

On the Bezdek–Pach Conjecture for Centrally Symmetric Convex Bodies

Dedicated to Ted Bisztriczky, on his sixtieth birthday.

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Abstract. The Bezdek–Pach conjecture asserts that the maximum number of pairwise touching positive homothetic copies of a convex body in \mathbb{R}^d is 2^d . Naszódi proved that the quantity in question is not larger than 2^{d+1} . We present an improvement to this result by proving the upper bound $3 \cdot 2^{d-1}$ for centrally symmetric bodies. Bezdek and Brass introduced the one-sided Hadwiger number of a convex body. We extend this definition, prove an upper bound on the resulting quantity, and show a connection with the problem of touching homothetic bodies.

1 Introduction

In this paper, \mathbb{R}^d denotes the d -dimensional Euclidean space, o is its origin, and \mathbb{S}^{d-1} is the unit sphere centered at o . A *convex body* K in \mathbb{R}^d is a compact, convex set with non-empty interior. A *positive homothetic copy* of K is a set of the form $\lambda K + t$ where $\lambda > 0$ and $t \in \mathbb{R}^d$. Two convex sets in \mathbb{R}^d are *non-overlapping* if their relative interiors are disjoint. Two non-overlapping convex sets with intersecting closures *touch* each other.

In 1971, C. M. Petty [8] proved that the maximum cardinality of a family of pairwise touching translates of a convex body K is at most 2^d with equality if, and only if, K is an affine image of a cube. As an extension of this problem, K. Bezdek and J. Pach [2] conjectured in 1988 that the maximum number of pairwise touching positive homothetic copies of a convex body K in \mathbb{R}^d is 2^d . They showed that any such family of homothetic copies has at most 3^d elements, and if C is a d -dimensional Euclidean ball, then the maximum is equal to $d + 2$. In 2006, M. Naszódi [7] improved the first estimate of Bezdek and Pach by proving the upper bound 2^{d+1} .

Our main goal is to investigate this problem for *o-symmetric* convex bodies, that is, for bodies K satisfying $K = -K$. First, we introduce some notions.

Definition 1.1 We call a set $S \subset \mathbb{R}^d$ *symmetrically antipodal* if for any distinct points $p, q \in S$, there is a hyperplane H such that $p \in H$, $q \in -H$, and both H and $-H$ support $\text{conv } S$.

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Definition 1.2 Let $K \in \mathbb{R}^d$ be an o -symmetric convex body and $S \subset K$. If the points of S are at pairwise distances 2 in the normed space with unit ball K , we say that S is a *diametral subset* of K .

H. Hadwiger [6] formulated the problem of determining the maximum number of pairwise non-overlapping translates of a convex body $K \subset \mathbb{R}^d$ touching K . This number is known as the *Hadwiger number* $H(K)$ of K . L. Fejes-Tóth [5] extended this concept to the notion of *generalized Hadwiger numbers* (see also [3]). Another variant is the *one-sided Hadwiger number* [1]. We combine the definitions of [1, 5] in the following manner.

Definition 1.3 Let $K \subset \mathbb{R}^d$ be an o -symmetric convex body and $\alpha > 0$. The *open* (resp. *closed*) *one-sided generalized Hadwiger number* $H_\alpha^+(K)$ (resp. $\overline{H}_\alpha^+(K)$) of K is the maximum number of pairwise non-overlapping translates of αK that touch K and whose translation vectors are in an open (resp. closed) half-space whose boundary contains the origin. Furthermore, $H_\infty^+(K)$ (resp. $\overline{H}_\infty^+(K)$) is the maximum number of pairwise non-overlapping translates of K that contain the origin and whose translation vectors are in an open (resp. closed) half-space whose boundary contains the origin.

Our results are summarized in Theorems 1.4, 1.5, and 1.6. Theorem 1.6 is an immediate consequence of Theorems 1.4 and 1.5.

Theorem 1.4 *The following statements are equivalent.*

- (i) *There is an o -symmetric convex body $K \subset \mathbb{R}^d$ which has n pairwise touching homothets.*
- (ii) *There is a symmetrically antipodal set $S \subset \mathbb{R}^{d+1}$ of n points such that $o \notin \text{conv } S$.*
- (iii) *There is an o -symmetric convex body $K \subset \mathbb{R}^{d+1}$ which contains a diametral set S of n points such that $o \notin \text{conv } S$.*
- (iv) *There is an o -symmetric convex body $B \subset \mathbb{R}^{d+1}$ with $H_\infty^+(B) \geq n$.*

Theorem 1.5 *Let $\alpha \in (0, \infty)$, $d \geq 2$, and K be an o -symmetric convex body in \mathbb{R}^d . Then*

$$(1.1) \quad \overline{H}_\alpha^+(K) \leq \frac{(1 + 2\alpha)^{d-1}(1 + 3\alpha) - 2}{2\alpha^d}$$

and

$$(1.2) \quad \overline{H}_\infty^+(K) \leq 3 \cdot 2^{d-2}.$$

Moreover, $\overline{H}_\infty^+(K) = 3 \cdot 2^{d-2}$ if and only if K is an affine d -cube.

Theorem 1.6 *The cardinality of a family of pairwise touching positive homothetic copies of an o -symmetric convex body in \mathbb{R}^d is strictly less than $3 \cdot 2^{d-1}$.*

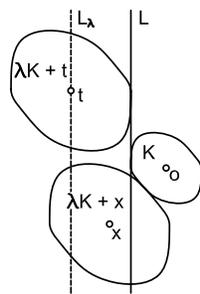


Figure 1

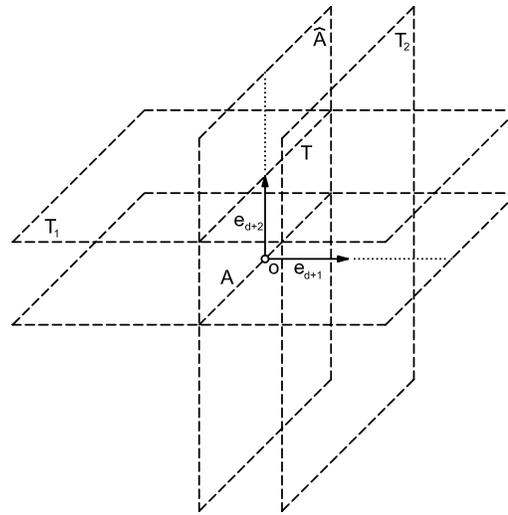


Figure 2

2 Proof of Theorem 1.4

We need the following, rather technical, definition and a lemma.

Definition 2.1 Let $K \subset \mathbb{R}^d$ be a convex body that contains o in its interior, and let $\lambda > 0$. Furthermore, let L be a hyperplane of \mathbb{R}^d and L^+ be one of the two closed half-spaces of \mathbb{R}^d bounded by L . Then we call the set $L_\lambda := \{t \in L^+ : \lambda K + t \text{ touches } L\}$ the λK -translate of L into L^+ .

We note that L_λ is a translate of L .

Lemma 2.2 Let $K \subset \mathbb{R}^d$ be an o -symmetric convex body, and $\lambda K + x$ be a positive homothet of K touching K . Let L be a supporting hyperplane of K , and L_λ be the λK -translate of L into the half-space that does not contain K . Then x is in the closed half-space that is bounded by L_λ and contains K (see Figure 1).

Proof Suppose that x is not in the closed half-space that is bounded by L_λ and contains K . Then by the definition of L_λ we have that $\lambda K + x$ is contained in the open half-space that is bounded by L and does not contain K . Hence, $\lambda K + x$ does not touch $\lambda K + x$, a contradiction. ■

Proof of Theorem 1.4 Let e_1, \dots, e_{d+2} be the standard basis of \mathbb{R}^{d+2} . To prove the equivalence of (i) and (ii), we regard \mathbb{R}^{d+1} and \mathbb{R}^d as the affine subspaces

$$\hat{A} := \{x \in \mathbb{R}^{d+2} : \langle x, e_{d+1} \rangle = 0\}$$

and

$$A := \{x \in \mathbb{R}^{d+2} : \langle x, e_i \rangle = 0 \text{ for } i = d + 1, d + 2\}$$

of \mathbb{R}^{d+2} , respectively. We also need the following affine subspaces:

$$\begin{aligned} T_1 &:= \{v \in \mathbb{R}^{d+2} : \langle v, e_{d+2} \rangle = 1\}; \\ T_2 &:= \{v \in \mathbb{R}^{d+2} : \langle v, e_{d+1} \rangle = 1\}; \\ T &:= \{v \in \mathbb{R}^{d+2} : \langle v, e_{d+2} \rangle = 1 \text{ and } \langle v, e_{d+1} \rangle = 0\}. \end{aligned}$$

Note that T_1 and T_2 are hyperplanes in \mathbb{R}^{d+2} , T is a d -flat, and $T \subset T_1$. (see Figure 2).

First, we prove that (i) implies (ii). Consider a family $\{\lambda_i K + x_i : i = 1, 2, \dots, n\}$ of pairwise touching homothets of the o -symmetric d -dimensional convex body K in \mathbb{R}^d . We let $K_i := \lambda_i K + x_i + e_{d+2}$ and note that $\{K_i : i = 1, 2, \dots, n\}$ is also a family of pairwise touching homothets of K , which is contained in T . We set

$$X' := \{x'_i := x_i + \lambda_i e_{d+1} + e_{d+2} \in \mathbb{R}^{d+2} : i = 1, 2, \dots, n\}.$$

Observe that $X' \subset T_1$.

For any pair of distinct indices i, j , let $L(i, j)$ be an affine $(d - 1)$ -flat which separates K_i and K_j in T . Furthermore, let $H_k(i, j) := \text{aff}(L(i, j) \cup \{x'_k\})$ for $k \in \{i, j\}$. Observe that $H_i(i, j), H_j(i, j) \subset T_1$, and they partition T_1 into four closed convex cones, exactly one of which contains both x'_i and x'_j . We denote this cone by $C(i, j)$.

By Lemma 2.2 and the construction of X' , we have that X' is a subset of $C(i, j)$. Thus, we have shown the following property of X' .

- (*) For every pair of distinct indices i, j there is a cone $C(i, j)$ in T_1 containing X' . The cone $C(i, j)$ is the intersection of two closed half-spaces Z_i and Z_j of T_1 , such that $x'_i \in \text{relbd } Z_i, x'_j \in \text{relbd } Z_j$, and $L(i, j) := (\text{relbd } Z_i) \cap (\text{relbd } Z_j)$ is an affine subspace of T with $\dim L(i, j) = d - 1$. Furthermore, $C(i, j)$ is symmetric about any line of the form $\{z + t e_{d+1} : t \in \mathbb{R}\}$, where $z \in L(i, j)$.

Let X'' denote the image of X' under the central projection from the origin o onto the hyperplane T_2 of \mathbb{R}^{d+2} . More specifically, let

$$X'' := \left\{ \frac{1}{\lambda_i} x_i + e_{d+1} + \frac{1}{\lambda_i} e_{d+2} : i = 1, \dots, n \right\}.$$

Note that X'' is contained in the open half-space $\{v \in T_2 : \langle v, e_{d+2} \rangle > 0\}$ of T_2 . Consider the image S of X'' under the orthogonal projection of T_2 onto the hyperplane $\hat{A} = \mathbb{R}^{d+1}$. From (*) and the previous remark, S is a symmetrically antipodal set in \mathbb{R}^{d+1} such that $o \notin \text{conv } S$.

Now we prove that (ii) implies (i). Let $S \subset \mathbb{R}^{d+1}$ be a symmetrically antipodal set of n points such that $o \notin \text{conv } S$. Reversing our consideration in the previous part, we may clearly construct a set $X' = \{x'_i : i = 1, 2, \dots, n\}$ such that $X' \subset T_1, \lambda_i := \langle x'_i, e_{d+1} \rangle > 0$, and X' satisfies (*).

Let $x_i := x'_i - \lambda_i e_{d+1}$ for $i = 1, 2, \dots, n$. Clearly, $x_i \in T$. Set

$$K := \text{conv} \left\{ \frac{x_i - x_j}{\lambda_i + \lambda_j} : i, j = 1, \dots, n \right\}$$

and note that K is an o -symmetric compact, convex set in $A = \mathbb{R}^d$.

Assume that $\dim K = d$. We want to show that $\{\lambda_i K + x_i - e_{d+2} : i = 1, 2, \dots, n\}$ is a family of pairwise touching positive homothets of the convex body K in $A = \mathbb{R}^d$. Clearly, it is sufficient to show that $K_i := \lambda_i K + x_i$ and $K_j := \lambda_j K + x_j$ touch in T for every $i \neq j$.

Since $x_i + \frac{\lambda_i}{\lambda_i + \lambda_j}(x_j - x_i) = x_j + \frac{\lambda_j}{\lambda_i + \lambda_j}(x_i - x_j)$, we obtain that K_i and K_j intersect. It remains to show that, say, K_1 and K_2 are separated by a $(d - 1)$ -dimensional affine subspace in T .

Let $L := L(1, 2)$ be the $(d - 1)$ -dimensional affine subspace in T described in (*). Then

$$L = \{x \in T : \langle x, u \rangle = 0\} = \{x \in T_1 : \langle x, u \rangle = \langle x, e_{d+1} \rangle = 0\}$$

for some vector $u \in A = \mathbb{R}^d$. We may assume that, say, $\langle u, x_1 \rangle > 0$. Then the symmetry of the cone $C := C(1, 2)$ in (*) yields that $\langle u, x_2 \rangle < 0$.

Let $u_i := u - \frac{\langle u, x_i \rangle}{\lambda_i} e_{d+1}$ for $i = 1, 2$. It is simple to check that $\langle u_i, x \rangle = 0$ for any $x \in L$, and $\langle u_i, x'_i \rangle = 0$. An easy computation yields that $\langle u_1, x'_2 \rangle < 0$ and $\langle u_2, x'_1 \rangle > 0$, and thus

$$C = \{x \in T_1 : \langle u_1, x \rangle \leq 0 \text{ and } \langle u_2, x \rangle \geq 0\}.$$

Observe that the symmetry of the cone implies that $\langle u, x_2 \rangle / \lambda_2 = -\langle u, x_1 \rangle / \lambda_1$.

From (*) it follows that $\langle u_1, x'_k \rangle \leq 0$ and $\langle u_2, x'_k \rangle \geq 0$ for $k = 1, 2, \dots, n$. After substituting the definitions of u_1 and u_2 , and $x'_k = x_k + \lambda_k e_{d+1}$, we have that

$$(2.1) \quad -\frac{\langle u, x_1 \rangle}{\lambda_1} \leq \frac{\langle u, x_k \rangle}{\lambda_k} \leq \frac{\langle u, x_1 \rangle}{\lambda_1}.$$

We show that L separates K_1 and K_2 . By symmetry, it is sufficient to show that $\langle u, x \rangle \geq 0$ for any $x \in K_1$. By (2.1), for every j, k we have

$$\begin{aligned} \left\langle u, x_1 + \frac{\lambda_1}{\lambda_j + \lambda_k}(x_j - x_k) \right\rangle &= \langle u, x_1 \rangle + \frac{\lambda_1}{\lambda_j + \lambda_k} (\langle u, x_j \rangle - \langle u, x_k \rangle) \\ &\geq \langle u, x_1 \rangle + \frac{\lambda_1}{\lambda_j + \lambda_k} \left(-\frac{\lambda_j}{\lambda_1} \langle u, x_1 \rangle - \frac{\lambda_k}{\lambda_1} \langle u, x_1 \rangle \right) = 0, \end{aligned}$$

which yields our statement if $\dim K = d$. If $\dim K < d$, a similar argument may be applied.

Now we show that (ii) implies (iii). Let $S \subset \mathbb{R}^{d+1}$ be a symmetrically antipodal set such that $o \notin \text{conv } S$. For any pair of distinct points $p, q \in S$, let $H_{p,q}$ denote a hyperplane such that $p \in H_{p,q}$, $q \in -H_{p,q}$, and both $H_{p,q}$ and $-H_{p,q}$ support S . Let $S_{p,q}$ be the strip bounded by $H_{p,q}$ and $-H_{p,q}$, and let C be an o -symmetric cube with $S \subset C$. Set

$$K := C \cap \left(\bigcap_{\substack{p,q \in S \\ p \neq q}} S_{p,q} \right).$$

Note that K is an o -symmetric convex body in \mathbb{R}^{d+1} , S is a diametral set in K , and $o \notin \text{conv } S$. Thus (ii) yields (iii). Since (iii) clearly implies (ii), we have that (ii) and (iii) are equivalent.

The equivalence of (iii) and (iv) is easy to show, hence we omit it. Observe that S in (iii) corresponds to the set of translation vectors in (iv). ■

3 Proof of Theorem 1.5

Our proof is based on the idea of the proof of the theorem of [1]. We begin with a variant of the lemma of [1].

Lemma 3.1 *Let f be a function on $[0, 1]$ with the properties $f(0) \geq 0$, f is positive and monotone increasing on $(0, 1]$, and $f(x) = (g(x))^k$ for some concave function g with $k > 0$. Then*

$$F(x) := \frac{1}{f(x)} \int_0^x f(z) \, dz$$

is strictly increasing on $(0, 1]$.

Proof of Lemma 3.1 Let $0 < x < y \leq 1$. We want to show that $F(x) < F(y)$ or, equivalently, that

$$\frac{f(y) - f(x)}{f(x)} \int_0^x f(z) \, dz < \int_x^y f(z) \, dz.$$

This inequality trivially holds if $f(x) = f(y)$, and thus we may assume that $f(x) < f(y)$. Let $L(z) := \frac{g(y)-g(x)}{y-x}(z-x) + g(x)$ for $z \in [0, y]$. Note that since $g(z)$ is concave and $k > 0$, we have $f(z) \geq (L(z))^k$ for every $z \in [x, y]$, and $f(z) \leq (L(z))^k$ for every $z \in (0, x]$. Thus, it is sufficient to prove that

$$(3.1) \quad T := \int_x^y (L(z))^k \, dz - \frac{(g(y))^k - (g(x))^k}{(g(x))^k} \int_0^x (L(z))^k \, dz > 0.$$

After integrating and simplifying, we obtain that

$$T = \frac{(g(y))^k(y-x)}{k+1} + \frac{((g(y))^k - (g(x))^k)(yg(x) - xg(y))^{k+1}}{(g(x))^k(k+1)(g(y) - g(x))(y-x)^k}.$$

Note that from $0 < f(z)$, $z \in (0, 1]$, we have that $L(0) \geq 0$, which yields that $0 < \frac{g(y)}{y} \leq \frac{g(x)}{x}$. Thus, the second member of T is nonnegative, and the inequality in (3.1) immediately follows. ■

Proof of Theorem 1.5. First we present a detailed proof of (1.2) and discuss the case of equality. Next, we outline the proof of (1.1), which is essentially the same as that of (1.2).

Let $v \in S^{d-1}$ and let $K+t_1, K+t_2, \dots, K+t_n$ be pairwise non-overlapping translates of K such that $o \in K+t_i$ and $a_i := \langle t_i, v \rangle \geq 0$ for $i = 1, 2, \dots, n$. Set

$$h(x) := \{p \in \mathbb{R}^d : \langle p, v \rangle = x\}.$$

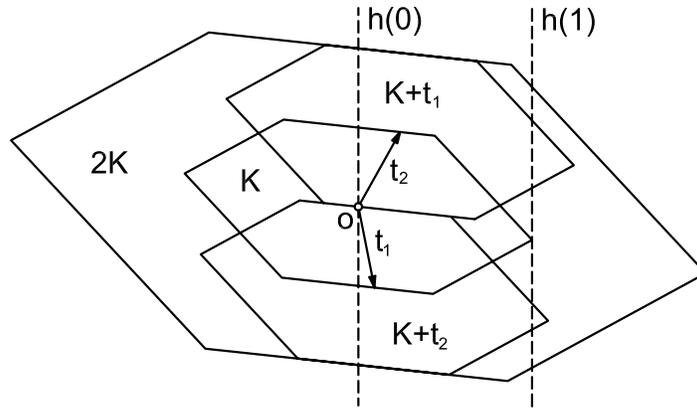


Figure 3

Without loss of generality, we may assume that $h(-1)$ and $h(1)$ are supporting hyperplanes of K . Note that $K + t_i$ is between $h(-1)$ and $h(2)$, and it is contained also in $2K$, for $i = 1, \dots, n$ (see Figure 3). Let $V_d(\cdot)$ (resp. $V_{d-1}(\cdot)$) denote d -dimensional (resp. $(d - 1)$ -dimensional) volume. Then

$$(3.2) \quad \int_{-1}^2 V_{d-1}\left(\left(\bigcup_{i=1}^n K + t_i\right) \cap h(x)\right) dx = nV_d(K).$$

We separate the above integral into two parts and estimate them separately. First note that

$$(3.3) \quad \int_0^2 V_{d-1}\left(\left(\bigcup_{i=1}^n K + t_i\right) \cap h(x)\right) dx \leq \int_0^2 V_{d-1}(2K \cap h(x)) dx = 2^{d-1}V_d(K).$$

Set $f(x) := V_{d-1}(K \cap h(x - 1))$, and observe that $f(0) \geq 0$, f is positive and monotone increasing on $(0, 1]$, and $f(\cdot)^{\frac{1}{d-1}}$ is concave by the Brunn–Minkowski inequality. By Lemma 3.1,

$$\begin{aligned} \int_{-1}^0 V_{d-1}\left(\left(\bigcup_{i=1}^n (K + t_i)\right) \cap h(x)\right) dx &= \sum_{i=1}^n \int_0^1 V_{d-1}(K \cap (h(x - 1) - t_i)) dx \\ &= \sum_{i=1}^n \int_0^{1-a_i} f(x) dx \leq \sum_{i=1}^n \int_0^1 f(x) dx \frac{f(1-a_i)}{f(1)} \\ &= \frac{V_d(K)}{2f(1)} \sum_{i=1}^n V_{d-1}(K \cap h(-a_i)) \end{aligned}$$

$$\begin{aligned} &= \frac{V_d(K)}{2f(1)} V_{d-1} \left(\left(\bigcup_{i=1}^n (K + t_i) \right) \cap h(0) \right) \leq \frac{V_d(K)}{2f(1)} V_{d-1}(2K \cap h(0)) \\ &= \frac{1}{2} V_d(K) \frac{V_{d-1}(2K \cap h(0))}{V_{d-1}(K \cap h(0))} = 2^{d-2} V_d(K). \end{aligned}$$

This inequality, combined with (3.2) and (3.3), yields (1.2).

Now we prove the equality part of the theorem. Assume that $n = 3 \cdot 2^{d-2}$. Then by Lemma 3.1, we have two possibilities. One is that $f(0) = 0$, and $a_i = 0$ or $a_i = 1$ for each $i = 1, \dots, n$. The other one is that $f(0) > 0$ and $a_i = 0$ for each $i = 1, \dots, n$.

If $f(0) > 0$, then $a_i = 0$ for each $i = 1, \dots, n$, and hence,

$$\{(K + t_i) \cap h(0) : i = 1, 2, \dots, n\}$$

is a family of pairwise non-overlapping translates of $K \cap h(0)$ in $h(0)$, each of which contains o . Thus, [8] implies that $n \leq 2^{d-1}$, which contradicts our assumption.

Assume that $f(0) = 0$, and $a_i = 0$ or $a_i = 1$ for $i = 1, \dots, n$. Observe that the family $\{(K + t_i) \cap h(0) : a_i = 0\}$ tiles $2K \cap h(0)$, hence $a_i = 0$ for 2^{d-1} values of i , and $K \cap h(0)$ is an affine $(d - 1)$ -cube. There is only the obvious way such that translates of $K \cap h(0)$ tile $2K \cap h(0)$, and so we obtain that $\{t_i : a_i = 0\}$ is an o -symmetric set. Moreover, $\{K \pm t_i : i = 1, 2, \dots, n\}$ is a family of pairwise non-overlapping translates of K , all containing the origin. By [8], the cardinality of this family is at most 2^d , with equality if and only if K is an affine d -cube. On the other hand, since $n = 3 \cdot 2^{d-2}$, we obtain that $\text{card}\{i : a_i = 1\} = 2^{d-2}$. Hence $\text{card}\{\pm t_i : i = 1, \dots, n\} = 2^{d-1} + 2 \cdot 2^{d-2} = 2^d$, and K is an affine d -cube.

Next, let $K := [-1, 1]^d$. Note that there are $3 \cdot 2^{d-2}$ vectors $(v_1, v_2, \dots, v_n) \in \{-1, 1\}^d$ whose coordinates satisfy the inequality $v_1 - v_2 \geq 0$. The translates of K by these vectors are pairwise non-overlapping, and each of them contains the origin. Thus, (1.2) is sharp for the cube.

Finally, we outline the proof of (1.1). Let $\alpha K + t_1, \alpha K + t_2, \dots, \alpha K + t_n$ be pairwise non-overlapping translates of K that touch K . Let $v, t_i, a_i, h(x)$ be as above. Note that $\alpha K + t_i$ is between $h(-\alpha)$ and $h(1 + 2\alpha)$, and it is contained also in $(1 + 2\alpha)K$, for $i = 1, \dots, n$. Thus,

$$(3.4) \quad \int_{-\alpha}^{1+2\alpha} V_{d-1} \left(\left(\bigcup_{i=1}^n \alpha K + t_i \right) \cap h(x) \right) dx = n\alpha^d V_d(K)$$

and

$$\begin{aligned} (3.5) \quad &\int_0^{1+2\alpha} V_{d-1} \left(\left(\bigcup_{i=1}^n \alpha K + t_i \right) \cap h(x) \right) dx \\ &\leq \int_0^{1+2\alpha} V_{d-1}((1 + 2\alpha)K \cap h(x)) dx - \int_0^1 V_{d-1}(K \cap h(x)) dx \\ &= \frac{(1 + 2\alpha)^d - 1}{2} V_d(K). \end{aligned}$$

Set $f(x) := V_{d-1}(\alpha K \cap h(x - \alpha))$, and observe that the conditions of Lemma 3.1 are satisfied by f . We may assume that $a_1, \dots, a_m \leq \alpha < a_{m+1}, \dots, a_n$. Then by Lemma 3.1,

$$\begin{aligned} & \int_{-\alpha}^0 V_{d-1} \left(\left(\bigcup_{i=1}^n (\alpha K + t_i) \right) \cap h(x) \right) dx \\ &= \sum_{i=1}^m \int_0^{\alpha - a_i} f(x) dx \leq \sum_{i=1}^m \int_0^{\alpha} f(x) dx \frac{f(\alpha - a_i)}{f(\alpha)} \\ &= \frac{\alpha^d V_d(K)}{2f(\alpha)} \sum_{i=1}^m V_{d-1}((\alpha K + t_i) \cap h(0)) \\ &= \frac{\alpha^d V_d(K)}{2f(\alpha)} V_{d-1} \left(\left(\bigcup_{i=1}^m (\alpha K + t_i) \right) \cap h(0) \right) \\ &\leq \frac{\alpha^d V_d(K)}{2f(\alpha)} [V_{d-1}((1 + 2\alpha)K \cap h(0)) - V_{d-1}(K \cap h(0))] \\ &= \frac{\alpha(1 + 2\alpha)^{d-1} - 1}{2} V_d(K). \end{aligned}$$

This inequality, combined with (3.4) and (3.5), yields (1.1). \blacksquare

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