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RATIONAL APPROXIMATIONS TO e

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Dedicated to Kurt Mahler on his 75th birthday

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Abstract

The greatest lower bound (in fact, $\frac{1}{2}$) is found of constants k such that

$$\left| e - \frac{p}{q} \right| < k \frac{\log \log q}{q^2 \log q}$$

for an infinity of rationals p/q.

Corresponding results are given for the inequality if e be replaced by $e^{\pm 2/t}$, where t is a positive integer.

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Problems of irrationality and transcendence have occupied mathematicians for a long time. Over the years many general results have been obtained, in particular for measures of irrationality or of transcendence for certain classes of numbers. The numbers e and π are of obvious interest and Mahler (1932) gave measures of transcendence for these numbers. Whilst these results yield measures of irrationality as special cases, it is to be expected that these are far from the real truth and, indeed, Mahler (1953) found a much improved result for π.

Apart from quadratic irrationalities, perhaps the simplest number to yield reasonably precise results is e. For this number the simple continued fraction expansion was found by Euler (1737) and a more general analysis covering Euler's procedure was developed by Hurwitz (1891, 1896) to yield the corresponding expansions for $e^{2/t}$ for any positive integer t. (Euler had already given the expansion for $e^{1/t}$ and the expansion for e^2 was discovered by Sundman (1895).) Independent derivations of these results were subsequently given by Davis (1945) and by Walters (1968).

Asymptotic results on rational approximations to e have been derived from the simple continued fraction expansion by Adams (1966) and general results on rational approximations to e^a $(a \in \mathbf{Q})$ by Bundschuh (1971). In particular, the latter author exhibits positive constants c_1, c_2 such that

$$\left| e - \frac{p}{q} \right| > c_1 \frac{\log\log q}{q^2 \log q}$$

for all rationals p/q, while

$$\left| e - \frac{p}{q} \right| < c_2 \frac{\log\log q}{q^2 \log q}$$

has infinitely many solutions in rationals p/q. Similar results are given for $e^{\pm 1/t}$, $t \in \mathbf{N}$. Results of this character are implicit in the paper of Davis (1945) mentioned above, since this includes explicit formulae for the p_n, q_n of the convergents p_n/q_n to the simple continued fractions for $e^{2/t}$, and asymptotic expressions for these and for $|e^{2/t} - p_n/q_n|$.

Bundschuh's work is based essentially on the application to certain confluent hypergeometric functions of Kummer's relation, yielding rational functions approximating e^x. These are in fact simply the Padé approximants to e^x lying on the principal diagonal of the usual Padé table (see Perron (1957)). It was proved in Davis (1945) that for $x = 1/t$ $(t \in \mathbf{N})$ these, together with certain adjacent approximants, give the set of convergents to the simple continued fraction for $e^{1/t}$. A similar, but slightly more complicated result holds for $e^{2/t}$ with t an odd integer.

In view of the continuing interest in the problem it may be appropriate to outline the derivation of these results, with a somewhat simpler method of establishing the convergents. For brevity, details are confined to the case of e itself; the methods used apply generally and the paper just cited may be consulted for the general case. Specifically, the following result is proved here.

THEOREM 1. *For any $\varepsilon > 0$ there is an infinity of solutions of the inequality*

$$\left| e - \frac{p}{q} \right| < (\tfrac{1}{2} + \varepsilon) \frac{\log\log q}{q^2 \log q} \tag{1}$$

in integers p, q. Further, there exists a number $q' = q'(\varepsilon)$ such that

$$\left| e - \frac{p}{q} \right| > (\tfrac{1}{2} - \varepsilon) \frac{\log\log q}{q^2 \log q} \tag{2}$$

for all integers p, q with $q \geqslant q'$.

We first indicate the basic idea. If $f(t)$ is a polynomial and $a>0$, and we write

$$Q=\int_0^\infty e^{-at}f(t)\,dt,$$

we have, on putting $\int_0^\infty=\int_0^1+\int_1^\infty$,

$$Q=I+e^{-a}P,\quad\text{that is}\quad e^a-\frac{P}{Q}=\frac{e^a I}{Q}=\frac{J}{Q^2},$$

where

$$I=\int_0^1 e^{-at}f(t)\,dt,\quad P=\int_0^\infty e^{-at}f(t+1)\,dt\quad\text{and}\quad J=e^a IQ.$$

Hereafter we take $a=1$. We define, for $n\geqslant 0$

$$Q_n=\frac{1}{n!}\int_0^\infty e^{-t}t^n(t-1)^n\,dt,\quad S_n=\frac{1}{n!}\int_0^\infty e^{-t}t^n(t-1)^{n+1}\,dt,$$

$$U_n=\frac{1}{n!}\int_0^\infty e^{-t}t^{n+1}(t-1)^n\,dt,$$

with further integrals P_n, R_n, T_n corresponding respectively to these as P does to Q above, and I_n, J_n having the obvious significance. Then we have $Q_0=1$, $S_0=0$, $U_0=1$, and $P_0=1$, $R_0=1$, $T_0=2$.

LEMMA 1. *For* $n\geqslant 1$,

$$S_n=2nQ_n+U_{n-1},\quad Q_n=U_{n-1}+S_{n-1},\quad U_{n-1}=S_{n-1}+Q_{n-1}.$$

PROOF. Writing $1=t-(t-1)$ gives $Q_{n-1}=U_{n-1}-S_{n-1}$. Putting $\varphi=t(t-1)$, so that $\varphi'=2t-1=t+(t-1)$, we have

$$Q_n=\frac{1}{n!}\int_0^\infty e^{-t}\varphi^n\,dt=\frac{1}{n!}\int_0^\infty e^{-t}n\varphi^{n-1}\varphi'\,dt=U_{n-1}+S_{n-1}.$$

Finally,

$$n!\,U_n=\int_0^\infty e^{-t}t\varphi^n\,dt=\int_0^\infty e^{-t}(\varphi^n+tn\varphi^{n-1}\varphi')\,dt$$

$$=n!\,Q_n+n\int_0^\infty e^{-t}t\{2(t-1)+1\}\varphi^{n-1}\,dt=n!\,Q_n+2n.n!\,Q_n+n.(n-1)!\,U_{n-1}.$$

Hence

$$S_n=U_n-Q_n=2nQ_n+U_{n-1}.$$

COROLLARY. $S_n, Q_n, U_n\in\mathrm{N}$, *except that* $S_0=0$.

LEMMA 2. *For* $n \geqslant 1$,

$$R_n = 2nP_n + T_{n-1}, \quad P_n = T_{n-1} + R_{n-1}, \quad T_{n-1} = R_{n-1} + P_{n-1}.$$

PROOF. Immediate deduction from Lemma 1 on noting the relation between the P, R, T and the Q, S, U.

COROLLARY. $R_n, P_n, T_n \in \mathbf{N}$.

LEMMA 3. $|I_n| \sim \dfrac{n!}{(2n+1)!} e^{-\frac{1}{2}}, \quad Q_n \sim \dfrac{(2n)!}{n!} e^{-\frac{1}{2}}, \quad |J_n| \sim \dfrac{1}{2n}.$

PROOF. Expanding the exponential in the integral for I_n and integrating term-by-term, which is trivially justified, we obtain

$$I_n = \frac{n!}{(2n+1)!} \sum_{\nu=s}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{(n+1)\dots(n+\nu)}{(2n+2)\dots(2n+\nu+1)} = \frac{n!}{(2n+1)!} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} k_n(\nu),$$

say. Since $0 < k_n(\nu) < 1$ for $n \geqslant 0$, the series converges uniformly and so approaches $e^{-\frac{1}{2}}$ as $n \to \infty$.

In the integral for Q_n, expanding the binomial, integrating term-by-term and using Tannery's theorem yields the stated result.

The last result follows immediately.

LEMMA 4. *For sufficiently large n, P_n/Q_n is a convergent to the simple continued fraction for e.*

PROOF. For then

$$\left| e - \frac{P_n}{Q_n} \right| = \frac{|J_n|}{Q_n^2} < \frac{1}{2Q_n^2},$$

and the results follows from Legendre's criterion.

LEMMA 5. *If $a_1, a_2, \dots$ are positive integers, a_0 an integer,*

$$x_\nu = a_\nu x_{\nu-1} + x_{\nu-2} \quad (\nu \geqslant 1), \quad x_{-1} = 0, \quad x_0 = 1,$$

$$y_\nu = a_\nu y_{\nu-1} + y_{\nu-2} \quad (\nu \geqslant 1), \quad y_{-1} = 1, \quad y_0 = a_0,$$

and a sub-sequence of y_ν/x_ν tends to a number α, then

(i) $\alpha = [a_0, a_1, a_2, \dots]$;

(ii) *the convergents to α are y_ν/x_ν $(\nu \geqslant 0)$.*

PROOF. The y_ν/x_ν are the convergents to $\beta = [a_0, a_1, a_2, \dots]$, and $\beta = \alpha$, since $y_{\nu_i}/x_{\nu_i} \to \alpha$ for some sequence $\{\nu_i\}$.

LEMMA 6. *If* $e = [a_0, a_1, a_2, \ldots]$ *is a simple continued fraction, with convergents* p_ν/q_ν, *then*

$$a_0 = p_0 = T_0 = 2, \quad q_0 = U_0 = 1$$

and, for $n \geqslant 1$,

$$a_{3n-2} = 1, \quad a_{3n-1} = 2n, \quad a_{3n} = 1;$$

$$p_{3n-2} = P_n, \quad p_{3n-1} = R_n, \quad p_{3n} = T_n;$$

$$q_{3n-2} = Q_n, \quad q_{3n-1} = S_n, \quad q_{3n} = U_n.$$

PROOF. The conditions of Lemma 5 are satisfied on taking

$$(x_{-1}, x_0, x_1, x_2, x_3, \ldots) = (S_0, U_0, Q_1, S_1, U_1, \ldots),$$

$$(y_{-1}, y_0, y_1, y_2, y_3, \ldots) = (R_0, T_0, P_1, R_1, T_1, \ldots).$$

Since $P_n/Q_n \to e$, the results follow.

PROOF OF THEOREM 1. Applying Stirling's formula to the asymptotic expression for Q_n given in Lemma 3, we find

$$Q_n \sim \left(\frac{4n}{e}\right)^n \sqrt{\left(\frac{2}{e}\right)}.$$

Hence $\log Q_n \sim n \log n$ and so $n \sim (\log Q_n)/\log\log Q_n$. Thus

$$\left| e - \frac{P_n}{Q_n} \right| = \frac{|J_n|}{Q_n^2} \sim \frac{1}{2nQ_n^2} \sim \frac{1}{2} \frac{\log\log Q_n}{Q_n^2 \log Q_n}, \tag{3}$$

and (1) of Theorem 1 is satisfied on taking $p = P_n$, $q = Q_n$, with $n \geqslant n_0(\varepsilon)$.

The relation (2) of Theorem 1 is certainly satisfied by any p/q which is not a convergent, since then $|e - p/q| > 1/q^2$. For any convergent which is not a P_n/Q_n, the next partial quotient is 1, and so in this case $|e - p/q| > 1/(3q^2)$ and again (2) is satisfied. The result (3) above shows that, for any $\varepsilon > 0$, (2) holds for $p/q = P_n/Q_n$ if n is sufficiently large. This completes the proof.

We state here the more general result which may be obtained using the same ideas.

THEOREM 2. *If* $a = \pm 2/t$, *where* $t \in \mathbb{N}$, *and*

$$c = \begin{cases} 1/t, & t \text{ even}, \\ 1/(4t), & t \text{ odd}, \end{cases}$$

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then, for any $\varepsilon > 0$, *the inequality*

$$\left| e^a - \frac{p}{q} \right| < (c+\varepsilon) \frac{\log \log q}{q^2 \log q}$$

has an infinity of solutions in integers p, q. *Further, there exists a number* q', *depending only on* ε *and* t, *such that*

$$\left| e^a - \frac{p}{q} \right| > (c-\varepsilon) \frac{\log \log q}{q^2 \log q}$$

for all integers p, q *with* $q \geqslant q'$.

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