11
Weak coupling

Perturbation theory forms one of the mainstays in the development of modern theoretical particle physics. Indeed, the successes of perturbative quantum electrodynamics lie at the heart of our nearly universal adoption of renormalizable quantum field theory as the framework with which to describe elementary particle interactions. As our space-time lattice represents a regulator for ultraviolet divergences, in principle all perturbative results could be reproduced in this formalism. The basic expansion parameter g_0^2 represents the temperature in the analog statistical system. At low temperatures the important degrees of freedom are low energy excitations involving gentle long-wavelength variations of the fields. In magnetic systems the analogous excitations are referred to as spin waves and perturbation theory is a spin wave expansion.

Perturbative analysis did not motivate the original formulation of lattice gauge theory. Highly developed methods for calculation already exist for other cutoff schemes such as that of Pauli and Villars (1949) or dimensional continuation (Ashmore, 1972; Bollini and Giambiagi, 1972; t'Hooft and Veltman, 1972). Because of this, perturbation theory on a lattice has received little attention and remains rather awkward. In this short chapter we merely sketch spin wave techniques for lattice gauge theory. We will only evaluate the lowest order contribution to the average plaquette. It is somewhat ironic that this weak coupling regime has played such a minor role in lattice gauge theory and yet it is exactly this region to which we must go for a continuum limit, as will be discussed in the next chapters. The main virtue of the lattice remains in non-perturbative analysis.

We limit this discussion to the pure gauge theory with partition function

$$Z = \int (\mathrm{d}U) \exp\left(-\beta \sum_{\square} (1-(1/n)\,\mathrm{Re}\,\mathrm{Tr}\, U_{\square})\right). \tag{11.1}$$

As the inverse coupling β becomes large, this integral is increasingly dominated by $U_{\square}$ near the identity. Perturbation theory begins with a saddle point approximation taken at this maximum of the exponentiated action. We parametrize the plaquette operators

$$U_{\square} = \exp(\mathrm{i}\lambda^{\alpha}\omega_{\square}^{\alpha}), \tag{11.2}$$

https://doi.org/10.1017/9781009290395.012 Published online by Cambridge University Press

where the matrices λ^α generate the group and are normalized as in eqs (6.6–7). To leading order we have

$$1-(1/n)\,\mathrm{Re}\,\mathrm{Tr}\,U_\square = (1/(4n))\,\omega_\square^\alpha\,\omega_\square^\alpha + O(\omega_\square^4), \tag{11.3}$$

and Z becomes

$$Z = \int (\mathrm{d}U)\exp\left(-(\beta/(4n))\,\omega_\square^\alpha\,\omega_\square^\alpha + O(\beta\omega^4)\right). \tag{11.4}$$

For large β the exponential is highly suppressed unless

$$\omega = O(\beta^{-\frac{1}{2}}) = O(g_0). \tag{11.5}$$

Thus the ω^4 terms in eq. (11.4) are of order the coupling constant squared.

To proceed we would like to evaluate the leading behavior of the integral in eq. (11.4) in the Gaussian approximation. Here we encounter a technical difficulty in that the integrand is not damped in all directions when considered as a function of the link variables U_{ij}. Indeed, a gauge transformation can arbitrarily alter any given link and yet leave the action unchanged. Gauge fixing is an essential first step in the perturbative analysis. Our integrand receives a Gaussian damping only for those directions which do not represent gauge degrees of freedom.

The details of the gauge choice will be unimportant to the discussion here. One possibility is to set all timelike links to the identity, i.e. work in the 'temporal' gauge, and then on the spacelike surface $t = 0$ to do the additional gauge fixing necessary to eliminate the freedom of time-independent gauge transformations. If we now select any particular link, its value will be driven to the identity when β goes to infinity. There is a non-uniformity to this limit because links far from the hypersurface $t = 0$ are less constrained than those near it. For this technical reason we impose an infrared cutoff by working on a finite lattice.

After the gauge fixing, one quarter of the links are no longer variables. The remaining links are driven to the identity, about which we can expand

$$U_{ij} = 1 + \mathrm{i}\lambda^\alpha\omega_{ij}^\alpha + O(\omega_{ij}^2), \tag{11.6}$$

$$\omega_\square^\alpha = \sum_{ij\in\square}\omega_{ij}^\alpha + O(\omega^2). \tag{11.7}$$

The integration measure in the vicinity of the identity takes the simple form

$$\mathrm{d}U_{ij} = (J + O(\omega_{ij}^2))\,d^{n_g}\omega_{ij}, \tag{11.8}$$

where the weight J will ultimately be absorbed as an irrelevant constant. Here n_g is the number of group generators. Now the partition function assumes the form

$$Z = K\int \prod_{\{ij\}} d^{n_g}\omega_{ij}\exp\left[(-\tfrac{1}{2}\beta\omega D^{-1}\omega) + O(\beta\omega^3)\right]. \tag{11.9}$$

Here K is an overall constant factor and D^{-1} is a large matrix operating in the space of the variables ω_{ij}. In this form the partition function looks much like that discussed for a free field in chapter 4. The operator D is the propagator for the gauge gluons and enters into the Feynman diagrams of the theory. The $O(\beta\omega^3)$ terms are of order the coupling constant. They generate the vertices of the perturbative expansion.

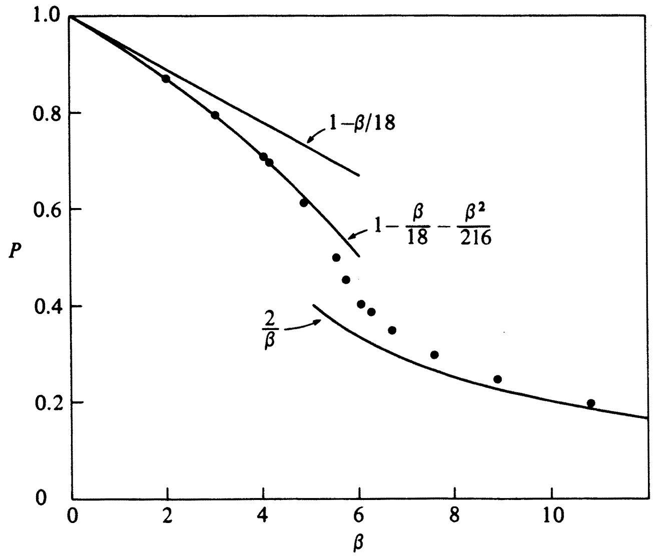

Fig. 11.1. The average plaquette for $SU(3)$ lattice gauge theory. The curves are the leading strong and weak coupling approximations and the points are from Monte Carlo analysis on 4^4 and 6^4 lattices.

For actual calculations these lattice propagators are quite cumbersome. However we can obtain some information on the average plaquette with very little effort. As our integral is now Gaussian, its value is a determinant

$$Z = K'\,|D/\beta|^{\frac{1}{2}}(1+O(\beta^{-1})). \qquad (11.10)$$

The matrix D has the dimensionality of the parameter space after gauge fixing; consequently, it is a square matrix of $3n_g N^4$ rows. Here the factor of 3 is the number of non-fixed links per site. Removing a factor of β from each row of the matrix, we find

$$Z = K'\,|D|^{\frac{1}{2}}\beta^{-3n_g N^4/2}(1+O(\beta^{-1})). \qquad (11.11)$$

For the average plaquette this implies

$$\begin{aligned} P &= -(1/(6N^4))(\partial/\partial\beta)\log Z \\ &= n_g/(4\beta)+O(\beta^{-2}). \end{aligned} \qquad (11.12)$$

This result has a simple interpretation in statistical mechanics. We have $3n_g N^4$ physical variables distributed over $6N^4$ plaquettes. If we give each degree of freedom $\frac{1}{2}kT = 1/(2\beta)$ average energy, then we obtain exactly eq. (11.12). This simple counting of variables receives corrections at higher temperatures where the non-linear interactions come into play.

In figure 11.1 we summarize the leading strong and weak coupling results for the gauge group $SU(3)$

$$\begin{aligned} P &= 1-\beta/18-\beta^2/216+O(\beta^3) \\ &= 2/\beta+O(\beta^{-2}). \end{aligned} \tag{11.13}$$

The points in the graph are the true values for the plaquette from Monte Carlo analysis.

Problems

1. Show that in the weak coupling regime the parameter b_3 of the last chapter behaves as

$$b_3(\beta) = 3(1-4/\beta+O(\beta^2)).$$

2. What is the leading weak coupling behavior for the average plaquette in Z_2 lattice gauge theory?

https://doi.org/10.1017/9781009290395.012 Published online by Cambridge University Press