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# HOPFIAN AND CO-HOPFIAN GROUPS

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The main results proved in this note are the following:

(i) Any finitely generated group can be expressed as a quotient of a finitely presented, centreless group which is simultaneously Hopfian and co-Hopfian.

(ii) There is no functorial imbedding of groups (respectively finitely generated groups) into Hopfian groups.

(iii) We prove a result which implies in particular that if the double orientable cover $N$ of a closed non-orientable aspherical manifold $M$ has a co-Hopfian fundamental group then $\pi_1(M)$ itself is co-Hopfian.

## 1. A CLASS OF CO-HOPFIAN GROUPS

DEFINITION 1: A group $G$ is said to be Hopfian (respectively co-Hopfian) if every surjective (respectively injective) endomorphism $f : G \to G$ is an automorphism.

We now recall the definition of a Poincaré duality group. Let $G$ be a group and $R$ a commutative ring. Let $n$ be an integer $\geqslant 0$. $G$ is said to be a duality group of dimension $n$ over $R$ (or an $R$-duality group of dimension $n$) if there exists a right $RG$-module $C$ such that one has *natural* isomorphisms $H^k(G; A) \simeq H_{n-k}\left(G; C \bigotimes_R A\right)$ for all $k \geqslant 0$ and all left $RG$-modules $A$. Here $C \bigotimes_R A$ is regarded as a right $RG$-module via $(x \otimes a)g = xg \otimes g^{-1}a$ for all $x \in C$, $a \in A$ and $g \in G$. The module $C$ occurring in the above definition is known as the dualising module. It turns out that in the above case $H^n(G; RG) = C$, the right $RG$ module structure on $RG$ yielding the right $RG$-module structure on $C$. In case $C \simeq R$ as a right *R-module*, $G$ will be called a Poincaré duality group of dimension $n$ over $R$. If further $C \simeq R$ with trivial right $G$-action then $G$ is called an orientable Poincaré duality group over $R$. By a Poincaré duality group (respectively orientable) we mean a Poincaré duality group (respectively orientable) over $\mathbb{Z}$. By a closed manifold we mean a compact manifold

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without boundary. By an aspherical space $X$ we mean a $K(\pi, 1)$ space for some group $\pi$.

A group $\pi$ is said to be of type $FP$ over $R$ if the trivial left $R\pi$-module $R$ admits a finitely generated projective resolution of finite length. We shall say that $\pi$ is of type $FP$ if it is of type $FP$ over $\mathbb{Z}$. It is well-known that a duality group $G$ over $R$ is of type $FP$ over $R$. [**2**] is an excellent reference for results on duality groups.

If $G$ is a group of type $FP$, the Euler characteristic $\chi(G)$ of $G$ can be defined as $\sum(-1)^k \dim H_k(G;\mathbb{Q})$ where $\mathbb{Q}$ is the trivial $G$-module $\mathbb{Q}$ and dimension of $H_k(G;\mathbb{Q})$ is as a vector space over $\mathbb{Q}$. If $G$ is of type $FP$ and $H$ any subgroup of $G$ of finite index then $H$ is also of type $FP$. In this case it is also known that $\chi(H) = [G:H]\chi(G)$ [**4, 2**, p.179].

**Proposition 1.** *Let $G$ be a group of type $FP$ satisfying the following conditions:*

(i) *Any subgroup $H$ of $G$ with $[G:H] = \infty$ is not isomorphic to $G$ and*
(ii) $\chi(G) \neq 0$.

*Then $G$ is co-Hopfian.*

Proof: Let $H$ be a proper subgroup of $G$. We need to show that $H$ is not isomorphic to $G$. If $[G:H] = \infty$, it is part of the hypothesis. Let $[G:H] < \infty$. Since $H \subsetneq G$, we have $[G:H] = k$ an integer $> 1$. It follows that $\chi(H) = k\chi(G)$. Since $\chi(G) \neq 0$ we see that $\chi(H) \neq \chi(G)$. Hence $H$ is not isomorphic to $G$. ∎

It is known that if $G$ is a duality group of dimension $n$ over $R$ then the cohomological dimension $cd_R G$ of $G$ over $R$ is $n$ [**2,** p.139]. A result of Strebel [**18**] asserts that if $G$ is a Poincaré duality group of dimension $n$ over $R$, then any subgroup $H$ of $G$ with $[G:H] = \infty$ satisfies $cd_R H \leqslant n-1$. Hence $H \not\simeq G$. As an immediate consequence of Proposition 1 we get the following (see [**2,** pp.178–179]).

**Proposition 2.** *Let $G$ be a Poincaré duality group satisfying $\chi(G) \neq 0$. Then $G$ is co-Hopfian.*

**Proposition 3.** *Let $M$ be a closed aspherical manifold satisfying $\chi(M) \neq 0$. Then $\pi_1(M)$ is co-Hopfian.*

By a surface we mean a compact, connected 2-dimensional manifold $N$ with or without boundary. In this case if $\delta N \neq \emptyset$ it is well-known that $\pi_1(N)$ is a free group of rank $\geqslant 0$ and that $\pi_1(N) = \{e\}$ if and only if $N \simeq D^2$. Any free group of rank $\geqslant 1$ is clearly not co-Hopfian. Denoting the closed orientable surface of genus $g$ by $\sum_g$ we have $\sum_0 \simeq S^2$. If we denote the non-orientable closed surface with $k$ cross-caps, $k \geqslant 1$ by $N_k$ we have $N_1 \simeq P^2$. It is well-known that $\sum_g$ for $g \geqslant 1$ and $N_k$ for $k \geqslant 2$

are all aspherical. Also $\chi\left(\sum_g\right) = 2 - 2g$ and $\chi(N_k) = 2 - k$. Thus $\chi\left(\sum_g\right) \neq 0$ for $g \geqslant 2$ and $\chi(N_k) \neq 0$ for $k \geqslant 3$. Also $\sum_1 \simeq S^1 \times S^1$ and $\pi_1\left(\sum_1\right) = \mathbb{Z} \oplus \mathbb{Z}$ is not co-Hopfian; $N_2 \simeq K$, the Klein bottle. It is known that $\sum_1$ covers itself non-trivially and $K$ covers itself non-trivially. In particular $\pi_1(K)$ is not co-Hopfian. One knows that $\pi_1(K) = \langle a, b;\ abab^{-1} = 1\rangle$. The subgroup $\mathrm{grp}\langle a^2, b\rangle$ is a proper subgroup of $\pi_1(K)$ which is isomorphic to $\pi_1(K)$. Clearly $\pi_1(S^2) = \{1\}$ and $\pi_1(P^2) \simeq \mathbb{Z}/2\,\mathbb{Z}$ are co-Hopfian. Thus Proposition 3 immediately yields the following:

**Theorem 1.**

(a) *The only closed surfaces $N$ with $\pi_1(N)$ non co-Hopfian are $S^1 \times S^1$ and $K$.*

(b) *The only surface $N$ with $\delta N \neq \emptyset$ and $\pi_1(N)$ co-Hopfian is $D^2$.*

Remarks. : (1) Theorem 1 could be known to experts. Since we can not find an explicit reference we have included a proof of it in our present paper. There is a lot of recent activity studying the co-Hopficity of fundamental groups of compact 3-manifolds [**6, 7, 8, 20**]. It is a well-known result that for any surface $N$, the fundamental group $\pi_1(N)$ is Hopfian [**1, 5, 10, 12**].

(2) If $G$ and $H$ are Poincaré duality groups it is well-known that $G \times H$ is a Poincaré duality group. If further $\chi(G) \neq 0 \neq \chi(H)$, from $\chi(G \times H) = \chi(G)\chi(H)$ we see that $\chi(G \times H) \neq 0$. Thus if $E = \{G \mid G$ a Poincaré duality group with $\chi(G) \neq 0\}$ then $E$ is closed under the formation of finite direct products and every $G \in E$ is co-Hopfian. In particular denoting $\pi_1\left(\sum_g\right) = \langle a_1, b_1, \ldots, a_g, b_g;\ \prod_{i=1}^{g}[a_i, b_i] = 1\rangle$ by $\Gamma_g$ and $\pi_1(N_k) = \langle a_1, a_2, \ldots\ \ldots, a_k;\ a_1^2 a_2^2 \ldots a_k^2 = 1\rangle$ by $E_k$ we see that for any choice of finite number of integers $g_i \geqslant 2$ and $k_j \geqslant 3$ the group $\Gamma_{g_1} \times \ldots \times \Gamma_{g_r} \times E_{k_1} \times \ldots \times E_{k_s}$ is co-Hopfian.

(3) A result of Gottlieb [**9**] asserts that if $X$ is a finite $K(G,1)$ complex with $\chi(X) \neq 0$, then $G$ is centreless. It follows that each one of the groups $\Gamma_{g_1} \times \ldots \times \Gamma_{g_r} \times E_{k_1} \times \ldots \times E_{k_s}$ where $g_i \geqslant 2$, $k_j \geqslant 3$ for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$ is centreless.

(4) Let $G$ be a torsion-free group and $H$ a subgroup of finite index in $G$. A result of Bieri [**3**] asserts that $G$ is a Poincaré duality group if and only if $H$ is. From $\chi(G) = [G : H]\chi(H)$ we see that $\chi(G) \neq 0 \Leftrightarrow \chi(H) \neq 0$. In the above situation if one of the groups is a Poincaré duality group with non-zero Euler characteristic then both the groups are co-Hopfian.

Recall that a group is said to be complete if it is centreless and its only automorphisms are inner automorphisms. Denote the cyclic group of order $m$ by $C_m$. Let $m$, $n$ be integers satisfying $m \geqslant 2$ and $n \geqslant 3$. In [**14**] Miller and Schupp prove the following:

**THEOREM.** *Any countable group $G$ is embeddable in a complete Hopfian group $H$ which is a quotient of $C_m * C_n$. In case $G$ is finitely presented, $H$ above is finitely presented. In case $G$ has no elements of order $m$ or $n$, the group $H$ constructed above will also be co-Hopfian.*

In their paper they give a specific construction for $H$. Of course $H$ will depend on $G$, $m$ and $n$. They also point out that if $G$ is the universal finitely presented group of Higman [**11**], $G$ can not be embedded in a finitely presented co-Hopfian group. When $G$ is this group of Higman, the construction of Miller and Schupp yields a finitely presented complete Hopfian group $H$ containing $G$, hence containing an isomorphic copy of every finitely presented group. Our earlier results allow us to prove the following:

**THEOREM 2.** *Let $G$ be any finitely generated group. Then $G$ can be expressed as a quotient of a finitely presented centreless group $H$ which is simultaneously Hopfian and co-Hopfian.*

PROOF: We can express $G$ as a quotient of a free group $F_g$ of rank $g$ for some integer $g \geqslant 2$.

Clearly $F_g$ is a quotient of $\Gamma_g$. If in the free group on $a_1, b_1, \dots \dots, a_g, b_g$ we set $b_1 = \dots = b_g = 1$ the relation $\prod_{i=1}^{g}[a_i, b_i] = 1$ is automatically satisfied and we get the free group on $a_1, \dots, a_g$ as a quotient of $\Gamma_g$. As already seen $\Gamma_g$ is a centreless, finitely presented group which is simultaneously Hopfian and co-Hopfian whenever $g \geqslant 2$. ∎

REMARK. (5) Evidently any direct factor of a co-Hopfian group is co-Hopfian. Using this observation it is easy to see that there exist finitely generated Abelian groups which are not quotients of co-Hopfian Abelian groups. In fact it is not true that an aribtrary finitely generated free Abelian group $A \neq 0$ ($\mathbb{Z}$, for instance) can be expressed as a quotient of an Abelian co-Hopfian group.

## 2. Orientation subgroup of index 2

Remark (4) in Section 1 applies to Poincaré duality groups with non-vanishing Euler characteristic. There are Poincaré duality groups $G$ with $\chi(G) = 0$ which are co-Hopfian, for instance $G = \pi_1(M)$ where $M$ is a closed 3-dimensional Haken manifold with positive Gromov invariant [**6, 7**]. Let $G$ be a non-orientable Poincaré duality group. Then the dualising module $H^n(G; \mathbb{Z}G)$ as an Abelian group is isomorphic to $\mathbb{Z}$ but the right $G$-action on $\mathbb{Z}$ will not be trivial. Hence $H = \{g \in G \mid m \cdot g = m$ for all $m \in \mathbb{Z}\}$ is a subgroup of index 2 in $G$. $H$ is an orientable Poincaré duality group of the same dimension $n$ as $G$. Every other subgroup of index 2 in $G$ is a non-orientable Poincaré duality group. In this section we shall show that if $H$ is co-Hopfian so is $G$. First we prove a general

**Lemma 1.** *Let $G$ be any group and $H$ a subgroup of $G$ satisfying the following conditions:*

(i) $[G:H]<\infty$

(ii) *for every injective endomorphism $\varphi : G \to G$ we have $\varphi(H) \subseteq H$.*

*Then if $H$ is co-Hopfian so is $G$.*

Proof: Since $[G:H]<\infty$, the only subgroup $E$ of $G$ satisfying $H \subseteq E$ and $[E:H]=[G:H]$ is $G$ itself. Let $\varphi : G \to G$ be any injective endomorphism. By assumption $\varphi(H) \subseteq H$, hence $\varphi \mid H : H \to H$ is an injective endomorphism. Since $H$ is a co-Hopfian $\varphi(H)=H$. Since $\varphi : G \to \varphi(G)$ is an isomorphism we get $[G:H]=[\varphi(G):\varphi(H)]=[\varphi(G):H]$. Our first observation implies that $\varphi(G)=G$. This proves the co-Hopficity of $G$. ▯

**Proposition 4.** *Let $G$ be a non-orientable Poincaré duality group and $H$ the orientable Poincaré duality subgroup of index 2 in $G$. Then for any injective endomorphism $\varphi : G \to G$ we have $\varphi(H) \subseteq H$.*

Proof: Let $\varphi$ be of dimension $n$. Then we know that $H$ is an orientable Poincaré duality group of dimension $n$. Hence $cd\ H=n$. Since $\varphi : H \simeq \varphi(H)$ we see that $cd\ \varphi(H)=n$. By the result of Strebel [**17**] we see that $[G:\varphi(H)]<\infty$. If $\varphi(H)\not\subseteq H$, then $\varphi(H)$ will be a non-orientable Poincaré duality group of dimension $n$. Hence $H_n(\varphi(H);\mathbb{Z})=0$ while $H_n(H;\mathbb{Z})=\mathbb{Z}$ [**2,** p.174]. Since $\varphi : H \to \varphi(H)$ is an isomorphism, this leads to a contradiction. ▯

**Theorem 3.** *Let $G$ be a non-orientable Poincaré duality group and $H$ the orientable Poincaré duality subgroup of index 2 in $G$. Then if $H$ is co-Hopfian so is $G$.*

Proof: Immediate consequence of Proposition 4 and Lemma 1. ▯

**Corollary 1.** *Let $M$ be a non-orientable closed aspherical manifold and $N \xrightarrow{p} M$ the orientable double cover of $M$. Then if $\pi_1(N)$ is co-Hopfian so is $\pi_1(M)$.*

Proof: Immediate consequence of Theorem 3. ▯

## 3. Embeddings into Hopfian groups

For the definition of an algebraically closed group the reader can refer to [**13,** Section 8, Chapter 14] or the paper of Scott [**16**] or [**19,** Section 3]. It was proved by W.R. Scott that any countable group $G$ can be embedded in a countable algebraically closed group. Practically the same proof yields the result that any infinite group $G$ can be embedded in an algebraically closed group having the same cardinality as $G$. A result of B.H. Neumann [**15**] asserts that any algebraically closed group is simple and not finitely generated. Since any simple group is clearly Hopfian we get the following:

**Proposition 5.** *Every group can be embedded in a Hopfian group.*

Let $\mathcal{G}$ denote the category of groups, $\mathcal{A}$, $\mathcal{C}$ certain collections of groups.

Definition 2: By a functorial embedding of groups from the class $\mathcal{A}$ into groups belonging to the class $\mathcal{C}$ we mean an embedding $G \overset{i_G}{\rightarrowtail} F(G)$ for every $G \in \mathcal{A}$ with $F(G) \in \mathcal{C}$ satisfying the following conditions:

(1) $F : \mathcal{A} \to \mathcal{C}$ is a covariant functor when we regard $\mathcal{A}$ and $\mathcal{C}$ as full subcategories of $\mathcal{G}$.

(2) For any morphism $f : G \to H$ in $\mathcal{A}$, the diagram

$$\begin{array}{ccc} G & \xrightarrow{i_G} & F(G) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F(f)} \\ H & \xrightarrow{i_H} & F(H) \end{array}$$

is commutative.

In [**19**] the second author has shown that there is no functorial embedding of groups into algebraically closed groups. Let $\mathcal{G}_{f \cdot g}$ (respectively $\mathcal{G}_{f \cdot p}$) denote the class of finitely generated (respectively finitely presented) groups. Let $\mathcal{H}$ denote the class of Hopfian groups and $\mathcal{H}_{f \cdot g} = \mathcal{H} \cap \mathcal{G}_{f \cdot g}$ , $\mathcal{H}_{f \cdot p} = \mathcal{H} \cap \mathcal{G}_{f \cdot p}$. In the present paper we shall show that there are no functorial embeddings of groups (respectively finitely generated groups) into Hopfian groups. Since $\mathcal{H}_{f \cdot g} \subset \mathcal{H}$, it follows that there is no functorial embedding of finitely generated groups into finitely generated Hopfian groups.

**Theorem 4.** *There is no functorial embedding of groups (respectively finitely generated groups) into Hopfian groups.*

Proof: Since $\mathcal{G}_{f \cdot g} \subset \mathcal{G}$, for proving Theorem 4, it suffices to show that there is no functorial embedding of finitely generated groups into Hopfian groups. If possible let $F : \mathcal{G}_{f \cdot g} \to \mathcal{H}$ be a functor and $i_G : G \rightarrowtail F(G)$ be an embedding for every $G \in \mathcal{G}_{f \cdot g}$ with $F(G) \in \mathcal{H}$ satisfying conditions (1) and (2) of Definition 2. In [**18**] Tyrer–Jones constructed a finitely generated group $G \neq \{e\}$ which is isomorphic to $G \times G$. Let $\theta : G \times G \to G$ be an isomorphism. Then $F(\theta) : F(G \times G) \to F(G)$ is an isomorphism. Let $j_i : G \to G \times G$, $p_i : G \times G \to G$ be the obvious inclusions and projections $(i = 1, 2)$, namely $j_1(g) = (g, e)$, $j_2(g) = (e, g)$, $p_1(g, h) = g$ and $p_2(g, h) = h$ for any $g \in G$, $h \in G$. From $p_1 \circ j_1 = 1d_G$ we get $F(p_1) \circ F(j_1) = 1d_{F(G)}$. Hence

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$F(p_1) : F(G \times G) \to F(G)$ is a surjective homomorphism. From the commutativity of

$$\begin{array}{ccc} G & \xrightarrow{i_G} & F(G) \\ \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle F(j_2)} \\ G \times G & \xrightarrow{i_{G\times G}} & F(G \times G) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle F(p_1)} \\ G & \xrightarrow[i_G]{} & F(G) \end{array}$$

we see that $F(j_2) \circ i_G(G) \subseteq \operatorname{Ker} F(p_1)$. Also $F(p_2) \circ F(j_2) = 1d_{F(G)}$ shows that $F(j_2) : F(G) \to F(G \times G)$ is an embedding. It follows that $\operatorname{Ker} F(p_1) \supseteq F(j_2) \circ i_G(G) \neq \{e\}$. Hence $F(G)$ is a proper quotient of $F(G \times G)$ and $F(\theta) : F(G \times G) \simeq F(G)$. This contradicts the Hopficity of $F(G \times G)$. ∎

REMARK. (6) It is not true that every Abelian group can be eimbedded in a Hopfian Abelian group. For instance any injective non-Hopfian Abelian group (for example, an infinite direct sum of copies of $\mathbb{Q}$; $\mathbb{Z}_{p^\infty}$ for any prime $p$) can not be embedded into a Hopfian Abelian group.

OPEN PROBLEMS.

(1) Does there exist a finitely presented group $G$ with $G$ isomorphic to $G \times G$? If such a group exists the proof of Theorem 4 shows that there is no functorial embedding of finitely presented groups into Hopfian groups.

(2) Is it true that every group can be expressed as a quotient of a suitable co-Hopfian group?

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