BULL. AUSTRAL. MATH. SOC. 32A35, 46E10
VOL. 45 (1992) [163-170]

ON A CONJECTURE OF Z. JIANZHONG

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Let φ be a nonnegative, nondecreasing and nonconstant function defined on $[0,\infty)$ such that $\Phi(t) = \varphi(e^t)$ is a convex function on $(-\infty,\infty)$. The Hardy-Orlicz space $H(\varphi)$ is defined to be the class of all those functions f holomorphic in the open unit disc of the complex plane $\mathbf{C}$ satisfying $\sup_{0<r<1} \int_{-\pi}^{\pi} \varphi(|f(re^{it})|)\,dt < \infty$. The subclass $H(\varphi)^+$ of $H(\varphi)$ is defined to be the class of all those functions $f \in H(\varphi)$ satisfying $\sup_{0<r<1} \int_{-\pi}^{\pi} \varphi(|f(re^{it})|)\,dt = \int_{-\pi}^{\pi} \varphi(|f^*(e^{it})|)\,dt$, where $f^*(e^{it}) = \lim_{r\to1} f(re^{it})$ for almost all points e^{it} of the unit circle. In 1990, Z. Jianzhong conjectured that $H(\varphi)^+ = H(\psi)^+$ if and only if $H(\varphi) = H(\psi)$. In the present paper we prove that it is true not only on the unit disc of $\mathbf{C}$ but also on the unit ball of $\mathbf{C}^n$.

1. INTRODUCTION

Let $n \geqslant 1$ be an integer. Let $H(B)$ denote the space of all holomorphic functions in the open unit ball B of the complex n-dimensional Euclidean space $\mathbf{C}^n$. We call a nonnegative real-valued function φ defined on $[0,\infty)$ a modulus function if it is a nondecreasing and nonconstant function such that $\Phi(t) = \varphi(e^t)$ is a convex function on $(-\infty,\infty)$. According to Deeb and Marzuq [1], for a given modulus function φ, the Hardy-Orlicz space $H(\varphi)$ is defined as

$$H(\varphi) = \{f \in H(B) : \sup_{0<r<1} \int_S \varphi(|f(r\zeta)|)\,d\sigma(\zeta) < \infty\},$$

where $S = \partial B$ is the unit sphere of $\mathbf{C}^n$ and σ is the rotation invariant positive Borel measure on S for which $\sigma(S) = 1$. Let

$$H^+(B) = \{f \in H(B) : \lim_{r\to1} f(r\zeta) = f^*(\zeta) almost everywhere [\sigma] on S\}.$$

The space $H(\varphi)^+$ is defined to be the class of all those functions $f \in H^+(B) \cap H(\varphi)$ satisfying the condition

$$\sup_{0<r<1} \int_S \varphi(|f(r\zeta)|)\,d\sigma(\zeta) = \int_S \varphi(|f^*|)\,d\sigma.$$

Received 4 March 1991

Let $N(B)$ and $N^+(B)$ denote the Nevanlinna class and the Smirnov class respectively; that is,

$$N(B) = \{f \in H(B) : \sup_{0<r<1} \int_S \log^+ |f(r\zeta)|\, d\sigma(\zeta) < \infty\},$$

$$N^+(B) = \{f \in N(B) : \lim_{r\to 1} \int_S \log^+ |f(r\zeta)|\, d\sigma(\zeta) = \int_S \log^+ |f^*|\, d\sigma\}.$$

In [4, p.32, Remark 4], Jianzhong conjectured (for dimension $n = 1$) that for two modulus functions φ and ψ, $H(\varphi)^+ = H(\psi)^+$ if and only if $H(\varphi) = H(\psi)$. The main purpose of this paper is to prove that Jianzhong's conjecture is true for any dimension $n \geqslant 1$.

2. Inclusion relation between the spaces $H(\varphi)$

To prove the Proposition 1 described below, we recall some notations used in Rudin [9]. For $0 < p < \infty$, the Lebesgue spaces $L^p(\sigma)$ have their customary meaning. $L^0(\sigma)$ stands for the set of all measurable functions u for which

$$\int_S \log^+ |u|\, d\sigma < \infty.$$

LSC denotes the set of all lower semicontinuous functions on S. The following theorem is proved in Rudin [9, pp.19-20].

Theorem R. *Suppose $u \in LSC \cap L^0(\sigma)$, $u > 0$ on S. Then there is an $f \in N^+(B)$ whose boundary values f^* satisfy*

$$|f^*(\zeta)| = u(\zeta)$$

almost everywhere $[\sigma]$ on S.

The following Proposition 1 is proved in Hasumi and Kataoka [3, Theorem 1.3] for the case $n = 1$.

Proposition 1. *Let φ and ψ be modulus functions. If*

$$\varlimsup_{t\to\infty} \frac{\varphi(t)}{\psi(t)} = \infty,$$

then there exists an $f \in H(\psi) \cap N^+(B)$ which dose not belong to $H(\varphi)$.

Proof: The proof for arbitrary dimension n closely follows that of Hasumi and Kataoka for $n = 1$ [3, Proof of Theorem 1.3]. Put $\Phi(t) = \varphi(e^t)$, $\Psi(t) = \psi(e^t)$ for $-\infty \leqslant t < \infty$. Then Φ and Ψ are nondecreasing nonconstant convex functions on $[-\infty, \infty)$, and

$$\varlimsup_{t\to\infty} \frac{\Phi(t)}{\Psi(t)} = \infty.$$

Hence we can choose a sequence $\{t_j\}$ such that $0 < t_1 < t_2 < t_3 < \dots$, $\lim_{j\to\infty} t_j = \infty$, $\Psi(t_j) > 2^j j^{-2}$ and $\Phi(t_j)/\Psi(t_j) > j$, $j = 1,2,3,\dots$. Set $\varepsilon_j = \{j^2\Psi(t_j)\}^{-1}$, for each j. Then we see that $\varepsilon_j < 2^{-j}$, $j = 1,2,3,\dots$, and so $\sum \varepsilon_j < 1$. Consequently, there is a sequence $\{E_j\}$ of disjoint open subsets of the unit sphere S of $\mathbf{C}^n$ such that $\sigma(E_j) = \varepsilon_j$, $j = 1,2,3,\dots$. We define a function u on S by

$$u = \sum_{j=1}^{\infty} t_j \chi_j,$$

where χ_j is the characteristic function of the set E_j. Since E_j is an open subset of S, χ_j is lower semicontinuous on S, that is, $\chi_j \in LSC$. Since each number t_j is positive, it follows that $u \in LSC$. The function $\Psi \circ u$ is Borel measurable on S, and it holds that

$$\begin{aligned}\int_S (\Psi \circ u)\, d\sigma &= \sum_j \Psi(t_j)\sigma(E_j) + \Psi(0)\{1 - \sum_j \sigma(E_j)\}\\ &\leqslant \sum_j \Psi(t_j)\varepsilon_j + \Psi(0) = \sum_j j^{-2} + \Psi(0) < \infty,\end{aligned}$$

so we have $\Psi \circ u \in L^1(\sigma)$. Since Ψ is convex, nondecreasing and nonconstant, $\Psi(t) \geqslant Ct$ for some constant $C > 0$ and for all sufficiently large t. Thus we see that $u \in L^1(\sigma)$. On the other hand, the same way as in the case of $\Psi \circ u$ gives that

$$\begin{aligned}\int_S (\Phi \circ u)\, d\sigma &= \sum_j \Phi(t_j)\varepsilon_j + \Phi(0)\left(1 - \sum_j \varepsilon_j\right)\\ &\geqslant \sum_j j\Psi(t_j)\varepsilon_j = \sum_j j^{-1} = \infty.\end{aligned}$$

This means that $\Phi \circ u$ does not belong to $L^1(\sigma)$.

Now we put $v = e^u$ on S. Since $u \in LSC \cap L^1(\sigma)$ and $0 \leqslant u < \infty$, it follows that $v \in LSC \cap L^0(\sigma)$ and $1 \leqslant v < \infty$. By Theorem R, there exists an $f \in N^+(B)$ whose boundary values f^* satisfy $|f^*| = v$ almost everywhere $[\sigma]$. Since $f \in N^+(B)$, we have $\log|f| \leqslant P[\log|f^*|]$ in B, where P is the Poisson kernel in B.(See for example, Stoll [**10**, Lemma 3.1.]. It follows from Jensen's inequality that

$$\Psi(\log|f|) \leqslant P[\Psi \circ \log|f^*|] = P[\Psi \circ \log v] = P[\Psi \circ u]$$

in B, because Ψ is convex and nondecreasing on $(-\infty, \infty)$. Since $\Psi \circ u \in L^1(\sigma)$, $P[\Psi \circ u]$ is harmonic in B. Noting $\Psi(\log|f|) = \psi(|f|)$, we see that $f \in H(\psi)$. Finally, we shall show that f does not belong to $H(\varphi)$. By Fatou's lemma, we have

$$\begin{aligned}\varliminf_{r\to 1}\int_S \varphi(|f(r\zeta)|)\,d\sigma(\zeta) &\geqslant \int_S \varphi(|f^*|)\,d\sigma \\ &= \int_S \Phi(\log|f^*|)\,d\sigma \\ &= \int_S (\Phi\circ u)\,d\sigma = \infty.\end{aligned}$$

Thus f is not in $H(\varphi)$. This completes the proof. ∎

Now we consider the converse of Proposition 1. In the case of $n = 1$, the following Proposition 2 is proved in Jianzhong [**4**, Proposition 5]. (See also [**3**, Theorem 1.3] and [**5**, Theorem 2.1.]. The proof is the same for any dimension $n \geqslant 1$.

PROPOSITION 2. *Let φ and ψ be modulus functions. If*

$$\varlimsup_{t\to\infty}\frac{\varphi(t)}{\psi(t)} < \infty,$$

then $H(\psi) \subset H(\varphi)$.

Proposition 1 and Proposition 2 give the following

THEOREM 1. *Suppose φ and ψ are two modulus functions. Then the following hold:*

(1) $H(\psi) \subset H(\varphi)$ *if and only if*

$$\varlimsup_{t\to\infty}\frac{\varphi(t)}{\psi(t)} < \infty.$$

(2) $H(\psi) = H(\varphi)$ *if and only if*

$$\varlimsup_{t\to\infty}\frac{\varphi(t)}{\psi(t)} < \infty$$

and

$$\varliminf_{t\to\infty}\frac{\varphi(t)}{\psi(t)} > 0.$$

(3) $H(\psi) \subset H(\varphi)$ *and* $H(\psi) \neq H(\varphi)$ *if and only if*

$$\varlimsup_{t\to\infty}\frac{\varphi(t)}{\psi(t)} < \infty$$

and

$$\varliminf_{t\to\infty}\frac{\varphi(t)}{\psi(t)} = 0.$$

We remark that this is a generalisation of a result of Hasumi and Kataoka. They proved this in the case of the dimension $n = 1$. See [**3**, Theorem 1.3 and Corollary 4.1].

https://doi.org/10.1017/S0004972700037114 Published online by Cambridge University Press

3. Proof that $H(\varphi) \cap N^+(B) = H(\varphi)^+$

This equality is conjectured in Jianzhong [**4**, Remark 4]. In the case of $n = 1$, Hasumi and Kataoka [**3**, Theorem 2.1] proved that the equality is valid. To prove the general case we need the following lemmas:

LEMMA 1. *Suppose $\{u_j : j = 1,2,3,\ldots\}$ is a sequence of nonnegative $L^1(\sigma)$ functions such that $\lim u_j(\zeta) = u(\zeta)$ almost everywhere $[\sigma]$ on S. Then $\{u_j\}$ is uniformly integrable if and only if*

$$\lim_{j\to\infty} \int_S u_j \, d\sigma = \int_S u \, d\sigma < \infty.$$

PROOF: See Priwalow [**6**, Satz 3.2]. He proved the lemma for a compact interval $[a,b]$ in place of the unit sphere S, but the proof is the same for S. □

LEMMA 2. *Let $f \in H(B)$. Suppose that there is a real function $u \in L^1(\sigma)$ such that $\log|f| \leqslant P[u]$ in B. Then we have $f \in N^+(B)$.*

PROOF: (see Hahn [**2**, Theorem 4]; Rudin [**7**, Theorem 3.3.5.]) Put $u^+ = \max\{u, 0\}$. Then $u^+ \geqslant 0, u \leqslant u^+$ on S, and $u^+ \in L^1(\sigma)$. Since $\log|f| \leqslant P[u]$ in B, it follows that $\log^+|f| \leqslant P[u^+]$ in B. This shows $f \in N(B)$. Put $v = P[u^+]$ in B. For $0 < r < 1$ and $\zeta \in S$, we define $v_r(\zeta) = v(r\zeta)$. Then v is a positive harmonic function in B and $\{v_r : 0 < r < 1\} \subset L^1(\sigma)$. Hence we have

$$\lim_{r\to 1} \int_S v_r \, d\sigma = v(0) = \int_S u^+ \, d\sigma.$$

By Fatou's theorem (see for example, Rudin [**8**, Theorem 5.4.8.]),

$$v^*(\zeta) = \lim_{r\to 1} v_r(\zeta) = u^+(\zeta)$$

almost everywhere on S. Since $v_r \geqslant 0$ on S $(0 < r < 1)$, it follows from Lemma 1 that $\{v_r\}$ is uniformly integrable. Note that $\log^+|f_r| \leqslant v_r$ on S $(0 < r < 1)$. We therefore see that $\{\log^+|f_r| : 0 < r < 1\}$ is uniformly integrable. Consequently, Lemma 1 gives

$$\lim_{r\to 1} \int_S \log^+|f_r| \, d\sigma = \int_S \log^+|f^*| \, d\sigma.$$

This completes the proof. □

LEMMA 3. *For every modulus function φ, $H(\varphi) \subset N(B)$.*

PROOF: Put $\Phi(t) = \varphi(e^t)$. Then Φ is a nonnegative nonconstant nondecreasing convex function on $[-\infty, \infty)$, and so $\Phi(t) \geqslant Ct$ for some positive constant C and for

https://doi.org/10.1017/S0004972700037114 Published online by Cambridge University Press

all sufficiently large t. Now we define $\psi(t) = \log^+ t$ for $0 \leqslant t < \infty$ and $\Psi(t) = \psi(e^t)$ for $-\infty \leqslant t < \infty$. Then ψ is a modulus function and $H(\psi) = N(B)$. Moreover, it holds that $\Phi(t) \geqslant C\Psi(t)$ for all sufficiently large t. Hence we have

$$\varlimsup_{t\to\infty} \frac{\psi(t)}{\varphi(t)} = \varlimsup_{t\to\infty} \frac{\Psi(t)}{\Phi(t)} \leqslant C^{-1} < \infty.$$

It follows from Proposition 2 that $H(\varphi) \subset H(\psi) = N(B)$. □

Now we prove the following

Theorem 2. *For every modulus function φ, it holds that*

$$H(\varphi) \cap N^+(B) = H(\varphi)^+.$$

Proof: (see Hasumi and Kataoka [3, Theorem 2.1]; Rudin [7, Theorem 3.4.2.]) Suppose that $f \in H(\varphi)^+$. Put $\Phi(t) = \varphi(e^t)$ for $-\infty \leqslant t < \infty$. Then we have

$$\sup_{0<r<1} \int_S \Phi(\log|f(r\zeta)|)\, d\sigma(\zeta) = \int_S \Phi(\log|f^*|)\, d\sigma < \infty.$$

Since Φ is nonnegative, nonconstant, nondecreasing and convex on $[-\infty, \infty)$, there exists a positive finite Borel measure μ on S such that $\varphi(|f|) = \Phi(\log|f|) \leqslant P[\mu]$ in B and $\|\mu\| = \int_S \Phi(\log|f^*|)\, d\sigma$. (See for example, Rudin [8, Theorem 5.6.2.]). We set $u = P[\mu]$ in B. By Fatou's theorem, u has radial limits

$$u^*(\zeta) = \lim_{r\to 1} u(r\zeta)$$

for almost all $\zeta \in S$ $[\sigma]$ and $d\mu = u^* d\sigma + d\nu$, where $u^* \in L^1(\sigma)$ and ν is a finite positive singular Borel measure on S. Since $f \in H(\varphi)$, Lemma 3 gives $f \in N(B)$. Since $\varphi(|f|) \leqslant u$ in B, we have $\varphi(|f^*|) \leqslant u^*$ almost everywhere $[\sigma]$ on S. Consequently,

$$\begin{aligned}\|\mu\| = \int_S \Phi(\log|f^*|)\, d\sigma = \int_S \varphi(|f^*|)\, d\sigma \leqslant \int_S u^*\, d\sigma \\ \leqslant \int_S u^*\, d\sigma + \int_S d\nu = \|\mu\|.\end{aligned}$$

This shows $\nu = 0$, and so $\Phi(\log|f|) \leqslant u = P[u^*]$ in B. Since Φ is nonnegative, nonconstant, nondecreasing and convex on $(-\infty, \infty)$, there are two positive constants C_1 and C_2 such that $t \leqslant C_1\Phi(t) + C_2$ for all real t. Thus we have

$$\log|f| \leqslant C_1\Phi(\log|f|) + C_2 \leqslant C_1 P[u^*] + C_2 = P[C_1 u^* + C_2]$$

in B. Since $C_1 u^* + C_2 \in L^1(\sigma)$, it follows from Lemma 2 that $f \in N^+(B)$.

Conversely, we suppose $f \in H(\varphi) \cap N^+(B)$. Then $\log|f| \leqslant P[\log|f^*|]$ in B. By Jensen's inequality, we have

$$\varphi(|f|) = \Phi(\log|f|) \leqslant P[\Phi \circ \log|f^*|] = P[\varphi(|f^*|)]$$

in B. Since $\varphi(|f|)$ is subharmonic in B, Fatou's lemma gives

$$\int_S \varphi(|f^*|)\,d\sigma \leqslant \varliminf_{r\to 1} \int_S \varphi(|f(r\zeta)|)\,d\sigma(\zeta) \leqslant P[\varphi(|f^*|)](0) = \int_S \varphi(|f^*|)\,d\sigma.$$

Hence it follows that

$$\sup_{0<r<1} \int_S \varphi(|f(r\zeta)|)\,d\sigma(\zeta) = \lim_{r\to 1} \int_S \varphi(|f(r\zeta)|)\,d\sigma(\zeta) = \int_S \varphi(|f^*|)\,d\sigma.$$

This completes the proof. □

4. Proof of the main result

Now we can prove the main result of the present paper:

Theorem 3. *Let φ and ψ be modulus functions. Then $H(\varphi)^+ = H(\psi)^+$ if and only if $H(\varphi) = H(\psi)$.*

Proof: If $H(\varphi) = H(\psi)$, we have $H(\varphi)^+ = H(\psi)^+$ as an immediate consequence of Theorem 2. Conversely, suppose $H(\varphi)^+ = H(\psi)^+$. If $\varlimsup_{t\to\infty} \varphi(t)/\psi(t) = \infty$, then it follows from Proposition 1 that there is an $f \in H(\psi) \cap N^+(B)$ such that $f \notin H(\varphi)$. Theorem 2 gives $f \in H(\psi)^+$, but $f \notin H(\varphi)^+$. This contradicts the assumption $H(\varphi)^+ = H(\psi)^+$. So we have $\varlimsup_{t\to\infty} \varphi(t)/\psi(t) < \infty$. Similarly, we have $\varlimsup_{t\to\infty} \psi(t)/\varphi(t) < \infty$. By Theorem 1, we can thus conclude that $H(\varphi) = H(\psi)$. The proof is complete. □

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