A HOMOMORPHISM IN EXTERIOR ALGEBRA

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1. Introduction. In the following, V is a vector space over an arbitrary field F, $\dim_F V = n$. Let $\{e^1, \ldots, e^n\}$ be a basis for V, and $\{f_1, \ldots, f_n\}$ be the dual basis for V^*, $\langle f_j, e^i \rangle = \delta^i_j$. If $u = e^1 \wedge \ldots \wedge e^p$ and $g = f_1 \wedge \ldots \wedge f_p$, then the operators $\epsilon(u)$ and $i(g)$ (exterior and inner multiplication by u and g respectively) set up an equivalence between the ideal $\mathfrak{J}$ = range of $\epsilon(u)$ and the sub-algebra $\mathfrak{A}$ = range of $i(g)$ considered as vector spaces. That is, $\epsilon(u)i(g)$ is the identity on $\mathfrak{J}$, $i(g)\,\epsilon(u)$ is the identity on $\mathfrak{A}$. Under this equivalence $\{u \wedge e^{i_1} \wedge \ldots \wedge e^{i_k}\}$ and $\{e^{i_1} \wedge \ldots \wedge e^{i_k}\}$ are corresponding bases of $\mathfrak{J}$ and $\mathfrak{A}$ respectively ($p < i_1 < \ldots < i_k \leqslant n$). While $\mathfrak{A}$ is a subalgebra of ΛV (namely ΛW, where $W \subset V$ is the space spanned by $e^{p+1}, \ldots, e^n$), $\mathfrak{J}$ is multiplicatively trivial, i.e., within $\mathfrak{J}$ all products vanish. Throughout ΛV is a generic relation for the exterior algebra over the vector space V and $\Lambda^p V$ for elements of degree p.

2. Below we establish that certain homomorphisms on ΛV induce homomorphisms on $\mathfrak{A} = \Lambda W$. Using the above equivalence of $\mathfrak{J}$ and $\mathfrak{A}$ we then establish a matrix identity ("Sylvester's identity") as a corollary.

LEMMA 1. *If $e \in V$ and $f \in V^*$, then*

$$i(f)\epsilon(e) + \epsilon(e)i(f) = \langle f, e \rangle I. \tag{1}$$

This is entirely standard. In fact

$$i(f)(x_1 \wedge \ldots \wedge x_p) = \sum_{i=1}^{p} (-1)^{i-1} \langle x_i, f \rangle x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_p$$

from which (1) follows when both sides are restricted, as operators, to decomposable elements $x_1 \wedge \ldots \wedge x_p$, $p = 1, \ldots, n$. The unrestricted validity of (1) then follows by linearity.

COROLLARY 1. *$i(f_k)$ and $\epsilon(e^j)$ anti-commute if $k \neq j$.*

LEMMA 2. *Any two of the following three statements imply the third, where P is a linear map on ΛV.*

(i) *P is a derivation.*

(ii) *The range of P is multiplicatively trivial (i.e. $Px \wedge Py = 0$ for all x, y).*

(iii) *$I - P$ is a homomorphism.*

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Proof.

(a) $(I - P)x \wedge (I - P)y = x \wedge y - (Px \wedge y + x \wedge Py) + Px \wedge Py.$

(b) $(I - P)(x \wedge y) = x \wedge y - P(x \wedge y).$

From (a) and (b) we have

$$(I - P)x \wedge (I - P)y - (I - P)(x \wedge y) = P(x \wedge y) - (Px \wedge y + x \wedge Py) + Px \wedge Py$$

from which the result is immediate.

COROLLARY 2. *If $e \in V$, $f \in V^*$, $\langle f, e\rangle \neq 0$, then*

$$\frac{i(f)\epsilon(e)}{\langle f, e\rangle}$$

is a homomorphism. In particular $i(f_k)\epsilon(e^k)$ is a homomorphism.

Proof. $\epsilon(e)\, i(f)$ is a derivation whose range is multiplicatively trivial. Use equation (1) and Lemma 2.

The next lemma replaces the e and f of Corollary 2 by u and g, decomposable elements in $\Lambda^p V$, $\Lambda^p V^*$, respectively.

LEMMA 3. *If $g \in \Lambda^p V^*$, $u \in \Lambda^p V$, g and u decomposable, and $\langle g, u\rangle = i(g)u \neq 0$, then $\lambda i(g)\epsilon(u)$ is a homomorphism on ΛV where the scalar λ is chosen so that $\lambda^{-1} = \langle g, u\rangle$.*

Proof. Let $u = u^1 \wedge \ldots \wedge u^p$. No non-zero element in the subspace of V^* determined by g can vanish on each of $u^1, \ldots, u^p$ (for then $\langle g, u\rangle$ would vanish), so there exist $g_1, \ldots, g_p$ such that $\langle g_k, u^j\rangle = \delta_k^j$ and $g = \lambda^{-1} g_1 \wedge \ldots \wedge g_p$. Further, since $\langle g, u\rangle = \langle \lambda^{-1} g_1 \ldots g_p, u^1 \ldots u^p\rangle$ we have $\lambda^{-1} = \langle g, u\rangle$. However,

$$\begin{aligned}\lambda i(g)\epsilon(u) &= i(g_1 \wedge \ldots \wedge g_p)\epsilon(u^1 \wedge \ldots \wedge u^p)\\ &= i(g_p) \ldots i(g_1)\epsilon(u^1) \ldots \epsilon(u^p).\end{aligned}$$

Using Corollary 1 and the anti-commutativity of $i(g_1), \ldots, i(g_p)$ among each other, we have $\lambda i(g)\epsilon(u) = i(g_1)\epsilon(u^1) i(g_2)\epsilon(u^2) \ldots i(g_p)\epsilon(u^p)$, which, by Corollary 2, is the product of p homomorphisms and hence a homomorphism, as desired.

The next theorem is an immediate consequence of Lemma 3.

THEOREM. *If A is a homomorphism of ΛV and g, u are decomposable elements of $\Lambda^p V^*$, $\Lambda^p V$ respectively, such that $\langle g, Au\rangle = \lambda^{-1} \neq 0$, then $\lambda i(g) A \epsilon(u)$ is a homomorphism of ΛV (and indeed, one which leaves the subalgebra $\mathfrak{A}$ invariant).*

Proof. $\lambda i(g) A\epsilon(u)(x \wedge y) = \lambda i(g)\epsilon(A(u))(Ax \wedge Ay)$ for x and y in ΛV. By Lemma 2 $\lambda i(g)\epsilon(A(u))$ is a homomorphism, so that

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$$\begin{aligned}\lambda i(g)\, A\epsilon(u)(x \wedge y) &= \lambda i(g)\ \epsilon(Au)Ax \wedge \lambda i(g)\ \epsilon(Au)Ay \\ &= \lambda i(g)\ A\epsilon(u)x \wedge \lambda i(y)\ A\epsilon(u)y\end{aligned}$$

as desired.

3. Application. Let $B_{\mathfrak{J}}$ denote the restriction to $\mathfrak{J}$ of $\epsilon(u)i(g)\lambda A$ and $B_{\mathfrak{A}}$ the restriction to $\mathfrak{A}$ of $i(g)\,\lambda A\epsilon(u)$. Then $B_{\mathfrak{J}} : \mathfrak{J} \to \mathfrak{J}$ and $B_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ are equivalent under the maps $i(g)$ and $\epsilon(u)$, i.e., there is commutativity in

$$\begin{array}{ccc} \mathfrak{J} & \xrightarrow{\quad B_{\mathfrak{J}}\quad} & \mathfrak{J} \\ i(g)\downarrow\uparrow\epsilon(u) & & \epsilon(u)\uparrow\downarrow i(g) \\ \mathfrak{A} & \xrightarrow[\quad B_{\mathfrak{A}}\quad]{} & \mathfrak{A} \end{array}$$

where λ is determined as in Lemma 3.

It follows that $B_{\mathfrak{J}}$ and $B_{\mathfrak{A}}$ have the same matrix

$$\lambda A\begin{pmatrix}1\dots p & i\\ 1\dots p & j\end{pmatrix} = B\begin{pmatrix}i\\ j\end{pmatrix}$$

with respect to the corresponding bases $\{u \wedge e^i\}$, $\{e^i\}$ $(i = p+1, \dots, n)$ in $\Lambda^{p+1} \cap \mathfrak{J}$ and $\Lambda^1 V \cap \mathfrak{A} = W$ respectively.

As a consequence of our theorem, $B_{\mathfrak{A}}$ is a homomorphism on $\mathfrak{A}$. We then have that $B_{\mathfrak{J}}$ on $\Lambda^{p+k} \cap \mathfrak{J}$ has the matrix

$$\lambda^k B\begin{pmatrix}i_1\dots i_k\\ j_1\dots j_k\end{pmatrix}$$

with respect to the $\mathfrak{J}$ basis $\{u \wedge e^{i_1} \wedge \dots \wedge e^{j_k}\}$ $p < i_1 < \dots < i_k \leqslant m$. Since the corresponding matrix for $B_{\mathfrak{A}}$ on $\Lambda^k V \cap \mathfrak{A}$ is

$$\lambda A\begin{pmatrix}1\dots p\, i_1\dots i_k\\ 1\dots p\, j_1\dots j_k\end{pmatrix},$$

we have

$$\lambda^{k-1}B\begin{pmatrix}i_1\dots i_k\\ j_1\dots j_k\end{pmatrix} = A\begin{pmatrix}1\dots p\, i_1\dots i_k\\ 1\dots p\, j_1\dots j_k\end{pmatrix}$$

or

$$B\begin{pmatrix}i_1\dots i_k\\ j_1\dots j_k\end{pmatrix} = \left(A\begin{pmatrix}1\dots p\\ 1\dots p\end{pmatrix}\right)^{k-1} A\begin{pmatrix}1\dots p\, i_1\dots i_k\\ 1\dots p\, j_1\dots j_k\end{pmatrix},$$

the well-known identity of Sylvester.

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