On the Attractions of Spherical and Ellipsoidal Shells.

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(*Received 1st June 1914. Read 13th June 1914.*)

So far as I am aware, the methods, described below, of finding the attraction of a uniform shell of matter on a particle placed at an external or an internal point, are new. They are particular applications of a method of finding the attraction of a thin shell of uniform volume density bounded by similar and similarly situated ellipsoidal surfaces (what some have called a *homothetic shell*, and Thomson and Tait an *elliptic homœoid*) which I explained in a paper, on the attraction of ellipsoidal shells and of solid ellipsoids, which was published in the *Philosophical Magazine* for April 1907. I give here also a short account of the more general problem with some additional notes and remarks. The solution depends on a geometrical theorem of some interest which occurred to me in thinking over the problem of the ellipsoidal shell, and its solution by Poisson by a laborious and somewhat difficult process of integration (*Mémoires de l'Institut*, t. xv., 1835).

1. Consider a thin uniform spherical shell, of radius a and mass σ per unit of area, attracting a particle of unit mass at a point P distant f from the centre C of the shell. (1) Let P be external to the shell as shown in Fig. 1. An element of area dS at E has mass σdS, and its attraction on the unit particle at P is $\kappa\sigma dS/r^2$, where r denotes the distance EP, and κ the gravitation constant. Resolving along the direction of the resultant attraction of the shell on the particle, putting θ for the angle APE, we obtain for the attraction F of the whole shell the equation

$$F = \kappa\sigma\int\frac{\cos\theta}{r^2}\,dS,\qquad\qquad(1)$$

where the integral is taken over the surface of the shell. Now, describing through P a surface concentric with the shell, producing CE to meet that surface in E′ and joining AE′, we see that $AE' = r$, and $\angle AE'E = \theta$. Moreover, by radial projection of dS from the centre C, we obtain an area dS' on the concentric surface such that $dS = dS' \,.\, a^2/f^2$. Replacing dS in equation (1) by this value we get, integrating over the concentric surface,

$$F = \kappa\sigma\frac{a^2}{f^2}\int\frac{\cos\theta}{r^2}\,dS' \qquad\ldots\ldots(2)$$

But, since $AE' = r$ and $\angle AE'E = \theta$, $dS'\cos\theta/r^2$ is the solid angle subtended at A by the element of surface dS'. The integral is

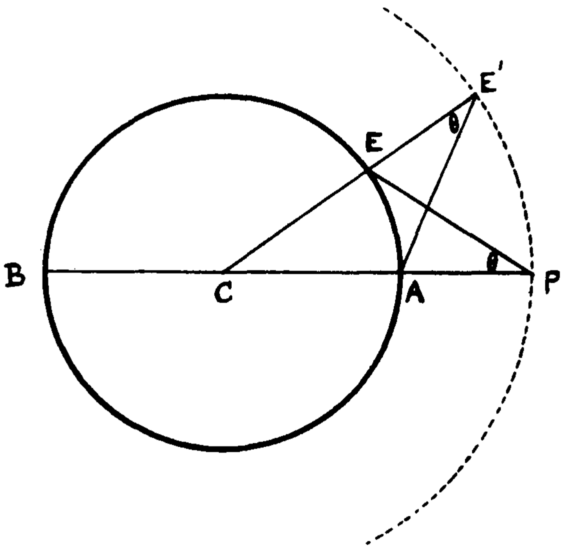

Fig. 1.

therefore the solid angle subtended by the whole concentric surface at the internal point A, or 4π. Hence

$$F = \kappa\frac{4\pi\sigma a^2}{f^2}, \qquad\ldots\ldots(3)$$

that is, the attraction is the same as if the whole mass were collected at C.

(2) Let P be an internal point, as shown in Fig. 2, and a concentric surface be described through P as before. Corre-

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sponding to dS there is an element of surface dS' as before on the concentric surface, and again $AE' = PE = r$, $\angle APE = \angle AE'E = \theta$. Thus, again,

$$F = \kappa \frac{a^2}{f^2} \int \sigma dS' \frac{\cos\theta}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Now, however, the integral is the solid angle subtended by the concentric surface at the *external* point A. This, of course, is zero, and therefore $F = 0$.

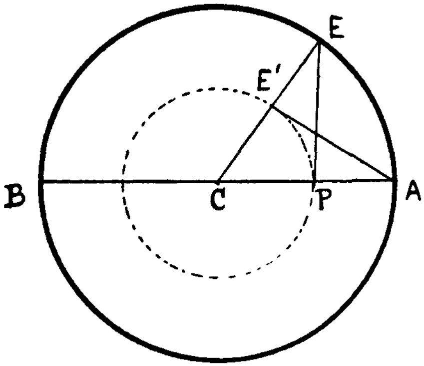

Fig. 2.

It will be noticed that, if the shell be incomplete, the attraction is in each case the product of $\kappa\sigma a^2/f^2$ by the proper value of the solid angle subtended at A by the part of the concentric surface corresponding to the incomplete shell. For example, let P be external to the shell, regarded as complete. It is obvious that if we divide the shell into two segments, by drawing the polar plane of P with respect to the shell, the solid angles subtended at A by the corresponding parts of the concentric surface are each 2π. Hence the shell is divided by the plane at right angles to CP drawn through the point P′, the inverse of P with respect to the shell, into two segments attracting equally the unit particle at P.

Of course this last result is obvious from another point of view. We have only to draw narrow cones through P so as to intersect the shell. The two elements intercepted by each cone have equal attractions on the particle at P. The elements fall into two sets, one set forming one of the two segments just specified the other set forming the other segment.

From the result we can easily find a solution of the problem of dividing a solid sphere into two parts which shall have equal attractions at any point P. Let the centre be C: on CP as diameter describe a sphere. This spherical surface divides the solid sphere into two parts which have equal attractions on a particle at P. These parts are lens-shaped, one, the farther from P, convexo-concave, the other convexo-convex. The spherical surface divides each of the concentric shells, of which the solid sphere may be regarded as built up, into two segments which join along the polar plane of P with respect to the shell.

This theorem holds whether the solid sphere be of uniform density or of density varying as a function of the distance from the centre. It also holds whether P be internal or external to the sphere. For the case of uniform density the theorem can be verified in this case at once, for the attractions of uniform spheres of different radii, but of the same density, on a unit particle in contact with the surface in each case, are proportional to the radii.

2. We now pass to the more general theorem. It was proved by Poisson, in the memoir referred to above, that the resultant attraction of an elliptic homœoid at an external point is directed along the internal axis, PQ, of the cone drawn from the external point as vertex to envelop the homœoid. The full significance of his theorem does not seem to have been perceived by Poisson. For that axis of the cone is the normal to the confocal ellipsoidal surface drawn through the external point, and the theorem shows that the family of external confocal ellipsoidal surfaces are the equi-potential surfaces of the homœoid, a fact of great importance in view of Green's theory of equivalent distribution. [This was contained in his celebrated *Essay* which was published privately in 1828, but remained unknown to the French mathematicians, and to the scientific world generally, until about twenty years later, when it was republished in Crelle's *Journal*.]

Refer to Fig. 3, which represents a section of the ellipsoidal shell, the attraction of which on a unit particle at P is to be considered, by a plane containing the internal axis PQ of the enveloping cone drawn from P as vertex. The confocal surface through P is indicated, and to this surface the axis in question is a normal. The equation of the confocal is

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=k\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and the equations of the outer and inner surfaces of the homœoid are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k-dk, \quad \ldots\ldots\ldots\ldots (6)$$

respectively.

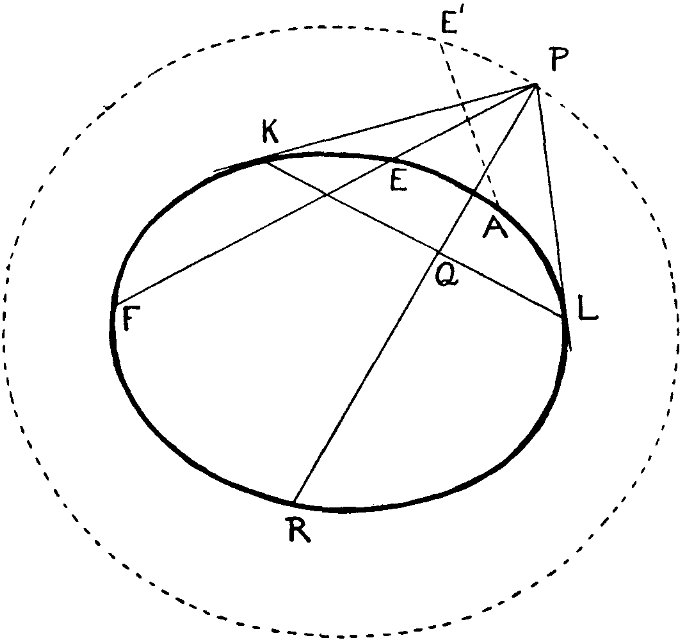

Fig. 3.

Imagine drawn from P as vertex a narrow cone of solid angle $d\omega$, intercepting two elements of the homœoid at E and F, and let the area of the element at E be dS. If the length of the perpendicular from the centre on the tangent plane at E be p, the mass of the element is $\frac{1}{2}p\rho dk d\mathrm{S}/k$. Now to dS we have by the theory of corresponding points on confocal surfaces a corresponding element at E′, the area of which, $d\Sigma$, is given by the equation

$$pdS = \frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\varpi d\Sigma \text{ ....}(7)$$

where ϖ is the length of the perpendicular from the centre on the tangent plane to the confocal at E′.

3. Before proceeding further it is necessary to explain and indicate a proof of the geometrical theorem by which the attraction integral is to be transformed. Let the point A on the surface of the shell correspond to P on the confocal. The point E corresponds to E′, and by the property of corresponding points AE′ = EP, that is, AE′ is also equal to r. Let ϖ_0 be the length of the perpendicular from the centre on the tangent plane to the confocal at P, and θ be the angle between the perpendicular (of length ϖ) from the centre on the tangent plane at E′ and the line AE′. Let the angle between the former perpendicular and the line EP be θ_0. [It is important to take the lines EP, AE′ in the senses indicated by the letters, and to take the perpendiculars both outward or both inward, to avoid any difficulty as to the signs of the cosines]. Then it can be proved that

$$\varpi\cos\theta_0 = \varpi_0\cos\theta \text{ ....} (8)$$

This is the geometrical theorem employed. I have not seen it referred to in works on geometry.

Of course the theorem also holds when the perpendiculars are those to the tangent planes which touch at the two points A, E on the shell. The proof in either case is easy: it is only necessary to express ϖ_0, ϖ, $\cos\theta_0$, $\cos\theta$, by means of the equation of the confocal and the coordinates of P and E′ when it is found that (8) is verified. The theorem of course is not confined to the particular case of *ellipsoids*.

Since an ellipsoid is confocal with itself, the theorem also holds when P coincides with A and E′ with E. Then EP coincides with EA and AE′ with AE. Hence if p_0, p be the lengths of the perpendiculars from the centre to the tangent planes at the extremities of a chord drawn to join any two points A, E on an ellipsoid, and

θ_0, θ be the angles between AE and p_0 and between EA and p respectively, we have again

$$p\cos\theta_0 = p_0\cos\theta. \qquad (8')$$

4. This latter case of the geometrical theorem enables us to prove Poisson's theorem of the direction of the resultant attraction, and to establish some other results. From P draw a narrow cone, of solid angle $d\omega$, intercepting elements of area dS_1, dS_2 at E, F on the shell, and let p_1, p_2 be the lengths of the perpendiculars from the centre on the tangent planes at E, F, θ_1, θ_2 the angles between these perpendiculars and EF, FE respectively. Then clearly we have $dS_1 = r_1^2 d\omega/\cos\theta_1$, $dS_2 = r_2^2 d\omega/\cos\theta_2$, where $r_1 = \text{PE}$, $r_2 = \text{PF}$. But by the statement made above, § 2, as to the mass of an element of the homœoid, the masses intercepted at E and F are

$$\tfrac{1}{2}p_1\rho dk dS_1/k, \quad \tfrac{1}{2}p_2\rho dk dS_2/k$$

which, by the values just found for dS_1, dS_2 are

$$\tfrac{1}{2}\rho dk r_1^2 d\omega p_1/k\cos\theta_1, \quad \tfrac{1}{2}\rho dk r_2^2 d\omega p_2/k\cos\theta_2.$$

The attractions on a particle of unit mass at P are therefore

$$\tfrac{1}{2}\rho dk d\omega p_1/k\cos\theta_1, \quad \tfrac{1}{2}\rho dk d\omega p_2/k\cos\theta_2,$$

and these are equal by (8′). It is clear, therefore, that the attracting elements fall into two sets, one set on the side of the polar plane towards P the other set on the side remote from P. The polar plane therefore divides the homœoid into two segments which exert equal attractions on a particle at P.

Moreover, for any narrow cone taken with its axis on one side of the axis of the enveloping cone, another cone of equal solid angle can be taken on the other side of the latter axis in the same plane with it, and equally inclined to it. By such pairs of cones the homœoids can be exactly exhausted, and so Poisson's theorem of the direction of the attraction at P is established.

A similar method to that just employed for the point P can be used to show that the attraction on a particle at any internal point is zero. It is only necessary to draw cones with their vertices at the point chosen, and apply the theorem (8′) as before. All reference to the homœoid as found by pure strain from a uniform spherical shell is thus rendered unnecessary. We shall obtain presently another proof of this theorem.

A proof of Poisson's theorem of the direction of the attraction was given by Steiner (*Crelle*, Bd. 12, 1834) after the *announcement* of the theorem in 1833. This, however, involved the assumption, to be justified by derivation of the homœoid from the spherical shell, that two elements intercepted by the same narrow cone, have masses in the ratio of the squares of their distances from the vertex of the cone. Steiner's proof is reproduced in the *Phil. Mag.* paper (*loc. cit.*).

5. We can now find the magnitude F of the resultant. We have, if r denote the distance EP, dS the area of an element at E, p the length of the perpendicular from the centre on the tangent plane at E, and θ_0 the angle EPQ,

$$\mathrm{F} = \tfrac{1}{2}\kappa\rho\frac{dk}{k}\int p\cos\theta_0\frac{d\mathrm{S}}{r^2} \quad \text{.........................(9)}.$$

The integration is taken over the homœoid. The difficulty in Poisson's investigation was the evaluation of this integral, and its accomplishment was a rather troublesome and lengthy process.

If in (9) we make the substitution given by (7), we get

$$\mathrm{F} = \tfrac{1}{2}\kappa\rho\frac{dk}{k}\;\frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\int\varpi\cos\theta_0\frac{d\Sigma}{r^2} \quad \text{.........(10)},$$

or by the relation (8), since ϖ_0 is the same for every element,

$$\mathrm{F} = \tfrac{1}{2}\kappa\rho\frac{dk}{k}\;\frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\varpi_0\int\frac{\cos\theta d\Sigma}{r^2}. \quad \text{.........(11)}$$

In (10) and (11) the integrations are to be taken over the confocal surface. Clearly the integral in (11) is the solid angle subtended at the point A by the surrounding confocal surface, and is therefore 4π. Hence we have

$$\mathrm{F} = 2\pi\kappa\rho\varpi_0\frac{dk}{k}\;\frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \quad \text{............(12)}$$

which is equivalent to Poisson's result. From this (see *Phil. Mag.*, *loc. cit. supra*) the potential of the homœoid at P, and at any other point is easily obtained.

We can proceed in exactly the same way when P is within the homœoid, by taking a confocal ellipsoidal surface through P. We have then for the equation of the confocal

$$\frac{x^2}{a^2-u}+\frac{y^2}{b^2-u}+\frac{z^2}{c^2-u}=k,$$

and get by the same process the same equation (12), except that u is replaced by $-u$. Here, however, the integral is the solid angle subtended by the closed surface of the confocal at the *external* point A, and is zero. Hence F is zero at P.

As in the case of the spherical shell, if the homœoid is incomplete its attraction is given by estimation of the corresponding solid angle subtended by the confocal at A.

Also, it can easily be shown that the parts of the confocal corresponding to the two segments, into which the shell is divided by the polar plane, subtend each an angle 2π at the point A, so that these segments exert equal attractions at the external point P, as proved more simply above.

6. It is clear that the surface of which the equation is

$$\frac{x^2-fx}{a^2}+\frac{y^2-gy}{b^2}+\frac{z^2-hz}{c^2}=0 \quad\text{.................}(13)$$

contains the intersections of all the polar planes of point P (coordinates f, g, h) with respect to the surfaces (common centre C) obtained by giving different values to k in the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k. \quad\text{............................}(14)$$

Hence if f, g, h be the coordinates of the point P at which the unit particle is situated, and an ellipsoid, of which the equation (referred to the same axes as are used for the homœoids) is (13), be described from the point of coordinates ($\frac{1}{2}f$, $\frac{1}{2}g$, $\frac{1}{2}h$) as centre, it will contain the intersections with the surfaces of all the polar planes of the point P with respect to the family of surfaces represented by (14). Thus it will divide each homœoid characterised by an assigned value of k into two segments which exert equal attractions on a particle at P.

Further, if a solid ellipsoid be constructed by placing together concentric homœoidal shells, of which the equations of the succes-

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sive surfaces are given by varying k, and which have each uniform volume density, varying in any manner from shell to shell, the ellipsoidal surface of which the equation is (13) will divide the solid ellipsoid into two parts which attract equally a unit particle at P. This holds even when P is within the solid ellipsoid. The centre of this surface is the mid-point of CP, and obviously CP is a diameter.

Thus the results stated in the first part of this paper for spherical distributions have their exact counterparts in the general case of ellipsoidal distributions.