 1$; *or*
- *(iii)* $n \equiv 3 \pmod 4$, G *contains* $\frac{1}{2}(n+1) + 1$ *involutions in three conjugacy classes, of sizes* 1, $\frac{1}{4}(n+1)$ *and* $\frac{1}{4}(n+1)$, *and* G *has two point (line) orbits of length* $\frac{1}{4}(n+1)$, n *of length* $\frac{1}{2}(n+1)$ *and* $\frac{1}{2}(n-1)$ *of length* $n + 1$.

Proof. If G contains only one involution then the centre of this homology is a fixed point, any point on its axis lies in an orbit of length $\frac{1}{2}(n+1)$, and every other point in an orbit of length $n + 1$.

Suppose $n \equiv 1 \pmod 4$ and G contains more than one involution. Then the centres of these involutions must lie on the fixed line f and the axes must pass through the fixed point F. Also, no centre lies on an axis, and no axis is a chord of $\underline{\underline{C}}$. But the chords of $\underline{\underline{C}}$ through F meet f in the points of an orbit of length $\frac{1}{2}(n+1)$, so these are the centres of the involutory homologies in G, and the remaining $\frac{1}{2}(n+1)$ points on f are the intersections with f of the axes, and form a single orbit. The assertions of *(ii)* now follow readily.

Finally consider the case where $n \equiv 3 \pmod 4$ and G contains more than one involution. If P is the intersection with the fixed line f of a chord of $\underline{\underline{C}}$ through the fixed point F, then $|P^G| = \frac{1}{2}(n+1)$ and so $|G_P| = 2$ and either G contains an involutory (F, f)-homology or P is

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a centre or PF is an axis. The third case is impossible since G acts regularly on $\underline{C}$. The second case is also impossible since $|P^G|$ is even, which implies that if P is centre of an involutory homology then its axis is also a chord through F . Thus G contains an involutory (F, f)-homology θ and, since θ is in the kernel of the representation of G on P^G , $\theta \in Z(G)$. The centre of any other involution in G lies on f in an orbit of length $\frac{1}{4}(n+1)$. Since the length of every point orbit other than $\{F\}$ is at least $\frac{1}{4}(n+1)$, there are two orbits of centres on f , each of length $\frac{1}{4}(n+1)$. The assertions of *(iii)* now follow readily.

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