DERIVATIONS AND PURE STATES

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In this note we point out and discuss a relationship between the following two properties which a C^*-algebra A may have:

(D) all derivations on A are inner;
(P) the set of pure states of A is w^*-compact.

The relationship is deduced from a comparison of two existing theorems ((I) and (II) below).

For general terminology concerning C^*-algebras, we refer to [**4**]. In particular, we denote by $\text{Prim}(A)$ (resp. $\hat{A}$) the set of primitive ideals of A (resp. the set of (unitary-equivalence-classes of) non-zero irreducible *-representations of A), topologized as in [**4**, 3.1.1, 3.1.5]. If A is postliminal, the canonical surjection $\pi \mapsto \ker \pi$ of $\hat{A}$ onto $\text{Prim}(A)$ is a homeomorphism [**4**, 3.1.6, 4.3.7]. For any positive integer n, we shall denote by $\text{Prim}_n(A)$ (resp. $\hat{A}_n$) the set of primitive ideals of A which are the kernels of n-dimensional irreducible *-representations of A (resp. the set of n-dimensional elements of $\hat{A}$). A will be called *homogeneous* if every irreducible *-representation of A is of the same *finite* dimension. A is said to have *continuous trace* if A is liminal, $\hat{A}$ is Hausdorff, and for each π_0 in $\hat{A}$, there is an element a in A and a neighbourhood U of π_0 in $\hat{A}$ such that, for all π in U, $\pi(a)$ is a projection of rank 1 ([**6**, 4.1], [**4**, 4.5]).

We shall consider the following properties, in addition to those above:

(P_1) property (P) holds, and in addition $\text{Prim}_1(A)$ is open in $\text{Prim}(A)$;
(P') the elements of the w^*-closure of the set of pure states of A are proportional to pure states of A;
(P_1') property (P') holds, and in addition $\text{Prim}_1(A)$ is open in $\text{Prim}(A)$.

If A has an identity, (P') reduces to (P); if A has no identity then (P) cannot hold [**4**, 2.12.13], and (P') is the appropriate substitute.

The existing theorems mentioned above may be stated as follows:

(I). ([**5**, Theorem 3], [**1**, Corollary 5.4]). *Let A be a separable postliminal C^*-algebra. Then the following two conditions are equivalent:*

(i) *Property (D) holds for A;*
(ii) *A is the direct sum of finitely many homogeneous C^*-algebras, each non-commutative homogeneous summand possessing an identity.*

REMARKS 1. Inspection of the proof in [**5**]—see also [**12**, Theorem 1]—shows that the implication (ii) ⇒ (i) is valid without restriction (a C^*-algebra satisfying (ii) is automatically postliminal, indeed liminal, but need not be separable).

2. The implication (i) ⇒ (ii) is proved in [**5**] for a separable liminal C^*-algebra, and

generalized to a separable postliminal C^*-algebra in [1], where it is shown that a separable postliminal C^*-algebra satisfying (i) is necessarily liminal.

(II). [13, Theorem 1]. *Let A be a C^*-algebra with identity. Then the following two conditions are equivalent:*

(i) *property (P_1) holds for A;*
(ii) *A is the direct sum of finitely many homogeneous C^*-algebras with identity.*

As an immediate deduction from (I) and (II) and Remark 1 following (I), we can state the following result.

THEOREM. *Let A be a C^*-algebra with identity. Then $(P_1) \Rightarrow (D)$. If also A is separable and postliminal, then $(D) \Rightarrow (P_1)$, and a fortiori $(D) \Rightarrow (P)$.*

We note immediately that the connecting property here—that A be the direct sum of finitely many homogeneous C^*-algebras with identity—is satisfied if and only if A has continuous trace and identity. That the direct sum of finitely many homogeneous C^*-algebras has continuous trace follows without difficulty from [6, Theorem 4.3]. Conversely, suppose that A has continuous trace and identity. Then every irreducible $*$-representation of A is finite-dimensional [4, 4.7.14.b], and so $\hat{A} = \bigcup_{n=1}^{\infty} \hat{A}_n$. Applying [6, Lemma 4.1] to the identity element of A, we see that the map $\pi \mapsto \dim \pi$ is continuous on $\hat{A}$. Hence, for each n, $\hat{A}_n$ is open and closed in $\hat{A}$. Since A has an identity, $\hat{A}$ is compact [4, 3.1.8], so that $\hat{A}_n$ is empty except for finitely many n. That A is the direct sum of finitely many homogeneous C^*-algebras now follows from [4, 3.2.2, 3.2.3].

We shall show that the various conditions in the Theorem cannot be relaxed. In doing so we shall make use of the following theorem characterizing property (P') (and therefore property (P) when A has an identity).

(III). [8, Theorem 6]. *Let A be a C^*-algebra. Then property (P') holds for A if and only if A is liminal,* Prim (A) *is Hausdorff, and every singular point in* Prim (A) *belongs to* $\text{Prim}_1(A)$.

Here a point y in Prim (A) is called *singular* if there is an element p in A with $p(x)$ (the canonical image of p in A/x) a projection for all x in some neighbourhood N of y, with rank $p(y) = 1$, and such that for each neighbourhood M of y contained in N, there is an x in M with rank $p(x) > 1$. We note that A has continuous trace if and only if A is liminal, Prim (A) $(\cong \hat{A})$ is Hausdorff, and there are *no* singular points in Prim(A). It is clear, in view of the remarks at the top of p. 601 of [8], that if A has continuous trace then A satisfies these three properties. Conversely, suppose that A satisfies these properties. Then given y in Prim(A) there exists an element p in A and a neighbourhood N of y with $p(x)$ a projection for all x in N, with rank $p(y) = 1$, and with rank $p(x) \leqq 1$ for all x sufficiently near y. In fact, since the map $x \mapsto \|p(x)\|$ is continuous on Prim (A) (which follows from [4, 3.3.9]), rank $p(x) = 1$ for all x sufficiently near y, and so A has continuous trace. Noting that if $\text{Prim}_1(A)$ is open in Prim (A) then no point in $\text{Prim}_1(A)$ can be singular, we see that if (P_1') holds then A has continuous trace. On the other hand, if A has continuous trace then (P')holds, and if also A

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has an identity then $\mathrm{Prim}_1(A)$ is open in $\mathrm{Prim}(A)$ (use [**6**, Lemma 4.1] as above). Hence if A has an identity then A has continuous trace if and only if (P_1) holds, confirming what we have already noted and showing that (II) can be deduced from (III). (If A has continuous trace but no identity, then $\mathrm{Prim}_1(A)$ need not be open in $\mathrm{Prim}(A)$; we shall return to this point later.)

Returning now to the Theorem, we show first that in the implication $(P_1)\Rightarrow(D)$, property (P_1) cannot be replaced by property (P). Let A be the C^*-algebra of all bounded sequences $m=\{m_n\}_{n=1,2,\ldots}$ of 2×2 complex matrices such that m_n converges to a matrix of the form

$$\begin{bmatrix}\lambda(m) & 0\\ 0 & \lambda(m)\end{bmatrix}$$

as $n\to\infty$. Then A is liminal and $\mathrm{Prim}(A)$ is Hausdorff: $\mathrm{Prim}(A)$ $(\cong\hat{A})$ is homeomorphic to the one-point compactification $\mathbf{N}\cup\{\infty\}$ of the set $\mathbf{N}$ of positive integers, each t in $\mathbf{N}$ corresponding to the 2-dimensional representation $m\mapsto m_t$ and ∞ to the 1-dimensional representation $m\mapsto\lambda(m)$. The point ∞ in $\mathrm{Prim}(A)$ is singular, but is the only singular point since each other point is isolated. Hence A (which has an identity) satisfies (P), but does not have continuous trace, and so does not satisfy (P_1)—in fact we can see directly that $\mathrm{Prim}_1(A)=\{\infty\}$ is not open in $\mathrm{Prim}(A)$. A does not satisfy property (D): for example, the derivation implemented by the sequence $\{b_n\}$, where $b_n=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ for each n, is outer [**12**, §5, Example (1)].

We show now that the conditions that A be separable and postliminal are both necessary for the validity of the implication $(D)\Rightarrow(P)$ in the Theorem. Let A be the C^*-algebra of all bounded sequences $m=\{m_n\}$ of 2×2 complex matrices such that $(m_n)_{12}$ and $(m_n)_{21}\to 0$ as $n\to\infty$. As is shown in [**12**, §5, Example (3)], A satisfies property (D), while $\mathrm{Prim}(A)$ $(\cong\hat{A})$ consists of $\mathbf{N}$ with two copies of $\beta\mathbf{N}\backslash\mathbf{N}$ adjoined (where $\beta\mathbf{N}$ denotes the Stone–Čech compactification of $\mathbf{N}$) and so is not Hausdorff. Hence A (which has an identity) does not satisfy (P). Here A is postliminal—indeed liminal—but not separable. On the other hand, suppose that A is a UHF C^*-algebra [**7**], so that A has an identity and is separable but not postliminal. A satisfies (D) (see for example [**11**], or [**9**, Corollary 3.3]) but not (P): indeed the w^*-closure of the set of pure states of A is the (strictly larger) set of all states of A [**7**, Theorem 2.8 and the paragraph following its proof]. We may perhaps mention here the C^*-algebra of all bounded linear operators on an infinite-dimensional Hilbert space: this C^*-algebra has an identity, is neither separable nor postliminal, satisfies (D) (for example by [**3**, p. 311, Théorème 1]) but not (P) [**10**, Theorem 2].

We conclude with some remarks about the situation when A has no identity. As we have noted, (P) cannot then hold, and is to be replaced by (P') (and (P_1) by (P_1')). Property (D) may however still hold (see for example [**1**, Example 7.6]), and of course (I) above applies to C^*-algebras with or without identity. Nevertheless, we may often usefully consider the following property instead (see [**1**]), which reduces to (D) when A has an identity:

(D') every derivation on A is determined by a multiplier.

In the absence of an identity, properties (P_1') and D' are still related via the continuous trace property, but not so straightforwardly as were (P_1) and (D) in the Theorem. We have seen that (P_1') implies that A has continuous trace, and that if A has continuous trace then A satisfies (P'); but the following example shows that A can have continuous trace without $\mathrm{Prim}_1(A)$ being open in $\mathrm{Prim}(A)$, and so without (P_1') being satisfied. Let A be the C^*-algebra of all bounded sequences $m = \{m_n\}$ of 2×2 complex matrices such that m_n converges to a matrix of the form $\begin{bmatrix}\lambda(m) & 0\\ 0 & 0\end{bmatrix}$ as $n\to\infty$. Then $\mathrm{Prim}(A)$ $(\cong\hat{A})$ is homeomorphic to the one-point compactification $\mathbf{N}\cup\{\infty\}$ of $\mathbf{N}$; A has continuous trace (this time the point ∞ in $\mathrm{Prim}(A)$ is not singular, as is shown by consideration of the constant sequence $\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\}$, which now belongs to A); but $\mathrm{Prim}_1(A) = \{\infty\}$ is not open in $\mathrm{Prim}(A)$.

If a C^*-algebra A has continuous trace and is also *quasicentral* (that is, no primitive ideal of A contains the centre Z of A), then indeed $\mathrm{Prim}_1(A)$ is open in $\mathrm{Prim}(A)$, so that (P_1') holds. In fact, taking an argument from the proof of [**2**, Theorem 3.2.2], we show as follows that $\hat{A}_1$ is open in $\hat{A}$. Let $\pi_0\in\hat{A}_1$. Since A has continuous trace, there exists p in A with $\pi(p)$ a projection of rank 1 for all π in some neighbourhood of π_0. In particular, $\pi_0(p) = I_{\pi_0}$ (the identity operator on the representation space H_{π_0}), and we need to show that $\pi(p) = I_\pi$ for all π in some neighbourhood of π_0. Since A is quasicentral, there is a positive element z in Z with $\pi_0(z) = I_{\pi_0}$. For any π in $\hat{A}$, $\pi(z)$ is a scalar multiple of I_π (by irreducibility), and $\pi(z)$ is positive; so $\pi(z) = \lambda_\pi I_\pi$ for some non-negative real number λ_π. Since A has continuous trace, $\hat{A}$ is Hausdorff, so the maps $\pi\mapsto\|\pi(z)\| = \lambda_\pi$ and $\pi\mapsto\|\pi(p-z)\|$ are continuous on $\hat{A}$ [**4**, 3.3.9]. Hence $\lambda_\pi\to 1$ as $\pi\to\pi_0$, and so

$$\|\pi(p)-I_\pi\|\leqq\|\pi(p)-\pi(z)\|+\|\pi(z)-I_\pi\| = \|\pi(p-z)\|+|\lambda_\pi-1|\to 0$$

as $\pi\to\pi_0$, which gives the desired result. The C^*-algebra in the last paragraph is easily seen not to be quasicentral: its centre consists of all sequences $m = \{m_n\}$ with each m_n a scalar multiple of the identity and $\lambda(m) = 0$, and so is contained in the kernel ('∞') of the 1-dimensional representation $m\mapsto\lambda(m)$. (We mention here that the full statement of [**2**, Theorem 3.2.2] is that if A is a C^*-algebra all of whose irreducible *-representations are finite-dimensional, then the following three conditions are equivalent:

(i) A is quasicentral and has continuous trace;
(ii) the map $\pi\mapsto\dim\pi$ is continuous on $\hat{A}$;
(iii) A is the $C^*(\infty)$-sum [**4**, 1.9.14] of a sequence $(A_n)_{n=1,2,\ldots}$ of C^*-algebras, where each A_n is either zero or homogeneous of degree n.

In particular, if A has an identity, we get again that A has continuous trace if and only if A is the direct sum of finitely many homogeneous C^*-algebras.)

As for the connection between property (D') and the continuous trace property, it is known that if A has continuous trace and $\mathrm{Prim}(A)$ $(\cong\hat{A})$ is paracompact, then (D') holds; the assumption of paracompactness here, which is automatically fulfilled if A has a countable approximate identity and *a fortiori* if A is separable, may be unnecessary [**1**, Theorem 3.2 and Remark 3.3]. If A is separable and postliminal and $\mathrm{Prim}(A)$ is quasi-separated [**1**, 4.1, 4.5] then property (D') implies that A has continuous trace [**1**, Theorem 4.3; see also 4.4].

The relationships between the derivation and pure state properties discussed in this paper arise indirectly, via the continuous trace property. It would be interesting to know if a more direct link exists.

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