CHROMATIC NUMBER AND TOPOLOGICAL COMPLETE SUBGRAPHS

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1. Introduction and terminology. A graph with $m(\geq 1)$ vertices, each pair of distinct vertices connected by an edge, and also a graph obtained from such a graph by the process of subdividing edges through the insertion of new vertices of valency 2, will be denoted by $\langle\langle m, o\rangle\rangle$. A graph obtained from a graph with $m(\geq 2)$ vertices in which each pair of distinct vertices are connected by an edge, by deleting n $(\leq m-1)$ edges incident with one and the same vertex, and also a graph obtained from such a graph by the process of subdividing edges through the insertion of new vertices of valency 2, will be denoted by $\langle\langle m, n\rangle\rangle$. Vertices with valency ≥ 3 are called branch-vertices of the $\langle\langle m, n\rangle\rangle$. An $\langle\langle m, o\rangle\rangle$ is known as a topological complete graph with m vertices; a $\langle\langle 3, o\rangle\rangle$ is a circuit. It is known that every 4-chromatic graph contains a $\langle\langle 4, 0\rangle\rangle$ as a subgraph [1] and that every graph without multiple edges with $N \geq 4$ vertices and $2N-2$ edges contains a $\langle\langle 4, 0\rangle\rangle$ [2], but so far only conjectures exist concerning conditions for the existence of $\langle\langle m, o\rangle\rangle$ with $m \geq 5$. For example, G. Hajos conjectured that every k-chromatic graph contains a $\langle\langle k, o\rangle\rangle$, and I conjectured that every graph without multiple edges with $N \geq 5$ vertices and $3N-5$ edges contains a $\langle\langle 5, 0\rangle\rangle$.

In this paper the method of distance-classes with respect to a selected vertex, recently used by K. Wagner to establish a general homomorphism theorem [3], will be employed to prove the following

THEOREM. If $k(m,n)$, where m and n are integers and $0 \leq n \leq m-3$, is such that every $k(m,n)$-chromatic graph contains an $\langle\langle m, n\rangle\rangle$, then every $(2k(m,o)-1)$-chromatic graph

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contains an <<m+1, m-3>>, and for $1 \leq n \leq m - 3$ every (2k(m,n)-1)-chromatic graph contains an <<m,n-1>>. Every 7-chromatic graph contains a <<5,1>> and every 13-chromatic graph contains a <<5,0>>. Corresponding to any integers m,n, with $0 \leq n$, m-3 there exists a finite k(m,n). Every [k(m+1,n)-1]-chromatic graph contains an <<m,n>>.

2. Distance classes in a connected graph. Let γ be a connected graph and let A_o be a vertex of γ. If A is any vertex of γ other than A_o, then the number of edges contained in a path connecting A_o and A having least possible number of edges is called the distance between A_o and A, and any such path is called a geodesic. For $i = 1,2,3,\ldots$ let V_i denote the set of vertices of γ whose distance from A_o is i and let γ_i denote the subgraph of γ consisting of V_i and the edges of γ joining two vertices of V_i. The vertices of γ other than A_o are clearly partitioned into the distance-classes $V_1, V_2, V_3, \ldots$. We say that V_i and V_{i+1} are neighbouring distance classes, and so are A_o and V_1. It is easy to see that:

(1) Any edge of γ either joins two vertices in the same distance class or it joins two vertices in neighbouring distance classes;

(2) Each vertex of γ is connected to A_o by a geodesic, and each edge of a geodesic joins two vertices belonging to neighbouring distance classes, and no two vertices of a geodesic belong to the same distance class.

The union of p distinct paths such that one vertex, called the focus, is end-vertex of all the paths, and any two of the paths have the focus and nothing else in common, is called a p-pencil, the vertices of valency 1 are called end-vertices of the pencil. 1-pencils and 2-pencils are paths.

(3) Any two distinct vertices in the same distance-class V_i are connected by a path contained in γ of which no edge and no vertex except its end-vertices belongs to γ_i.

For each of the two vertices is connected to A_o by a geodesic, and the union of two such geodesics contains a path with the required property, by (2).

(4) Any three distinct vertices in the same distance-class V_i are the end-vertices of a 3-pencil contained in γ of which no edge and no vertex except its end-vertices belongs to γ_i.

For each of the three vertices is connected to A_o by a geodesic and the union of three such geodesics contains a 3-pencil with the required property, by (2).

We deduce from (3) and (4) that

(5) If γ_i contains an $<<m, o>>$ with $m \geq 3$ then γ contains an $<<m+1, m-3>>$, and if γ_i contains an $<<m, n>>$ with $1 \leq n \leq m-3$ then γ contains an $<<m, n-1>>$.

For let M_1, M_2, M_3 be any three branch-vertices of an $<<m, o>>$ in γ_i. By (4), γ contains a 3-pencil with end-vertices M_1, M_2, M_3 having nothing but M_1, M_2, M_3 in common with the $<<m, o>>$. The union of the $<<m, o>>$ and the 3-pencil is an $<<m+1, m-3>>$, the focus of the 3-pencil being the (m+1)-th branch vertex. In the second case let N_1 and N_2 be two branch-vertices of the $<<m, n>>$ having valency $< m-1$ in the $<<m, n>>$. By (3), N_1 and N_2 are connected by a path contained in γ having nothing but N_1, N_2 in common with the $<<m, n>>$. The union of the $<<m, n>>$ and the path is an $<<m, n-1>>$. (5) is now proved.

K. Wagner [3] pointed out that

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(6) If the chromatic number of γ is at least $2k-1$ then at least one of $\gamma_1, \gamma_2, \gamma_3, \ldots$ has chromatic number $\geq k$.

For otherwise colouring the vertices of $\gamma_1, \gamma_3, \gamma_5, \ldots$ from the stock of colours $1, \ldots, k-1$ and A_0 and the vertices of $\gamma_2, \gamma_4, \gamma_6, \ldots$ from the stock $k, \ldots, 2k-2$ would, by (1), furnish a colouring of γ with at most $2k-2$ colours.

3. <u>The proof of the theorem completed</u>. Suppose that γ is $(2k(m,o)-1)$-chromatic. By (6) at least one of $\gamma_1, \gamma_2, \gamma_3, \ldots$ has chromatic number $\geq k(m,o)$ and therefore contains an <<m, o>>. Hence by (5), γ contains a <<m+1, m-3>>. Similarly, for $1 \leq n \leq m-3$ every $(2k(m,n)-1)$-chromatic graph contains an <<m, n-1>>. Therefore, since every 4-chromatic graph contains a <<4, 0>> [1], it follows that every 7-chromatic graph contains a <<5, 1>>, and consequently every 13-chromatic graph contains a <<5, 0>>. Hence every 25-chromatic graph contains a <<6, 2>>, every 49-chromatic graph contains a <<6, 1>> and every 97-chromatic graph contains a <<6, 0>> etc.; corresponding to any integers m, n with $0 \leq n \leq m-3$ there exists a finite integer $k(m,n)$ such that every $k(m,n)$-chromatic graph contains a <<m, n>>. Needless to say, the bounds obtainable by this method are very far from best possible.

Let δ denote any $[k(m+1,n)-1]$-chromatic graph. Take a vertex not in δ and join it by an edge to every vertex of δ. The graph so obtained is $k(m+1,n)$-chromatic and therefore contains a <<m+1, n>>, hence δ contains a <<m, n>>.

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