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Fundamental local equivalences in quantum geometric Langlands

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Abstract

In quantum geometric Langlands, the Satake equivalence plays a less prominent role than in the classical theory. Gaitsgory and Lurie proposed a conjectural substitute, later termed the *fundamental local equivalence*. With a few exceptions, we prove this conjecture and its extension to the affine flag variety by using what amount to Soergel module techniques.

1. Introduction

1.1 In the early days of the geometric Langlands program, experts observed that the fundamental objects of study deform over the space of *levels* κ for the reductive group G. For example, if G is simple, this is a one-dimensional space. Moreover, levels admit duality as well: a level κ for G gives rise to a *dual* level[1] $\check{\kappa}$ for the Langlands dual group $\check{G}$. This observation suggested the existence of a *quantum* geometric Langlands program, deforming the usual Langlands program.

The first triumph of this idea appeared in the work of Feigin and Frenkel [FF91], where they proved duality of affine $\mathcal{W}$-algebras: $\mathcal{W}_{G,\kappa} \simeq \mathcal{W}_{\check{G},\check{\kappa}}$. We emphasize that this result is quantum in nature: the level κ appears. For the *critical* level $\kappa = \kappa_c$, which corresponds to classical geometric Langlands, Beilinson and Drinfeld [BD99] used Feigin–Frenkel duality to give a beautiful construction of Hecke eigensheaves for certain irreducible local systems.

1.2 The major deficiency of the quantum geometric Langlands was understood immediately: the Satake equivalence is more degenerate.

For instance, in the classical case where $\kappa = \kappa_c$ is critical, the compatibility of global geometric Langlands with geometric Satake essentially characterizes the equivalence. Concretely, this means that for an irreducible $\check{G}$-local system σ on a smooth projective curve, one expects there to exist a canonical Hecke eigensheaf $\mathcal{A}_\sigma$ with eigenvalue σ.

This does not hold in the quantum setting. For instance, if κ is *irrational*, then the Satake category is equivalent to Vect and the Hecke eigensheaf condition is vacuous. For *rational* κ, one hopes for a neutral gerbe of irreducible Hecke eigensheaves. This is known for a torus, but the

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[1] We include a translation by critical levels in the definition of $\check{\kappa}$; see § 2.7 for our conventions on levels.

gerbe is not[2] canonically trivial; cf. [Lys15, § 5.2]. For general reductive G, we are not aware of any conjecture explicitly describing the relevant gerbe.

We remark that quantum geometric Langlands for rational κ is closely tied to the theory of automorphic forms on metaplectic groups; cf. [GL18]. It is well known that Hecke eigenvalues form too coarse a decomposition in the metaplectic setting. Indeed, already in the first announcement [Hec36] of his eponymous operators, Hecke himself made this observation.

> *Es sei zum Schlauß noch erwähnt, daß für die Formen halbzahlinger Dimension, wie die einfachen Thetareihen und deren Potenzprodukte, sich eine ähnliche Theorie nicht aufbauen läßt. Da nämlich für diese die Zuordnung zu einer Stufe und zu einer Kongruenzgruppe bekanntlich nicht mehr so einfach wie bei ganzzahliger Dimension ist, so kann man die Operatoren T_m nur für Quadratzahlen $m = n^2$ definieren, und man erhält so einen Zusammenhang nur zwischen den Koeffizienten $a(N)$ und $a(Nn^2)$.*
>
> *(In conclusion, it should be mentioned that a similar theory cannot be developed for forms of half-integral weight, such as simple theta functions and monomials in them. Since the assignment of a level and a congruence group is not as easy as for integer dimensions, the operators T_m can only be defined for square numbers $m = n^2$, and a relationship is obtained only between the coefficients $a(N)$ and $a(Nn^2)$.)*

In the study of metaplectic automorphic forms, one often repeatedly finds the guiding principle that *it is more fruitful to study Whittaker coefficients than Hecke eigenvalues*. Indeed, above Hecke exactly observes the gap that appears between the two in the metaplectic context, and that Whittaker coefficients provide the finer information. See [GP80] for a classical application of these ideas.

We refer to [GGW18] for further discussion and more recent perspectives.

1.3 In the geometric setting, Gaitsgory made significant advances in applying the above perspective. In [Gai08b] and [Gai16b], Gaitsgory, pursuing unpublished ideas he developed jointly with Lurie, formulated a series of conjectures regarding κ-twisted Whittaker D-modules on the affine Grassmannian $\mathrm{Gr}_G = \mathfrak{L}G/\mathfrak{L}^+G$ and the affine flag variety $\mathfrak{L}G/I$.

Let κ be a nondegenerate level for G and let $\check{\kappa}$ denote the dual level for $\check{G}$. Let $\mathsf{Whit}_\kappa^{\mathrm{sph}}$ denote the DG category of κ-twisted Whittaker D-modules on Gr_G, and let $\mathsf{Whit}_\kappa^{\mathrm{aff}}$ denote the similar category for $\mathfrak{L}G/I$.

We write $\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}$ for the affine Lie algebra associated to $\check{G}$ and $\check{\kappa}$, and let

$$\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\mathfrak{L}^+\check{G}} \quad \text{and} \quad \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{I}}$$

denote the DG categories of Harish-Chandra modules for the pairs $(\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}, \mathfrak{L}^+\check{G})$ and $(\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}, \check{I})$, respectively. We refer to § 2 for further clarification of the notation.

Conjecture 1.3.1 (Gaitsgory–Lurie [Gai08b, Conjecture 0.10]). *There is an equivalence of DG categories*

$$\mathsf{Whit}_\kappa^{\mathrm{sph}} \simeq \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\mathfrak{L}^+\check{G}}. \tag{1.1}$$

Conjecture 1.3.2 (Gaitsgory [Gai16b, Conjecture 3.11]). *There is an equivalence of DG categories*

$$\mathsf{Whit}_\kappa^{\mathrm{aff}} \simeq \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{I}}. \tag{1.2}$$

[2] However, in this abelian setting, recent work [Lys21] of Lysenko indicates that the gerbe is *sometimes* canonically trivial. More specifically, this is true in the absence of certain 2-torsion. Whether this phenomenon has a non-abelian counterpart remains unclear.

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We now state a preliminary version of our main result and then discuss the hypothesis that appears in it.

THEOREM. *Suppose that κ is good in the sense of § 3.4.4. Then Conjectures 1.3.1 and 1.3.2 are true.*

1.3.3 Briefly, a level κ is good if and only if after restriction to every simple factor of $\mathfrak{g}$, κ either is irrational or is a rational level whose denominator is coprime to the bad primes of the root system. We recall that the latter always lie in $\{2, 3, 5\}$. For G of type A, there are no bad primes, so every level is good in this case. For an explicit description for general G, see Table 1.

1.4 Related works

As emphasized by Gaitsgory in the initial papers [Gai08b] and [Gai16b], Conjectures 1.3.1 and 1.3.2 provide quantum analogues of theorems in classical local geometric Langlands, i.e. for κ or $\check{\kappa}$ at the critical level. We presently review these statements.

1.4.1 For $\kappa = \kappa_c$ critical, Conjecture 1.3.1 is known from work of Frenkel, Gaitsgory, and Vilonen and is a variant of the geometric Satake equivalence.

In this case, heuristically we have $\check{\kappa} = \infty$; to interpret this carefully, we refer to [Zha17] for details. Standard arguments then give

$$\widehat{\check{\mathfrak{g}}}_{\infty}\text{–}\mathsf{mod}^{\mathfrak{L}^+\check{G}} = \mathsf{QCoh}(\check{\mathfrak{g}}[[t]]\,dt/\mathfrak{L}^+\check{G}) = \mathsf{QCoh}(\mathbb{B}\check{G}) = \mathsf{Rep}(\check{G}).$$

Here the action of $\mathfrak{L}^+\check{G}$ on $\check{\mathfrak{g}}[[t]]\,dt$ is the gauge action.

Then, by [FGV01], the composition

$$\mathsf{Rep}(\check{G}) \to D(\mathrm{Gr}_G)^{\mathfrak{L}^+G} \to \mathsf{Whit}(\mathrm{Gr}_G) \tag{1.3}$$

is an equivalence, where the first functor is the geometric Satake functor [MV07] and the second functor is given by convolution on the unit object of the right-hand side.

We emphasize that, unlike with the Satake equivalence, the equivalence $\mathsf{Rep}(\check{G}) \xrightarrow{\sim} \mathsf{Whit}(\mathrm{Gr}_G)$ is an equivalence of derived categories, not merely abelian categories. This amounts to the *cleanness* property of spherical Whittaker sheaves from [FGV01] and the geometric Casselman–Shalika formula from *loc. cit.*

Remark 1.4.2. That (1.3) is an equivalence is special to integral levels. That is, at non-integral levels, the spherical Hecke category produces only a small part of the spherical Whittaker category. This failure, especially at rational levels, accounts for part of our interest in Theorem 1.3.

1.4.3 For $\kappa = \kappa_c$ critical, a version of Conjecture 1.3.2 is the main result of [AB09]. This deep work of Arkhipov and Bezrukavnikov was one of the most significant breakthroughs in geometric Langlands and underlies seemingly countless advances in the area since.

1.4.4 By the above discussion, for $\kappa = \kappa_c$, Conjecture 1.3.1 is a sort of variant of the geometric Satake theorem. One of Gaitsgory's key insights is that in quantum geometric Langlands, Hecke operators play a diminished role, while Conjecture 1.3.1 plays the fundamental role that Satake plays in the classical theory.

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1.4.5 For $\check{\kappa} = \check{\kappa}_c$ critical, Conjecture 1.3.1 is the main result of [FG09b], where Frenkel and Gaitsgory construct an equivalence

$$\widehat{\check{\mathfrak{g}}}_{\mathrm{crit}}\text{–}\mathsf{mod}^{\mathfrak{L}^+\check{G}} \simeq \mathsf{QCoh}(\mathrm{Op}_G^{\mathrm{unr}}).$$

Here the right-hand side is the ind-scheme of *unramified* (or *monodromy-free*) *opers* for G; this category is the $\kappa \to \infty$ limit of $\mathsf{Whit}_\kappa(\mathrm{Gr}_G)$ as above, and is defined in *loc. cit.* For the centrality of this result in the geometric Langlands program, see [Gai15, § 11].

Conjecture 1.3.2 in this case is a folklore extension of the main results of [FG09a] and [FG06a], but whose complete proof is not recorded in the literature.

1.4.6 Finally, let us discuss the previously known cases of Conjectures 1.3.1 and 1.3.2. In the original paper [Gai08b], Gaitsgory proved Conjecture 1.3.1 for κ irrational, and in fact a stronger version of it, as we describe below. As far as we are aware, no other cases of the conjectures have been obtained.

1.5 Factorization

In fact, Gaitsgory conjectured more, related to the factorization of the Beilinson–Drinfeld affine Grassmannian.

In Conjecture 1.3.1, he conjectured an equivalence of *factorization categories*; cf. [Gai08b] and [Ras15a]. Similarly, in Conjecture 1.3.2, it is expected that the equivalence should be one of the factorization modules for the (conjecturally equivalent) factorization categories appearing in Conjecture 1.3.1.

When $\kappa = \kappa_c$, these goals are implicit in the original work. In the spherical case, this is spelled out in [Ras21a, Theorem 6.36.1]. In the Iwahori case, a weak version of the compatibility with factorization module structures was shown in [Ras16, Theorem 10.8.1].

1.6 The role of this paper

In the decade since their formulation, Gaitsgory has been advancing an ambitious program to establish the fundamental local equivalences. We refer to [ABC$^+$18] for an overview of this project; [Gai08b], [BG08], and [BG08] for early work on it; and [Gai16a], [Gai17], [Gai18a], [Gai18b], [Gai19], [GL19], and [GL18] for some of Gaitsgory's recent advances in this project.

Gaitsgory's program, though still incomplete, represents a new paradigm for Kac–Moody algebras, quantum groups, and quantum geometric Langlands. It is full of lovely, innovative constructions and numerous breakthroughs. It is also quite sophisticated, as seems always to be the case when working with factorization algebras.

Our work is not intended to supersede the eventual conclusion of Gaitsgory's project. Rather, we regard the equivalences of Conjectures 1.3.1 and 1.3.2 (i.e. forgetting factorization) to be interesting results in geometric representation theory and geometric Langlands.

For example, as discussed above, the $\kappa = \kappa_c$ analogues of our results include the geometric Casselman–Shalika formula [FGV01] and the deep work of Arkhipov and Bezrukavnikov [AB09]. In fact, as we hope to explain elsewhere, the geometric part of our study of $\mathsf{Whit}_\kappa^{\mathrm{sph}}$, suitably adapted to the function-field setting, should imply some new function-theoretic results on the metaplectic Casselman–Shalika formula.

Moreover, while our present results are expected to be interesting outcomes of Gaitsgory's methods, we find it desirable to have a more direct argument.

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1.7 Methods

Our techniques are remarkably elementary in comparison to the above work of Gaitsgory or, for example, [AB09]. Our main input consists of classical methods developed by Soergel and his school.

1.7.1 In his initial work [Soe90], Soergel showed that a block of Category $\mathcal{O}$ for $\mathfrak{g}$ can be reconstructed from the Weyl group of G. Fiebig [Fie06] extended this work to Kac–Moody algebras. As a consequence of Fiebig's work, the category $\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{I}}$ can be completely recovered from the combinatorial datum of the root datum of $\check{G}$ and the level $\check{\kappa}$.

To prove Conjecture 1.3.2, we provide a similar Coxeter-theoretic description of $\mathsf{Whit}_{\kappa}^{\mathrm{aff}}$. We do this by relating $\mathsf{Whit}_{\kappa}^{\mathrm{aff}}$ to $\widehat{\mathfrak{g}}_{\kappa}\text{–}\mathsf{mod}^{I}$,[3] which allows us to apply Fiebig's results directly to $\mathsf{Whit}_{\kappa}^{\mathrm{aff}}$.

We then prove Conjecture 1.3.2 by matching Langlands dual combinatorics. Here we draw the reader's attention to Theorem 3.5.6, which is a combinatorial shadow of quantum Langlands duality.

It is striking that these fundamental conjectures of Gaitsgory have been open for over a decade, but admit a solution that almost could have been given at the time.

Remark 1.7.2. In fact, Theorem 3.5.6, combined with the description of twisted Hecke categories as Soergel bimodules, obtained in finite type recently in [LY20], should yield quantum Langlands duality for affine Hecke categories. Therefore, Soergel's methods, as applied in the present paper, should suffice to prove the local quantum geometric Langlands correspondence for categorical representations generated by Iwahori invariant vectors.

Remark 1.7.3. Because of our reliance on [Fie06], our construction is a little non-canonical. Indeed, in [Fie06] there is a choice of projective cover of simple objects. With that said, hewing closer to the Koszul dual picture as in [LY20] would provide canonical equivalences.

Remark 1.7.4. After completing this paper, we learned of the thesis of Chris Dodd [Dod11], which reproves the results of Arkhipov and Bezrukavnikov [AB09] by a Soergel module argument. Our argument may be thought of as a quantum deformation of Dodd's approach. We thank Roman Bezrukavnikov for bringing this to our attention.

1.8 Comparison

In short, we relate Langlands dual categories using Fiebig's combinatorial description of blocks of affine Category $\mathcal{O}$.

Gaitsgory's program compares these categories via a factorization algebra Ω_q (and some of its cousins), which may also be constructed directly from the root datum of G; cf. [Gai08a], [ABC$^+$18], and [Gai19].

It would be quite interesting to find a direct relationship between these two perspectives.

---

[3] In finite type, an analogous result appears in the work of Milicic and Soergel [MS97]. Their techniques are not available in the affine setting, so our methods differ.

We use the perspective of loop group actions on categories to study Kac–Moody representations. We convolve by an explicit object, constructed from $\mathcal{W}$-algebras.

In contrast, in the finite-type setting, [MS97] relies on good properties of Harish-Chandra bimodules with generalized central characters. The theory of affine Harish-Chandra bimodules is in its infancy and is much more difficult than in finite type. As we explain in [CD21], our methods are sufficient to *establish* similar properties of a suitable category of Harish-Chandra bimodules in affine type. However, it should also be possible to prove this equivalence directly by a Soergel module argument; cf. Remark 3.6.2.

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Remark 1.8.1. We highlight one point of departure in our perspective from Gaitsgory's. At negative levels, our equivalence is t-exact by construction and matches highest-weight structures. This was previously anticipated in the spherical case by Gaitsgory, but was ambiguous in the affine case. After we told Gaitsgory about our results, he found an argument showing that a similar property must hold for the equivalence he is working on.

In our approach, these properties are key in deducing the parahoric version of the theorem from the Iwahori version.

2. Preliminary material

In this section, we collect standard definitions and notation. We invite the reader to skip to the next section and refer back as needed.

2.1 Notation for groups

Let G be a reductive group over $\mathbb{C}$.[4]

2.1.1 We fix once and for all a pinning (T, B, ψ) of G. That is, we fix $T \subset B \subset G$ where T is a Cartan and B a Borel subgroup of G. In addition, for the unipotent radical N of B, we fix a nondegenerate character $\psi : N \to \mathbb{G}_a$.

2.1.2 Given the above data, there is a canonically defined Langlands dual group $\check{G}$, which also comes with a pinning. In particular, we have Cartan and Borel subgroups $\check{T} \subset \check{B} \subset \check{G}$. Again, $\check{N}$ denotes the unipotent radical of $\check{B}$.

2.1.3 The data $T \subset B \subset G$ determine a Borel B^- opposite to B, so $B^- \cap B = T$. We denote its radical by N^-. The same applies for $\check{G}$.

2.1.4 We denote the appropriate Lie algebras by $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}, \mathfrak{t}, \mathfrak{b}^-, \mathfrak{n}^-, \check{\mathfrak{g}}, \check{\mathfrak{b}}, \check{\mathfrak{n}}, \check{\mathfrak{t}}, \check{\mathfrak{b}}^-$, and $\check{\mathfrak{n}}^-$.

2.1.5 We write $\check{\Lambda}_G$ for the lattice of coweights of G, i.e. the cocharacter lattice of T, and Λ_G for the lattice of weights of G, i.e. the character lattice of T. We denote the root lattice, i.e. the integral span of the roots, by

$$Q \subset \Lambda_G.$$

In other words, $Q = \Lambda_{G^{\mathrm{ad}}}$ where G^{ad} is the adjoint group of G. Similarly, we let $\check{Q} \subset \check{\Lambda}_{\check{G}}$ denote the coroot lattice, and one has $\check{Q} = \Lambda_{\check{G}^{\mathrm{ad}}}$.

2.1.6 We let $\mathcal{I}$ denote the set of nodes of the Dynkin diagram of G. For $i \in \mathcal{I}$, the corresponding simple roots and coroots are denoted by

$$\alpha_i \in \Lambda_G \quad \text{and} \quad \check{\alpha}_i \in \check{\Lambda}_G.$$

2.1.7 Our choice of pinning defines a standard involutive anti-homomorphism[5]

$$\tau : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}.$$

[4] Our arguments apply more generally to any split reductive group over a field k of characteristic 0. That is, the cohomology that appears is purely de Rham, never étale or Betti.

[5] One sometimes finds this involution called the Chevalley involution, or the *Cartan* involution, but the latter terminology is potentially misleading.

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More precisely, for $i \in \mathcal{I}$ let $e_i \in \mathfrak{n}$ be the unique vector of weight α_i such that $\psi(e_i) = 1$. Then τ is the unique involution such that $\tau([x, y]) = -[\tau(x), \tau(y)]$, $\tau|_{\mathfrak{t}} = \mathrm{id}_{\mathfrak{t}}$, and

$$[e_i, \tau(e_i)] = \check{\alpha}_i \quad \text{for } i \in \mathcal{I}.$$

We observe that τ lifts to an involutive anti-homomorphism on G, which we also denote by τ.

2.2 Loops and arcs

2.2.1 For any affine variety Z of finite type, we let $\mathfrak{L}Z$ denote its algebraic loop space and $\mathfrak{L}^+Z$ its algebraic arc space. The former is an ind-scheme, while the latter is an affine scheme. There is a canonical evaluation map $\mathfrak{L}^+Z \to Z$ given by evaluation of a jet at the origin.

For an affine algebraic group $Z = H$, $\mathfrak{L}H$ is a group ind-scheme, with $\mathfrak{L}^+H \subset \mathfrak{L}H$ a group subscheme.

2.2.2 The Borel subgroup $B \subset G$ defines the *Iwahori* subgroup

$$I := \mathfrak{L}^+G \times_G B \subset \mathfrak{L}^+G \subset \mathfrak{L}G.$$

Dually, we have a preferred Iwahori subgroup $\check{I} \subset \mathfrak{L}\check{G}$. We denote the pro-unipotent radical of I by $\mathring{I}$ and note the canonical isomorphism

$$T \simeq I/\mathring{I}.$$

2.3 Weyl groups

The combinatorics of affine Weyl groups plays an important role in this paper. We recall notation and fundamental constructions below.

2.3.1 We let W_f denote the Weyl group of G and recall that W_f is also the Weyl group of $\check{G}$.

2.3.2 The extended affine Weyl group of G is the semidirect product

$$\tilde{W} := W_f \ltimes \check{\Lambda}_G.$$

The subgroup of $\tilde{W}$ given by

$$W := W_f \ltimes \check{Q}$$

is the affine Weyl group, where we recall that $\check{Q}$ is the coroot lattice.

2.3.3 Let $S_f \subset W_f$ denote the set of simple reflections s_i for $i \in \mathcal{I}$. Let $S \subset W$ denote the union of S_f with the set of simple affine reflections as in [Kac90, § 7]. We remind the reader that the simple affine reflections are indexed by simple factors of G.

The pairs (W_f, S_f) and (W, S) are Coxeter systems, i.e. Coxeter groups with preferred choices of simple reflections. We recall that the Bruhat order and length function on W each extend in a standard way to $\tilde{W}$.

2.4 Categories

We repeatedly work with DG categories and their symmetries. In this setting, we use the following conventions.

2.4.1 We let $\mathrm{DGCat}_{\mathrm{cont}}$ denote the symmetric monoidal ∞-category of cocomplete DG categories and continuous DG functors as defined in [GR17, § I.1.10]. We denote the binary product underlying the symmetric monoidal structure by $- \otimes -$; this is the *Lurie tensor product.*

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For simplicity, we sometimes refer to ∞-categories as *categories*, and similarly for DG categories.

2.4.2 Given a t-structure on a DG category $\mathcal{C}$, we let $\mathcal{C}^{\leqslant 0}$ and $\mathcal{C}^{\geqslant 0}$ denote the subcategories of connective and coconnective objects. That is, we use cohomological indexing notation.

We denote the heart of such a t-structure by

$$\mathcal{C}^{\heartsuit} := \mathcal{C}^{\leqslant 0} \cap \mathcal{C}^{\geqslant 0}.$$

2.5 D-modules

We make essential use of categories of D-modules on ind-pro-finite-type schemes such as $\mathfrak{L}G$, as developed in [Ber17] and [Ras15b]. For an ind-scheme X, we denote by $D(X)$ what in *loc. cit.* is denoted by $D^*(X)$.

2.5.1 By functoriality, $D(\mathfrak{L}G)$ carries a canonical convolution monoidal structure. We denote the corresponding (∞-)category of DG categories equipped with an action of $D(\mathfrak{L}G)$ by

$$D(\mathfrak{L}G)\text{–}\mathsf{mod} := D(\mathfrak{L}G)\text{–}\mathsf{mod}(\mathrm{DGCat}_{\mathrm{cont}}).$$

We use similar notation for other group (ind-)schemes such as $\check{I}$ and $\mathfrak{L}N$.

2.6 Invariants and coinvariants

2.6.1 Given a group ind-scheme H and a $D(H)$-module $\mathcal{C}$, we denote its categories of invariants and coinvariants by

$$\mathcal{C}^H := \mathsf{Hom}_{D(H)\text{–}\mathsf{mod}}(\mathsf{Vect}, \mathcal{C}) \quad \text{and} \quad \mathcal{C}_H := \mathsf{Vect} \underset{D(H)}{\otimes} \mathcal{C},$$

respectively. See [Ber17] for further discussion.

2.6.2 Similarly, for a multiplicative D-module χ on H, we denote the corresponding categories of twisted invariants and twisted coinvariants by

$$\mathcal{C}^{H,\chi} \quad \text{and} \quad \mathcal{C}_{H,\chi}.$$

Our multiplicative D-modules will be obtained by one of the following two procedures.

2.6.3 First, any character

$$\lambda \in \mathrm{Hom}(H, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{C}$$

determines a character D-module 't^λ' of H. In this case, we denote twisted invariants by $\mathcal{C}^{H,\lambda}$. We apply this construction particularly for $H = T$ and $H = I$.

2.6.4 Similarly, given an additive character

$$\psi : H \to \mathbb{G}_a,$$

we obtain a character D-module 'e^ψ' on H. Here we denote twisted invariants by $\mathcal{C}^{H,\psi}$. We apply this construction for $H = \mathfrak{L}N$.

2.6.5 Suppose H is an affine group scheme with pro-unipotent radical H^u. We suppose that H/H^u is of finite type.

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By [Ber17], the canonical forgetful map $\mathcal{C}^H \to \mathcal{C}$ admits a continuous right adjoint Av_*^H. Moreover, there is a canonical equivalence $\mathcal{C}_H \simeq \mathcal{C}^H$ fitting into the following commutative diagram.

$$\begin{array}{ccc} \mathcal{C} & & \\ \downarrow & \searrow^{\mathrm{Av}_*^H} & \\ \mathcal{C}_H & \longrightarrow & \mathcal{C}^H \end{array} \tag{2.1}$$

The same is true in the presence of a multiplicative D-module χ.

2.6.6 Now suppose $\mathcal{C} \in D(\mathfrak{L}G)\text{–}\mathsf{mod}$. Let $\psi : \mathfrak{L}N \to \mathbb{G}_a$ denote the Whittaker character of $\mathfrak{L}N$.

In this case, [Ras21b, Theorem 2.1.1] provides a canonical equivalence

$$\mathcal{C}_{\mathfrak{L}N,\psi} \simeq \mathcal{C}^{\mathfrak{L}N,\psi}. \tag{2.2}$$

We highlight that this case is more subtle than that of an affine group scheme considered above.

2.6.7 As a final piece of notation, for an ind-scheme X with an action of a group ind-scheme H, we use the notation

$$D(X/H,\chi) \coloneqq D(X)_{H,\chi}. \tag{2.3}$$

By [Ras15b, Proposition 6.7.1], this notation is unambiguous.

We remark that in the setting of either (2.1) or (2.2), these coinvariants coincide with invariants.

2.7 Levels

Recall that a level κ for G is a G-invariant symmetric bilinear form

$$\kappa : \mathrm{Sym}^2(\mathfrak{g}) \to \mathbb{C}.$$

2.7.1 We let $\kappa_{\mathfrak{g},c}$ denote the critical level for G, i.e. $-\frac{1}{2}$ times the Killing form of G. Where G is unambiguous, we simply write κ_c.

2.7.2 Suppose G is simple. A level κ is rational if κ is a rational multiple of the Killing form and irrational otherwise. We say that κ is positive if $\kappa - \kappa_c$ is a positive rational multiple of the Killing form. We say that a level κ is negative if κ is not positive or critical. In particular, any irrational level is negative.

For a general reductive G, we say that a level κ is rational, irrational, positive, or negative if its restrictions to each simple factor are so.

2.7.3 A level κ is nondegenerate if $\kappa - \kappa_c$ is nondegenerate as a bilinear form. For such κ, the dual level $\check{\kappa}$ for $\check{G}$ is the unique nondegenerate level such that the restriction of $\check{\kappa} - \check{\kappa}_{\check{\mathfrak{g}},c}$ to $\mathfrak{t}^*$ and the restriction of $\kappa - \kappa_{\mathfrak{g},c}$ to $\mathfrak{t}$ are dual symmetric bilinear forms.

2.7.4 For a simple Lie algebra $\mathfrak{g}$, the basic level $\kappa_{\mathfrak{g},b} = \kappa_b$ is the unique positive level such that the short coroots have squared length 2, i.e.

$$\min_{i \in \mathcal{I}} \kappa_b(\check{\alpha}_i, \check{\alpha}_i) = 2.$$

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2.8 Twisted D-modules

Given a level κ, there is a canonical monoidal DG category of twisted D-modules, $D_\kappa(\mathfrak{L}G)$; see, for example, [Ras21b, § 1.29]. We again use the notation

$$D_\kappa(\mathfrak{L}G)\text{–mod} := D_\kappa(\mathfrak{L}G)\text{–mod}(\mathrm{DGCat}_{\mathrm{cont}}).$$

2.8.1 We recall that the multiplicative twisting defined by κ is canonically trivialized on $\mathfrak{L}^+G$ and $\mathfrak{L}N$. In particular, for

$$\mathcal{C} \in D_\kappa(\mathfrak{L}G)\text{–mod} \tag{2.4}$$

we can make sense of invariants and coinvariants of $\mathfrak{L}^+G$ and $\mathfrak{L}N$ with coefficients in $\mathcal{C}$. The same is true for any subgroup, in particular for $I \subset \mathfrak{L}^+G$. Also, the same applies with a twisting, such as in the Whittaker setup for $\mathfrak{L}N$. We remark that the identification (2.2) of Whittaker invariants and coinvariants [Ras21b] was proved more generally for $D_\kappa(\mathfrak{L}G)$–mod.

Example 2.8.2. Because $D_\kappa(\mathfrak{L}G)$ carries commuting actions of $D(\mathfrak{L}N)$ and $D(I)$, following the convention of § 2.6.7 we use the notation

$$D_\kappa(\mathfrak{L}N, \psi\backslash\mathfrak{L}G/I)$$

for the appropriate invariants = coinvariants category.

2.9 Affine Lie algebras

2.9.1 Let $\mathfrak{L}\mathfrak{g}$ denote the Lie algebra of $\mathfrak{L}G$, considered with its natural inverse limit topology, that is,

$$\mathfrak{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t)) = \lim_n \mathfrak{g} \otimes \mathbb{C}((t))/t^n\mathbb{C}[\![t]\!].$$

Given a level κ, one obtains a continuous 2-cocycle $\mathfrak{L}\mathfrak{g} \otimes \mathfrak{L}\mathfrak{g} \to \mathbb{C}$ given by

$$\xi_1 \otimes \xi_2 \mapsto \operatorname{Res} \kappa(d\xi_1, \xi_2).$$

Here d is the exterior derivative and Res is the residue. We denote the corresponding central extension by

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{L}\mathfrak{g} \to 0.$$

2.9.2 We denote by $\widehat{\mathfrak{g}}_\kappa\text{–mod}^\heartsuit$ the abelian category of smooth representations of $\widehat{\mathfrak{g}}_\kappa$ on which the central element $\mathbf{1}$ acts via the identity.

We let $\widehat{\mathfrak{g}}_\kappa$–mod denote the DG category introduced by Frenkel and Gaitsgory in [FG09a, §§ 22 and 23]. This DG category is compactly generated and carries a canonical t-structure with heart $\widehat{\mathfrak{g}}_\kappa\text{–mod}^\heartsuit$. However, there are some nonzero objects in

$$\widehat{\mathfrak{g}}_\kappa\text{–mod}^{-\infty} := \bigcap_n \widehat{\mathfrak{g}}_\kappa\text{–mod}^{\leqslant -n},$$

so $\widehat{\mathfrak{g}}_\kappa$–mod is not the derived category of $\widehat{\mathfrak{g}}_\kappa\text{–mod}^\heartsuit$.

2.9.3 In [Ras20, §§ 10 and 11], a $D_\kappa(\mathfrak{L}G)$-module structure on $\widehat{\mathfrak{g}}_\kappa$–mod was constructed, enhancing previous constructions of Beilinson and Drinfeld [BD99, § 7] and of Frenkel and Gaitsgory [FG06b, § 22].

2.9.4 Let H be a sub-group scheme of $\mathfrak{L}^+G$ of finite codimension, and consider the corresponding category of equivariant objects

$$\widehat{\mathfrak{g}}_\kappa\text{–mod}^H.$$

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By [Ras21b, Lemma A.35.1], the bounded-below category

$$\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^{H,+}$$

canonically identifies with the bounded-below derived category of its heart $\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^{H,\heartsuit}$, which consists of Harish-Chandra modules for the pair $(\widehat{\mathfrak{g}}_\kappa, H)$. Moreover, $\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^{H}$ is compactly generated by inductions of finite-dimensional H-modules.

2.9.5 Let κ be a level. With an abuse of notation we let κ also denote the map $\kappa : \mathfrak{t} \to \mathfrak{t}^*$.

The dot action of $\tilde{W}$ on $\mathfrak{t}^*$ is defined by having W_f act through the usual dot action and having $\check{\Lambda}_G$ act through translations via $-(\kappa - \kappa_c)$; that is, writing ρ for the half-sum of the positive roots, for any $\lambda \in \mathfrak{t}^*$ we have

$$w \cdot \lambda := w(\lambda + \rho) - \rho, \quad w \in W_f \subset \tilde{W},$$

$$\check{\mu} \cdot \lambda := \lambda - (\kappa - \kappa_c)(\check{\mu}), \quad \check{\mu} \in \check{\Lambda} \subset \tilde{W}.$$

2.9.6 To discuss integral Weyl groups, we need some more standard facts about this dot action. Recall that the affine real coroots of $\widehat{\mathfrak{g}}_\kappa$ form a subset of $\mathfrak{t} \oplus \mathbb{C}\mathbf{1}$. In particular, to such a coroot $\check{\alpha}$ we may associate its classical part $\check{\alpha}_{\mathrm{cl}}$, i.e. its projection to $\mathfrak{t}$. This is a coroot of $\mathfrak{g}$, and in particular we may associate a classical root $\alpha_{\mathrm{cl}} \in \mathfrak{t}^*$.

2.9.7 Via a standard construction, $\check{\alpha}$ acts as an affine linear functional on $\mathfrak{t}^*$, which we denote by $\langle \check{\alpha}, - \rangle$, in such a way that, writing $s_{\check{\alpha}}$ for the associated reflection in $\tilde{W}$, one has

$$s_{\check{\alpha}} \cdot \lambda = \lambda - \langle \check{\alpha}, \lambda + \rho \rangle \alpha_{\mathrm{cl}}, \quad \lambda \in \mathfrak{t}^*.$$

Briefly, this arises via restricting a linear action of $\tilde{W}$ on $(\mathfrak{t} \oplus \mathbb{C}\mathbf{1})^*$ to the affine hyperplane of functionals whose pairing with $\mathbf{1}$ is 1. We refer the reader to [Dhi21, §§ 3.1 and 3.4], where this is reviewed in greater detail.

3. Fundamental local equivalences

In this section, we suppose that G is simple of adjoint type. Under this assumption, we prove the main theorem, i.e. the fundamental local equivalence, for good *negative* levels. In § 4 we deduce the same result for general G and general good levels from this case.

3.1 Overview of the argument

The proof of the main theorem requires fine arguments involving combinatorics of affine Lie algebras. To help the reader understand what follows, we begin with an overview of the main ideas. This inherently requires referring to concepts that have not been introduced yet, so the reader may safely skip this material and refer back as necessary.

We omit some technical considerations at this point in the discussion. For example, we do not carefully distinguish here between abelian and derived categories.

3.1.1 Suppose $\lambda \in \mathfrak{t}^*$ is a weight of $\mathfrak{g}$. Let us denote the block of Category $\mathcal{O}$ for $\widehat{\mathfrak{g}}_\kappa$ containing the Verma module M_λ by

$$\mathcal{O}_{\kappa,\lambda} \subset \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^{\mathring{I}}.$$

In § 3.2.3, we recall that λ determines a subgroup $W_\lambda \subset W$, its integral Weyl group.[6] This is a Coxeter group, i.e. it comes equipped with a set of simple reflections.

[6] In spite of the notation, W_λ depends also on κ.

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We make important use of Theorem 3.5.2, which is due to Fiebig [Fie06], following earlier work of Soergel [Soe90]. This result asserts that if λ is antidominant,[7] $\mathcal{O}_{\kappa,\lambda}$ is determined as a category by the data of (i) the Coxeter group W_λ and (ii) the subgroup

$$W_\lambda^\circ \subset W_\lambda \tag{3.1}$$

stabilizing λ under the dot action of W on $\mathfrak{t}^*$; cf. § 2.9.5.

Remark 3.1.2. For us, the most important case is where λ is integral. Here we write $W_{\mathfrak{g},\kappa}$ in place of W_λ. In this case, $W_{\mathfrak{g},\kappa}$ contains the finite Weyl group W_f. If κ is irrational, the two are equal, and the simple reflections in $W_{\mathfrak{g},\kappa} = W_f$ are the usual ones determined by our fixed Borel. If κ is rational, there is one additional simple reflection; we provide an explicit formula for it in Lemma 3.4.3.

In Theorem 3.2.7, we find a block decomposition for the Whittaker category $\mathsf{Whit}_\kappa^{\mathrm{aff}}$ on the affine flag variety for G. These blocks are indexed by $W_{\mathfrak{g},\kappa}$-orbits in $\check{\Lambda}_G = \Lambda_{\check{G}}$.

In Theorem 3.3.3, we show that up to varying κ by an integral translate (which does not affect the Whittaker category), its neutral block is equivalent to an integral block of Category $\mathcal{O}$ for $\widehat{\mathfrak{g}}_{-\kappa}$. We generalize this to general blocks at good[8] levels in Corollary 3.4.8. These identifications preserve the natural highest-weight structures on both sides. From now on, we assume that κ is good.

3.1.4 For each block of $\mathsf{Whit}_\kappa^{\mathrm{aff}}$, we explicitly compute the combinatorial datum (3.1) of the corresponding block of Category $\mathcal{O}$ for the corresponding Kac–Moody algebra[9] and provide a form of Langlands duality for this datum.

Essentially by construction, for any such block the corresponding integral Weyl group is $W_{\mathfrak{g},\kappa}$, with simple reflections as indicated above. In Theorem 3.5.6, we construct an isomorphism $W_{\mathfrak{g},\kappa} \simeq W_{\check{\mathfrak{g}},\check{\kappa}}$ preserving simple reflections, and that is the identity on W_f.[10]

Note that for any block of $\mathcal{O}$ for the Kac–Moody algebra, the corresponding integral Weyl group canonically acts on the set of isomorphism classes of simple objects in this block. Therefore, the above considerations provide an action of $W_{\check{\mathfrak{g}},\check{\kappa}}$ on the set of isomorphism classes of simple objects in $\mathsf{Whit}_\kappa^{\mathrm{aff}}$, which is canonically identified with $\check{\Lambda}_G = \Lambda_{\check{G}}$. Again, by construction, this action coincides with the dot action of § 2.9.5, but for $\check{G}$ rather than G.

In Corollary 3.2.11 and in the proof of Theorem 3.6.1, we check that simple objects of $\mathsf{Whit}_\kappa^{\mathrm{aff}}$ corresponding to antidominant weights of $\check{G}$ are also standard objects for the highest-weight structure on this category. These observations amount to matching the data (3.1) with the data of affine category $\mathcal{O}$ for $\check{G}$, completing the proof of Theorem 1.3 in the Iwahori case.

3.1.5 In § 3.7, under the hypotheses of this section, we deduce the parahoric version of Theorem 1.3 from the Iwahori version. In particular, this includes the spherical version of the theorem.

We do this by identifying the parahoric categories as full[11] subcategories of the corresponding Iwahori categories and then identifying the essential images under the isomorphism of Theorem 3.6.1.

[7] This is in the sense of affine Kac–Moody algebras. In particular, the definition depends on κ.

[8] In our actual exposition, the definition of a good level is essentially rigged so that such a result holds. The content is rather in Proposition 3.4.5, which provides a concrete description of good levels.

[9] Again, this Kac–Moody algebra is essentially $\widehat{\mathfrak{g}}_{-\kappa}$, except that we may need to replace $-\kappa$ by an integral translate.

[10] We are not aware of the identification of Theorem 3.5.6 having appeared previously in the literature.

[11] This fully faithfulness only holds at the abelian categorical level.

3.2 Block decomposition for the Whittaker category

We begin by decomposing the Whittaker category into blocks.

3.2.1 To do so, it will be useful to simultaneously consider the case of twisted Whittaker categories. Thus, we fix $\lambda \in \mathfrak{t}^*$ and a level κ and consider

$$\mathsf{Whit}_\lambda^{\mathrm{aff}} := D_\kappa(\mathfrak{L}N, \psi\backslash\mathfrak{L}G/I, -\lambda).$$

While these DG categories depend on both λ and κ, we will study them with fixed κ and varying λ, and for this reason we suppress κ for ease of notation.

3.2.2 For indexing reasons, it will be convenient to rewrite this category as follows. Consider the automorphism of $\mathfrak{L}G$ given by

$$g \mapsto \tau(g)\check{\rho}(t^{-1}),$$

where τ is as in § 2.1.7. This induces an equivalence

$$D_\kappa(\mathfrak{L}N, \psi\backslash\mathfrak{L}G/I, -\lambda) \simeq D_{-\kappa}(I, \lambda\backslash\mathfrak{L}G/\mathfrak{L}N^-, \psi) \tag{3.2}$$

where the ψ on the right-hand side denotes a nondegenerate character of $\mathfrak{L}N^-$ of conductor 1. In the following discussion, we will think of $\mathsf{Whit}_\lambda^{\mathrm{aff}}$ via the latter expression.

3.2.3 It will be important for us to consider $\mathsf{Whit}_\lambda^{\mathrm{aff}}$ as a module for the following version of the affine Hecke category.

Let W_λ denote the integral Weyl group of λ.[12] Recall that this is the subgroup of W that generated the reflections $s_{\check{\alpha}}$ corresponding to affine coroots $\check{\alpha}$ satisfying

$$\langle \check{\alpha}, \lambda \rangle \in \mathbb{Z},$$

where the pairing is via the action at level $-\kappa$ as in § 2.9.7. For $w \in W_\lambda$, write $j_{w,!*}$ for the intermediate extension of the simple object of

$$D_{-\kappa}(I, \lambda\backslash IwI/I, -\lambda)^\heartsuit.$$

We define $\mathcal{H}_\lambda$ to be the full subcategory of

$$D_{-\kappa}(I, \lambda\backslash\mathfrak{L}G/I, -\lambda) \tag{3.3}$$

generated under colimits and shifts by the objects

$$j_{w,!*} \quad \text{for } w \in W_\lambda.$$

By [LY20, § 4],[13] $\mathcal{H}_\lambda$ is closed under convolution and so admits a unique monoidal DG structure for which its embedding into (3.3) is monoidal. In addition, *loc. cit.* shows that $\mathcal{H}$ is the neutral block of (3.3), i.e. it is the minimal direct summand containing the identity element.

3.2.4 Recall that the Iwasawa decomposition provides $\mathfrak{L}G$ with a stratification by the double cosets

$$Iw\mathfrak{L}N^- \quad \text{for } w \in \tilde{W}.$$

Let $\tilde{W}^f \subset \tilde{W}$ denote the subset of elements of minimal length in their left W_f-cosets. We now describe the affine Whittaker category for a single stratum.

---

[12] Later in the paper, the twist will be fixed at $\lambda = 0$, but the group and level will vary. We will accordingly denote this integral Weyl group by $W_{\mathfrak{g},-\kappa}$ instead. We hope this does not cause confusion and will reintroduce this change in notation when it first occurs.

[13] While [LY20] is written ostensibly over a finite field and in the finite-type setting, the arguments we cite from it straightforwardly adapt to the present situation.

Lemma 3.2.5. *For $w \notin \tilde{W}^f$, the affine Whittaker category on $Iw\mathfrak{L}N^-$ vanishes, i.e.*

$$D_{-\kappa}(I, \lambda\backslash Iw\mathfrak{L}N^-/\mathfrak{L}N^-, \psi) \simeq 0.$$

For $w \in \tilde{W}^f$, !-restriction to any closed point gives an equivalence

$$D_{-\kappa}(I, \lambda\backslash Iw\mathfrak{L}N^-/\mathfrak{L}N^-, \psi) \simeq \mathsf{Vect}.$$

Proof. The twistings corresponding to both κ and λ are trivializable on a double coset. Moreover, ψ is trivial on the stabilizer in $\mathfrak{L}N^-$ of the point $w \in I\backslash\mathfrak{L}G$ if and only if $w \in \tilde{W}^f$, which implies the desired identities. □

For $w \in \tilde{W}^f$ we denote the corresponding standard, simple, and costandard objects of $\mathsf{Whit}^{\mathrm{aff}}_\lambda$ by

$$j^\psi_{w,!}, \quad j^\psi_{w,!*}, \quad \text{and} \quad j^\psi_{w,*}.$$

Explicitly, under the identification with Vect above, they correspond to the relevant extensions of $\mathbb{C}[-\ell(ww_\circ)] \in \mathsf{Vect}$, where ℓ denotes the length function on $\tilde{W}$ and $w_\circ$ the longest element of W_f.

3.2.6 We next obtain the block decomposition of $\mathsf{Whit}^{\mathrm{aff}}_\lambda$. To state it, for any double coset

$$W_\lambda y W_f$$

we write $\mathsf{Whit}^{\mathrm{aff}}_{\lambda,y}$ for the full subcategory of $\mathsf{Whit}^{\mathrm{aff}}_\lambda$ generated under colimits and shifts by the objects

$$j^\psi_{x,*} \quad \text{for } x \in W_\lambda y W_f \cap \tilde{W}^f.$$

Theorem 3.2.7. *Each $\mathsf{Whit}^{\mathrm{aff}}_{\lambda,y}$ is preserved by the action of $\mathcal{H}_\lambda$, and the direct sum of inclusions yields an $\mathcal{H}_\lambda$-equivariant equivalence*

$$\bigoplus_{y \in W_\lambda\backslash\tilde{W}/W_f} \mathsf{Whit}^{\mathrm{aff}}_{\lambda,y} \to \mathsf{Whit}^{\mathrm{aff}}_\lambda. \tag{3.4}$$

Proof. For $w \in \tilde{W}$ consider the corresponding standard, simple, and costandard objects

$$j_{w,!}, \quad j_{w,!*}, \quad \text{and} \quad j_{w,*} \quad \text{of} \quad D_{-\kappa}(I, w\cdot\lambda\backslash\mathfrak{L}G/I, -\lambda)^\heartsuit.$$

We use $- \overset{I}{\star} -$ to denote the convolution functor

$$D_{-\kappa}(\mathfrak{L}G/I, -\lambda) \otimes D_{-\kappa}(I, \lambda\backslash\mathfrak{L}G) \to D_{-\kappa}(\mathfrak{L}G)$$

and the induced functor

$$D_{-\kappa}(I, w\cdot\lambda\backslash\mathfrak{L}G/I, -\lambda) \otimes D_{-\kappa}(I, \lambda\backslash\mathfrak{L}G/\mathfrak{L}N^-, \psi) \to D_{-\kappa}(I, w\cdot\lambda\backslash\mathfrak{L}G/\mathfrak{L}N^-, \psi). \tag{3.5}$$

For any $w \in \tilde{W}$, if with an abuse of notation we denote its image in $\tilde{W}^f \xrightarrow{\sim} \tilde{W}/W_f$ again by w, we claim that there exist equivalences

$$j_{w,!} \overset{I}{\star} j^\psi_{e,*} \simeq j^\psi_{w,!} \quad \text{and} \quad j_{w,*} \overset{I}{\star} j^\psi_{e,*} \simeq j^\psi_{w,*}. \tag{3.6}$$

Note that in (3.6), the object $j^\psi_{e,*}$ belongs to $\mathsf{Whit}^{\mathrm{aff}}_\lambda$, whereas $j^\psi_{w,!}$ and $j^\psi_{w,*}$ belong to $\mathsf{Whit}^{\mathrm{aff}}_{w\cdot\lambda}$.

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We split the proof of (3.6) into several cases. First, suppose w lies in $\tilde{W}^f$. Then the second identity in (3.6) follows from the observation that the convolution map

$$IwI \overset{I}{\times} I\mathfrak{L}N^- \to Iw\mathfrak{L}N^- \tag{3.7}$$

is an isomorphism. The first identity for such w then follows by the cleanness of $j^{\psi}_{e,*}$ and the ind-properness of the multiplication map

$$\mathfrak{L}G \overset{I}{\times} \mathfrak{L}G \to \mathfrak{L}G.$$

Next, suppose $w = s$ is a simple reflection of W_f. The image of the convolution map as in (3.7) is contained in the locally closed sub-ind-scheme of $\mathfrak{L}G$ corresponding to the strata $I\mathfrak{L}N^- \cup Is\mathfrak{L}N^-$. As the only stratum of its closure that supports Whittaker sheaves is $I\mathfrak{L}N^-$, it is enough to compute the !-restrictions of our convolutions to the identity

$$i : e \to \mathfrak{L}G.$$

Let us write ι for the involution $g \mapsto g^{-1}$ on $\mathfrak{L}G$. By base change we may compute the !-fibre as

$$i^!(j_{s,*} \overset{I}{\star} j^{\psi}_{e,*}) \simeq \Gamma_{dR}(I\backslash\mathfrak{L}G, \iota_* j_{s,*} \overset{!}{\otimes} j^{\psi}_{e,*}) \simeq \Gamma_{dR}(\mathbb{P}^1 \setminus \{0, \infty\}, \text{`}e^t\text{'}[-\ell(w_0)]) \simeq \mathbb{C}[-\ell(w_\circ)],$$

as desired. A similar calculation on $\mathbb{P}^1$ yields the first identity in (3.6) for $w = s$.

Finally, for general $w \in \tilde{W}$, we write $w = w^f w_f$ for $w^f \in \tilde{W}^f$ and $w_f \in W_f$. Choosing a reduced expression for w_f and the corresponding factorizations of $j_{w,!}$ and $j_{w,*}$, we reduce to the cases considered above for twists in $\tilde{W} \cdot \lambda$. Given (3.6), the assertions of the theorem follow by the same argument as for [LY20, Proposition 4.11]. □

3.2.8 Having obtained the block decomposition of $\mathsf{Whit}^{\mathrm{aff}}_\lambda$, we now record some properties of each block as an $\mathcal{H}_\lambda$ module. We begin with some relevant combinatorics.

Recall that W_λ is a Coxeter group. Write $\check{\Phi}^+$ for the positive real coroots and consider the subset

$$\check{\Phi}^+_\lambda = \{\check{\alpha} \in \check{\Phi}^+ : \langle \check{\alpha}, \lambda \rangle \in \mathbb{Z}\}.$$

With this, a reflection $s_{\check{\alpha}}$ of W_λ is simple if and only if $\check{\alpha}$ is not expressible as a sum of two other elements of $\check{\Phi}^+_\lambda$.

Lemma 3.2.9. *Each double coset $W_\lambda y W_f$ contains a unique element that is minimal with respect to the Bruhat order. In particular, each W_λ orbit on $\tilde{W}/W_f$ contains a unique element yW_f that is minimal with respect to the Bruhat order. Moreover, its stabilizer*

$$yW_f y^{-1} \cap W_\lambda \tag{3.8}$$

is a parabolic subgroup of W_λ.

Proof. Let y be any element of minimal length in $W_\lambda y W_f$. We first claim that $y \leqslant w_\lambda y$ for every $w_\lambda \in W_\lambda$. Indeed, for any reflection $s_{\check{\alpha}}$ of W_λ with corresponding positive coroot $\check{\alpha}$, it follows from the minimal length of y that

$$s_{\check{\alpha}} y > y, \tag{3.9}$$

i.e. $y^{-1}(\check{\alpha}) > 0$. The claimed minimality now follows by induction on the length of an element of W_λ with respect to its Coxeter generators.

We next show that $y \leqslant w_\lambda y w_f$ for any w_f in W_f. However, it is clear that $y \leqslant ys$ for any simple reflection s of W_f. This implies that for any $w \in \tilde{W}$, we have $y \leqslant w$ if and only if $y \leqslant ws_j$. Applying this and a straightforward induction on the length to $w = w_\lambda y$ yields the claim.

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It remains to show that (3.8) is a parabolic subgroup of W_λ. To see this, consider $W_{y^{-1}\lambda} = y^{-1}W_\lambda y$. The positivity property (3.9) of y implies that conjugation by y^{-1} defines an isomorphism of Coxeter systems

$$W_\lambda \simeq W_{y^{-1}\lambda},$$

i.e. exchange of their simple reflections. It is therefore enough to show that $W_f \cap W_{y^{-1}\lambda}$ is a parabolic subgroup of $W_{y^{-1}\lambda}$, which is straightforward; cf. the proof of Lemma 8.8 in [Dhi21]. □

3.2.10 Combining Lemma 3.2.9 and Theorem 3.2.7, we obtain the following result.

Corollary 3.2.11. *For a double coset $W_\lambda y W_f$ with minimal element y, the corresponding object of $\mathsf{Whit}^{\mathrm{aff}}_{\lambda,y}$ is clean, i.e.*

$$j^\psi_{y,!} \simeq j^\psi_{y,!*} \simeq j^\psi_{y,*}.$$

In the following proposition, we collect some properties of the action of $\mathcal{H}_\lambda$ on $j^\psi_{y,!}$. To state them, let us denote the set of elements of W_λ of minimal length in their left cosets with respect to the parabolic subgroup (3.8) by

$$W^f_\lambda. \tag{3.10}$$

We additionally recall that $\mathsf{Whit}^{\mathrm{aff}}_\lambda$ carries a canonical t-structure; namely, an object $\mathcal{M}$ is coconnective, i.e. lies in $\mathsf{Whit}^{\mathrm{aff},\geqslant 0}_\lambda$, if and only if

$$\mathrm{Hom}(j^\psi_{w,!}, \mathcal{M}) \in \mathsf{Vect}^{\geqslant 0} \quad \text{for } w \in \tilde{W}^f.$$

Such a t-structure may be seen directly via gluing of t-structures stratum by stratum, and coincides with the general construction in § 5.2 and Appendix B of [Ras21b]; cf. *loc. cit.* Remark B.7.1.

Proposition 3.2.12. *For y as in Corollary 3.2.11 and for any w in W_λ, there are equivalences*

$$j_{w,!} \overset{I}{\star} j^\psi_{y,*} \simeq j^\psi_{wy,!} \quad \text{and} \quad j_{w,*} \overset{I}{\star} j^\psi_{y,*} \simeq j^\psi_{wy,*}. \tag{3.11}$$

Moreover, if w lies in W^f_λ, then convolution with $j^\psi_{y,*}$ yields an isomorphism of lines

$$\mathrm{Hom}_{D_{-\kappa}(I,\lambda\backslash \mathfrak{L}G/I,-\lambda)^\heartsuit}(j_{w,!}, j_{w,*}) \simeq \mathrm{Hom}_{\mathsf{Whit}^{\mathrm{aff}}_\lambda}(j^\psi_{wy,!}, j^\psi_{wy,*}). \tag{3.12}$$

Proof. Both assertions follow from (3.6) by standard arguments. Specifically, the proof of Proposition 5.2 in [LY20] yields (3.11). It follows from (3.11) that convolution with $j^\psi_{y,*}$ is t-exact.

Let us show that the latter observation implies (3.12). For any $u \in W_\lambda$, recall that $j_{u,!*}$ denotes the simple quotient of $j_{u,!}$. By t-exactness, we obtain a surjection

$$j^\psi_{uy,!} \overset{(3.11)}{\simeq} j_{u,!} \overset{I}{\star} j^\psi_{y,*} \to j_{u,!*} \overset{I}{\star} j^\psi_{y,*} \to 0. \tag{3.13}$$

For an element w of W^f_λ, consider the tautological exact sequence

$$0 \to K \to j_{w,!} \to j_{w,!*} \to 0.$$

By definition, K admits a filtration by intermediate extensions $j_{u,!*}$ for $u < w$ in the Bruhat order on W_λ. Convolving with $j^\psi_{y,*}$, by t-exactness we again obtain an exact sequence

$$0 \to K \overset{I}{\star} j^\psi_{y,*} \to j^\psi_{wy,!} \to j_{w,!*} \overset{I}{\star} j^\psi_{y,*} \to 0.$$

By our assumption on w, it follows from (3.13) that $[K \overset{I}{\star} j^\psi_{y,*} : j^\psi_{wy,!*}] = 0$. In particular, $j_{w,!*} \overset{I}{\star} j^\psi_{y,*}$ is nonzero.

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This nonvanishing and the t-exactness of convolution with $j^{\psi}_{y,*}$ imply that the intermediate extension sequence $j_{w,!} \twoheadrightarrow j_{w,!*} \hookrightarrow j_{w,*}$ yields a sequence of nonzero maps

$$j_{w,!} \overset{I}{\star} j^{\psi}_{y,*} \twoheadrightarrow j_{w,!*} \overset{I}{\star} j^{\psi}_{y,*} \hookrightarrow j_{w,*} \overset{I}{\star} j^{\psi}_{y,*}.$$

This verifies (3.12) (and also shows that $j_{w,!*} \overset{I}{\star} j^{\psi}_{y,*} \simeq j^{\psi}_{wy,!*}$). □

Remark 3.2.13. One can show that the clean objects $j^{\psi}_{y,!*}$ constructed above are the only clean extensions in $\mathsf{Whit}^{\mathrm{aff}}_{\lambda}$. Since we do not use this fact, we only sketch the proof. Specifically, write $\leqslant$ for the Bruhat order on W^f_λ. One can in fact show that for any $w, x \in W^f_\lambda$, the space of intertwining operators

$$\mathrm{Hom}_{\mathsf{Whit}^{\mathrm{aff},\heartsuit}_{\lambda}}(j^{\psi}_{wy,!}, j^{\psi}_{xy,!})$$

vanishes unless $w \leqslant x$, in which case it is one-dimensional and consists of embeddings. This may be deduced from (3.11) and the analogous classification of intertwining operators between standard objects of $\mathcal{H}_\lambda$. The latter may be proved directly or deduced via localization from the Kac–Kazhdan theorem on homomorphisms between Verma modules.

3.3 Whittaker singular duality

We now use the above-noted properties of the $\mathcal{H}_\lambda$-action to produce, under necessary hypotheses, equivalences between the neutral block of $\mathsf{Whit}^{\mathrm{aff}}_{\lambda}$ and a block of Category $\mathcal{O}$ for $\widehat{\mathfrak{g}}_{-\kappa}$.

3.3.1 Consider the category of (I, λ)-integrable modules for $\widehat{\mathfrak{g}}_{-\kappa}$, which we denote by

$$\widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}^{I,\lambda}. \tag{3.14}$$

Recall that this is compactly generated by the Verma modules M_θ for $\theta \in \lambda + \Lambda_G$, and note that the integral Weyl group of any such θ is W_λ.

3.3.2 We identify the desired blocks by matching their combinatorics to that of $\mathsf{Whit}^{\mathrm{aff}}_{e,\lambda}$ as follows. Suppose M_Θ is an irreducible Verma module in (3.14) such that the stabilizer of Θ under the dot action of W_λ is given by

$$W_\lambda \cap W_f.$$

Consider the corresponding block of (3.14), i.e. the full subcategory compactly generated by M_θ for $\theta \in W_\lambda \cdot \Theta$, and denote it by

$$\widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}^{I,\lambda}_{\Theta}.$$

THEOREM 3.3.3. *There is a canonical $\mathcal{H}_\lambda$-equivariant and t-exact equivalence*

$$\mathsf{Whit}^{\mathrm{aff}}_{e,\lambda} \simeq \widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}^{I,\lambda}_{\Theta}. \tag{3.15}$$

Remark 3.3.4. For emphasis, by definition of Θ, the integral Weyl group of this block is W_λ, and the stabilizer of Θ in W_λ is $W_\lambda \cap W_f$. In particular, on both sides of (3.15) one finds highest-weight categories whose highest weights are indexed by

$$W_\lambda/(W_\lambda \cap W_f) \simeq W^f_\lambda;$$

for $\mathsf{Whit}^{\mathrm{aff}}_{e,\lambda}$ compare (3.10) and (3.8) and recall that we are considering the neutral block, i.e. specializing to $y = e$ in (3.8).

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Remark 3.3.5. The minus sign in front of the level is an artifact of § 3.2.2. As we will discuss in greater detail in § 3.4.1, we potentially need to apply an integral translation to $\kappa \rightsquigarrow \kappa + \kappa'$ before finding Θ as above; in practice, this translation in particular ensures that $-\kappa - \kappa'$ is negative.

Example 3.3.6. Theorem 3.3.3 is always applicable when the twist λ is trivial; namely, by an integral translation, we may assume that $-\kappa$ is sufficiently negative, in which case we may take Θ to be $-\rho$.

Proof of Theorem 3.3.3. To produce an $\mathcal{H}_\lambda$-equivariant functor, we will construct a $D_{-\kappa}(\mathfrak{L}G)$-equivariant functor and then pass to (I,λ)-equivariant objects. After further projecting onto a block of (3.14), we will show that this is the sought-for equivalence.

We begin by constructing an $D_{-\kappa}(\mathfrak{L}G)$-equivariant functor

$$\mathsf{F} : D_{-\kappa}(\mathfrak{L}G/\mathfrak{L}N^-, \psi) \to \widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}.$$

To do so, recall that for any $D_{-\kappa}(\mathfrak{L}G)$-module $\mathcal{C}$ one has a canonical equivalence

$$\mathsf{Hom}_{D_{-\kappa}(\mathfrak{L}G)\text{–}\mathsf{mod}}(D_{-\kappa}(\mathfrak{L}G/\mathfrak{L}N^-, \psi), \mathcal{C}) \simeq \mathcal{C}^{\mathfrak{L}N^-,\psi}. \tag{3.16}$$

Explicitly, if we write $\operatorname{ins}\delta$ for the insertion of the delta function at the identity of $\mathfrak{L}G$ into Whittaker coinvariants, the equivalence (3.16) is given by evaluation at $\operatorname{ins}\delta$. Applying this to $\mathcal{C} \simeq \widehat{\mathfrak{g}}_{-\kappa}$–$\mathsf{mod}$, we obtain via the affine Skryabin equivalence (cf. [Ras21b])

$$\mathsf{Hom}_{D_{-\kappa}(\mathfrak{L}G)\text{–}\mathsf{mod}}(D_{-\kappa}(\mathfrak{L}G/\mathfrak{L}N^-, \psi), \widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}) \simeq \widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}^{\mathfrak{L}N^-,\psi} \simeq \mathcal{W}_{-\kappa}\text{–}\mathsf{mod}. \tag{3.17}$$

Therefore, to produce F we must specify a module for the $\mathcal{W}_{-\kappa}$-algebra.

We do so as follows. Write $\mathrm{Zhu}(\mathcal{W}_{-\kappa})$ for the Zhu algebra of $\mathcal{W}_{-\kappa}$ and $Z\mathfrak{g}$ for the center of the universal enveloping algebra of $\mathfrak{g}$. Let us normalize the identification

$$\mathrm{Zhu}(\mathcal{W}_{-\kappa}) \simeq Z\mathfrak{g}$$

as in [Dhi21]. Writing $\chi(\Theta)$ for the character of $Z\mathfrak{g}$ corresponding to Θ, i.e. via its action on the Verma module for $\mathfrak{g}$ with highest weight Θ, consider the associated local cohomology $Z\mathfrak{g}$-module $Ri^!_{\chi(\Theta)}Z\mathfrak{g}$. We will take F to be associated to the corresponding $\mathcal{W}_{-\kappa}$-module

$$\mathrm{pind}_{\mathrm{Zhu}(\mathcal{W}_{-\kappa})}^{\mathcal{W}_{-\kappa}} Ri^!_{\chi(\Theta)}Z\mathfrak{g}, \tag{3.18}$$

where pind denotes the standard induction from Zhu algebra modules to vertex algebra modules, i.e. the left adjoint to the functor of taking singular vectors.

Passing to (I,λ)-equivariant objects, we obtain a $D_{-\kappa}(I,\lambda\backslash\mathfrak{L}G/I,-\lambda)$-equivariant functor

$$\mathsf{F} : D_{-\kappa}(I,\lambda\backslash\mathfrak{L}G/\mathfrak{L}N^-, \psi) \to \widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}^{I,\lambda}.$$

Let us determine $\mathsf{F}(j^{\psi}_{e,*})$. To do so, write $\mathbb{C}_\psi$ for the one-dimensional representation of $\mathfrak{n}^-$ associated to the additive character ψ of $\mathfrak{L}N^-$. As we will explain in more detail below, we then have

$$\mathrm{Av}^{I,\lambda}_* \, \mathrm{pind}_{\mathrm{Zhu}(\mathcal{W}_{-\kappa})}^{\mathcal{W}_{-\kappa}} Ri^!_{\chi(\Theta)}Z\mathfrak{g} \simeq \mathrm{Av}^{I,\lambda}_* \, \mathrm{pind}_{\mathfrak{g}}^{\widehat{\mathfrak{g}}_{-\kappa}}(Ri^!_{\chi(\Theta)}Z\mathfrak{g} \underset{Z\mathfrak{g}}{\otimes} \mathrm{ind}_{\mathfrak{n}^-}^{\mathfrak{g}} \mathbb{C}_\psi) \tag{3.19}$$

$$\simeq \mathrm{pind}_{\mathfrak{g}}^{\widehat{\mathfrak{g}}_{-\kappa}} \mathrm{Av}^{B,\lambda}_*(Ri^!_{\chi(\Theta)}Z\mathfrak{g} \underset{Z\mathfrak{g}}{\otimes} \mathrm{ind}_{\mathfrak{n}^-}^{\mathfrak{g}} \mathbb{C}_\psi) \tag{3.20}$$

$$\simeq \bigoplus_{\xi \in (W_f\cdot\Theta)\cap\lambda+\Lambda_G} M_\xi \otimes \det(\mathfrak{b}^*[-1]). \tag{3.21}$$

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To see the first identification, (3.19), one may use that (3.18) is an iterated extension of Verma modules and in particular arises from the first step of the adolescent Whittaker filtration of $\mathcal{W}_{-\kappa}$–mod; cf. [Ras21b].

For the second identification, (3.20), if we write K_1 for the first congruence subgroup of $\mathfrak{L}^+G$, one has a corresponding factorization

$$\mathrm{Av}_*^{I,\lambda} \simeq \mathrm{Av}_*^{K_1,\lambda} \circ \mathrm{Av}_*^{B,\lambda}.$$

The second identification then follows from the K_1-integrability of a parabolically induced module, the pro-unipotence of K_1 (cf. Theorem 4.3.2 of [Ber17]), and the $D(G)$-equivariance of $\mathrm{pind}_{\mathfrak{g}}^{\widehat{\mathfrak{g}}_{-\kappa}}$.

For the third identification, (3.21), for any $\xi \in \lambda + \Lambda_G$ let us write $M_{\xi,\mathfrak{g}}$ for the corresponding Verma module for $\mathfrak{g}$. One has by adjunction

$$\mathrm{Hom}_{\mathfrak{g}\text{–}\mathsf{mod}^{B,\lambda}}(M_{\xi,\mathfrak{g}}, \mathrm{Av}_*^{B,\lambda} Ri^!_{\chi(\Theta)} Z\mathfrak{g} \underset{Z\mathfrak{g}}{\otimes} \mathrm{ind}_{\mathfrak{n}^-}^{\mathfrak{g}} \mathbb{C}_\psi) \simeq \mathrm{Hom}_{\mathfrak{g}\text{–}\mathsf{mod}}(M_{\xi,\mathfrak{g}}, Ri^!_{\chi(\Theta)} Z\mathfrak{g} \underset{Z\mathfrak{g}}{\otimes} \mathrm{ind}_{\mathfrak{n}^-}^{\mathfrak{g}} \mathbb{C}_\psi).$$

The latter vanishes unless $\xi \in W_f \cdot \Theta$, in which case we continue the above to

$$\simeq \mathrm{Hom}_{\mathfrak{g}\text{–}\mathsf{mod}}(M_{\xi,\mathfrak{g}}, \mathrm{ind}_{\mathfrak{n}^-}^{\mathfrak{g}} \mathbb{C}_\psi) \simeq \mathrm{Hom}_{\mathfrak{b}\text{–}\mathsf{mod}}(\mathbb{C}_\xi, \mathrm{Res}_{\mathfrak{g}}^{\mathfrak{b}} \mathrm{ind}_{\mathfrak{n}^-}^{\mathfrak{g}} \mathbb{C}_\psi) \simeq \det(\mathfrak{b}^*[-1]),$$

where in the last step one uses the canonical identification

$$U(\mathfrak{b}) \simeq \mathrm{Res}_{\mathfrak{g}}^{\mathfrak{b}} \mathrm{ind}_{\mathfrak{n}^-}^{\mathfrak{g}} \mathbb{C}_\psi.$$

By our assumption on Θ, each $M_{\xi,\mathfrak{g}}$ for $\xi \in (W_f \cdot \Theta) \cap \lambda + \Lambda_G$ generates a block of $\mathfrak{g}$–$\mathsf{mod}^{B,\lambda}$ equivalent to Vect; thus we have shown that

$$\mathrm{Av}_*^{B,\lambda} Ri^!_{\chi(\Theta)} Z\mathfrak{g} \underset{Z\mathfrak{g}}{\otimes} \mathrm{ind}_{\mathfrak{n}^-}^{\mathfrak{g}} \mathbb{C}_\psi \simeq \bigoplus_{\xi \in (W_f \cdot \Theta) \cap \lambda + \Lambda_G} M_{\xi,\mathfrak{g}} \otimes \det(\mathfrak{b}^*[-1]).$$

The identity (3.21) then follows by applying $\mathrm{pind}_{\mathfrak{g}}^{\widehat{\mathfrak{g}}_{-\kappa}}$.

The projection of the sum of Verma modules (3.21) onto $\widehat{\mathfrak{g}}_{-\kappa}$–$\mathsf{mod}_\Theta^{I,\lambda}$ picks out the summand

$$M_\Theta \otimes \det(\mathfrak{b}^*[-1]);$$

cf. [Dhi21, Lemma 8.7]. Accordingly, we consider the composition

$$D_{-\kappa}(I, \lambda\backslash\mathfrak{L}G/\mathfrak{L}N^-, \psi) \xrightarrow{\mathsf{F} \otimes \det(\mathfrak{b}[1])[\dim N^-]} \widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}^{I,\lambda} \to \widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}_\Theta^{I,\lambda}, \tag{3.22}$$

where the latter map is projection onto the block. Note that while the blocks of (3.14) are in general not preserved by $D_{-\kappa}(I, \lambda\backslash\mathfrak{L}G/I, -\lambda)$, they are preserved by $\mathcal{H}_\lambda$. In particular, by construction the composition (3.22), which we denote by Ψ, is an $\mathcal{H}_\lambda$-equivariant functor which sends $j^\psi_{e,*}$ to M_Θ.

It remains to show that Ψ is an equivalence. By (3.11) and $\mathcal{H}_\lambda$-equivariance, we obtain identifications

$$\Psi(j^\psi_{w,!}) \simeq \Psi(j_{w,!} \overset{I}{\star} j^\psi_{e,*}) \simeq j_{w,!} \overset{I}{\star} \Psi(j^\psi_{e,*}) \simeq M_{w\cdot\Theta} \quad \text{for } w \in W_\lambda,$$

where the last identity is a standard consequence of Kashiwara–Tanisaki localization at a negative level [KT96].[14] Similarly, if for $\theta \in \lambda + \Lambda$ we denote the contragredient dual of M_θ by A_θ, we obtain identifications

$$\Psi(j^\psi_{w,*}) \simeq \Psi(j_{w,*} \overset{I}{\star} j^\psi_{e,*}) \simeq j_{w,*} \overset{I}{\star} \Psi(j^\psi_{e,*}) \simeq A_{w\cdot\Theta} \quad \text{for } w \in W_\lambda.$$

[14] It also is straightforward to prove directly; see Lemma 4.0.8 of [CD21].

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Recall that the (co)standard objects $j^{\psi}_{w,!}$ and $j^{\psi}_{w,*}$ of $\mathsf{Whit}_{e,\lambda}$ are indexed by the minimal-length coset representatives of W_λ with respect to its parabolic subgroup $W_\lambda \cap W_f$, i.e.

$$W^f_\lambda,$$

and either collection compactly generates $\mathsf{Whit}_{e,\lambda}$.

Similarly, by our assumption on the stabilizer of Θ and the irreducibility of M_Θ, the collections of (co)standard objects $M_{w\cdot\Theta}$ and $A_{w\cdot\Theta}$, for $w \in W^f_\lambda$, each compactly generates $\widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}^{I,\lambda}_{\Theta}$.

It is therefore enough to check that for any $y, w \in W^f_\lambda$, the map

$$\Psi : \mathrm{Hom}_{\mathsf{Whit}^{\mathrm{aff}}_\lambda}(j^{\psi}_{y,!}, j^{\psi}_{w,*}) \to \mathrm{Hom}_{\widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}^{I,\lambda}}(M_{y\cdot\Theta}, A_{y\cdot\Theta})$$

is an equivalence. This again follows from (3.12), as desired. □

3.3.7 While Theorem 3.2.7 realizes the neutral block of $\mathsf{Whit}^{\mathrm{aff}}_\lambda$, this may be applied to other blocks as follows. Fix a double coset

$$y \in W_\lambda\backslash\tilde{W}/W_f$$

with associated minimal element y and block $\mathsf{Whit}^{\mathrm{aff}}_{\lambda,y}$.

Proposition 3.3.8. *Convolution with the clean object*

$$j_{y,*} \in D_{-\kappa}(I, \lambda\backslash\mathfrak{L}G/I, -y^{-1}\cdot\lambda)$$

yields a t-exact equivalence

$$\mathsf{Whit}^{\mathrm{aff}}_{y^{-1}\cdot\lambda,e} \xrightarrow{\sim} \mathsf{Whit}^{\mathrm{aff}}_{\lambda,y}.$$

Proof. The argument for Proposition 5.2 in [LY20] applies *mutatis mutandis.* □

3.4 Application of Whittaker singular duality and the classification of good levels

3.4.1 We would like to relate an arbitrary block $\mathsf{Whit}^{\mathrm{aff}}_{\lambda,y}$ of $\mathsf{Whit}^{\mathrm{aff}}_\lambda$ to Kac–Moody representations. Via Proposition 3.3.8, we should apply Theorem 3.2.7 to $\mathsf{Whit}^{\mathrm{aff}}_{y^{-1}\cdot\lambda,e}$. Therefore, we would like to produce a suitable Verma module in

$$\widehat{\mathfrak{g}}_{-\kappa}\text{–}\mathsf{mod}^{I,y^{-1}\cdot\lambda}. \tag{3.23}$$

It will be useful to expand the collection of available Verma modules. For example, if $-\kappa$ is positive rational and the twist $y^{-1}\cdot\lambda$ is trivial, there are no irreducible Verma modules in (3.23). To address this, note that for any integral level κ' for G one has a tautological t-exact equivalence

$$D_{-\kappa}(I, y^{-1}\cdot\lambda\backslash\mathfrak{L}G/\mathfrak{L}N^-, \psi) \simeq D_{-\kappa+\kappa'}(I, y^{-1}\cdot\lambda\backslash\mathfrak{L}G/\mathfrak{L}N^-, \psi).$$

We may further increase our supply of integral levels and characters as follows. Write G_s for the simply connected form of G and I_s for the Iwahori subgroup of its loop group. Then the tautological embedding

$$D_{-\kappa}(I_s, y^{-1}\cdot\lambda\backslash\mathfrak{L}G_s/\mathfrak{L}N^-, \psi) \to D_{-\kappa}(I, y^{-1}\cdot\lambda\backslash\mathfrak{L}G/\mathfrak{L}N^-, \psi)$$

induces an equivalence of neutral blocks.

3.4.2 To analyze when we may find the desired Verma modules, we will need some basic properties of the relevant integral Weyl group. So for an arbitrary level $\kappa_\circ$ let us denote by $W_{\mathfrak{g},\kappa_\circ}$

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the integral Weyl group of $0 \in \mathfrak{t}^*$ at level $\kappa_\circ$, i.e. what was denoted by W_0 in the notation of § 3.2.3.

To describe the simple reflections in $W_{\mathfrak{g},\kappa_\circ}$, recall the canonical identification of W with the semidirect product

$$W_f \ltimes \check{Q}.$$

For any finite coroot $\check{\alpha}$ we denote by $s_{\check{\alpha}}$ the corresponding reflection in W_f, and for an element $\check{\lambda}$ of the coroot lattice $\check{Q}$ we write $t^{\check{\lambda}}$ for the corresponding translation in W. In addition, let us write $\check{\theta}_s$ for the short dominant coroot, $\check{\theta}_l$ for the long dominant coroot, and r for the lacing number of $\mathfrak{g}$.

Lemma 3.4.3. *For any $\kappa_\circ$ and integral level κ', there are canonical identifications*

$$W_{\mathfrak{g},\kappa_\circ} \simeq W_{\mathfrak{g},-\kappa_\circ} \simeq W_{\mathfrak{g},\kappa_\circ+\kappa'} \tag{3.24}$$

intertwining their inclusions into W. If $\kappa_\circ$ is irrational, then $W_{\mathfrak{g},\kappa_\circ} \simeq W_f$, i.e. they coincide as subgroups of W. If $\kappa_\circ$ is rational, write it as

$$\kappa_\circ = \left(-h^\vee + \frac{p}{q}\right) \kappa_b,$$

where $h^\vee$ is the dual Coxeter number, p and q are coprime integers, and κ_b is the basic level. Then $W_{\mathfrak{g},\kappa_\circ}$ has simple reflections given by the simple reflections of W_f and the additional reflection

$$s_{0,\kappa_\circ} = \begin{cases} s_{\check{\theta}_s} t^{q\cdot\check{\theta}_s} & \text{if } (q,r) = 1, \\ s_{\check{\theta}_l} t^{q/r\cdot\check{\theta}_l} & \text{if } (q,r) = r. \end{cases} \tag{3.25}$$

Proof. Recall the standard enumeration of the affine real coroots as

$$\check{\Phi} \simeq \check{\Phi}_f \times \mathbb{Z},$$

where $\check{\Phi}_f$ denotes the finite coroots. With this enumeration, the element $\check{\alpha}_n \in \mathfrak{t} \oplus \mathbb{C}\mathbf{1}$, for $\check{\alpha} \in \check{\Phi}_f$ and $n \in \mathbb{Z}$, is given by

$$\check{\alpha} + n\frac{\kappa_\circ(\check{\alpha},\check{\alpha})}{2}\,\mathbf{1}.$$

In particular, $\check{\alpha}_n$ belongs to $W_{\mathfrak{g},\kappa_\circ}$ if and only if

$$\langle \check{\alpha}_n, 0\rangle = n\frac{\kappa_\circ(\check{\alpha},\check{\alpha})}{2}$$

is an integer, which straightforwardly implies the claims of the lemma. □

3.4.4 Having explicitly identified the Coxeter generators of the integral Weyl group, we will now obtain for most levels highest weights with prescribed stabilizers within it. Let us formulate this problem precisely. If we write Λ_{G_s} for the weight lattice and recall that I_s denotes the Iwahori subgroup of $\mathfrak{L}G_s$, then

$$\widehat{\mathfrak{g}}_{\kappa_\circ}\text{–}\mathsf{mod}^{I_s}$$

is compactly generated by the Verma modules M_λ for $\lambda \in \Lambda_{G_s}$. In particular, this category has highest weights consisting of the weight lattice. Let us say that a level $\kappa_\circ$ is *good* if for any finite parabolic subgroup $W_\circ$ of $W_{\mathfrak{g},\kappa_\circ}$ there exists an integral level κ' and a simple Verma module

$$M_\nu \in \widehat{\mathfrak{g}}_{\kappa_\circ+\kappa'}\text{–}\mathsf{mod}^{I_s}$$

whose highest weight has stabilizer $W_\circ$. Let us classify the good levels.

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Table 1. Bad primes for each simple Lie algebra.

$\mathfrak{g}$	$n(\mathfrak{g})$
A_n	1
B_n	2
C_n	2
D_n	2
E_6	$2\cdot 3$
E_7	$2\cdot 3$
E_8	$2\cdot 3\cdot 5$
F_4	$2\cdot 3$
G_2	$2\cdot 3$

Proposition 3.4.5. *Every irrational level is good. A rational level*

$$\kappa_\circ = \left(-h^\vee + \frac{p}{q}\right)\kappa_b,$$

where p and q are coprime integers, is good if and only if q is coprime to the number $n(\mathfrak{g})$ associated to $\mathfrak{g}$ in Table 1.

Proof. For $\kappa_\circ$ irrational, the claim is clear as $W_{\mathfrak{g},\kappa_\circ} \simeq W_f$. For $\kappa_\circ$ rational, which we may take to be negative, a weight $\lambda \in \Lambda_{G_s}$ is antidominant if and only if

$$\langle \lambda+\rho, \check{\alpha}_i\rangle \leqslant 0 \quad \text{for } i \in \mathfrak{I} \quad \text{and} \quad \begin{cases} \langle \lambda+\rho, \check{\theta}_s\rangle \geqslant -p & \text{if } (q,r)=1,\\ \langle \lambda+\rho, \check{\theta}_l\rangle \geqslant -p & \text{if } (q,r)=r,\end{cases}$$

as follows from (3.25). Let us write ω_i, with $i \in \mathfrak{I}$, for the fundamental weights and write the dominant coroots as sums of simple coroots

$$\check{\theta}_s = \sum_{i\in\mathfrak{I}} n_i\check{\alpha}_i, \quad \check{\theta}_l = \sum_{i\in\mathfrak{I}} m_i\check{\alpha}_i \quad \text{for } n_i, m_i \in \mathbb{Z}^{\geqslant 0}.$$

Recall the standard correspondence between finite parabolic subgroups $W_\circ$ of $W_{\mathfrak{g},\kappa_\circ}$ and nonempty faces of the above alcove, which associates to a face the stabilizer of any interior point. It follows that, after the transformation $\lambda \mapsto -\lambda-\rho$, we are looking for points of Λ_{G_s} within the alcove with vertices at

$$0 \quad \text{and} \quad \begin{cases} \dfrac{p}{n_i}\,\omega_i & \text{for } i\in\mathfrak{I} \quad \text{if } (q,r)=1,\\ \dfrac{p}{m_i}\,\omega_i & \text{for } i\in\mathfrak{I} \quad \text{if } (q,r)=r.\end{cases}$$

Recalling that we are free to replace p by any element of $p+q\mathbb{Z}$, it is straightforward to see that we can find points of Λ_{G_s} in the interior of every face of the alcove if and only if for each $i \in \mathfrak{I}$ one has

$$\begin{cases} p \in (n_i, q) & \text{for } i\in\mathfrak{I} \quad \text{if } (q,r)=1,\\ p\in(m_i,q) & \text{for } i\in\mathfrak{I} \quad \text{if } (q,r)=r.\end{cases}$$

To see this, note that these conditions are tautologically equivalent to being able to realize each vertex of the alcove as a point of Λ_{G_s}, so they are necessary. To see that they are sufficient, suppose they are satisfied. Via this assumption, for any positive integer N we may replace p

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with an element of $p + q\mathbb{Z}$ so that

$$\frac{p}{n_i} \in \mathbb{Z}^{\geqslant N} \quad \text{for all } i \in \mathcal{I}.$$

In particular, we may assume that N is greater than the number of vertices of the alcove, i.e. $N > |I| + 1$. In this case, every face of the alcove contains an interior point expressible as a convex combination of the vertices with coefficients in $(1/N)\mathbb{Z}$. As such a convex combination is a point of Λ_{G_s}, we are done.

Finally, recalling the n_i and m_i for each type (cf. Plates I–IX of [Bou02]) yields the entries of Table 1. Specifically, if n_i is prime, then $p \in (n_i, q)$ if and only if $(n_i, q) = 1$; and if n_i is composite, then each of its prime divisors occurs as another $n_{i'}$. □

3.4.6 In what follows, we will be most concerned with the Whittaker category on $\mathfrak{L}G/I$, i.e. $\mathsf{Whit}^{\mathrm{aff}}_{\lambda}$ for $\lambda = 0$. In this case, we replace λ by the level κ.

In other words, $\mathsf{Whit}^{\mathrm{aff}}_{\kappa} := \mathsf{Whit}^{\mathrm{aff}}_{0}$. We similarly denote the summands $\mathsf{Whit}^{\mathrm{aff}}_{0,y}$ of this category considered in Theorem 3.2.7 by $\mathsf{Whit}^{\mathrm{aff}}_{\kappa,y}$.

3.4.7 Let us obtain for good levels $\kappa_\circ$ the Kac–Moody realization of blocks of the Whittaker category. Fix a double coset, whose minimal-length element we denote by y, in

$$W_{\mathfrak{g},\kappa_\circ}\backslash \tilde{W}/W_f.$$

Corollary 3.4.8. *If $\kappa_\circ$ is good, then for any y as above, the corresponding block $\mathsf{Whit}^{\mathrm{aff}}_{\kappa,y}$ admits an equivalence with a block of $\widehat{\mathfrak{g}}_{\kappa_\circ+\kappa'}\text{–}\mathsf{mod}^{I,y^{-1}\cdot 0}$ for some integral level κ'.*

Proof. By Proposition 3.3.8 and Theorem 3.2.7, it is enough to produce a simple Verma module in

$$\widehat{\mathfrak{g}}_{\kappa_\circ+\kappa'}\text{–}\mathsf{mod}^{I,y^{-1}\cdot 0}$$

whose (antidominant) highest weight Θ has stabilizer under the dot action of W given by

$$y^{-1}W_{\mathfrak{g},\kappa_\circ}y \cap W_f.$$

By (3.9), it is equivalent to produce a simple Verma module in

$$\widehat{\mathfrak{g}}_{\kappa_\circ+\kappa'}\text{–}\mathsf{mod}^{I}$$

whose highest weight $y \cdot \Theta$ has stabilizer given by

$$W_{\mathfrak{g},\kappa_\circ} \cap yW_f y^{-1}. \tag{3.26}$$

The latter is provided by Proposition 3.4.5, as desired. □

Remark 3.4.9. As in Remark 3.3.4, and at the risk of redundancy, we emphasize that by construction, the integral Weyl group of the above block is identified as a Coxeter system with $W_{\mathfrak{g},\kappa}$, and the stabilizer of Θ in $W_{\mathfrak{g},\kappa}$ is $W_{\mathfrak{g},\kappa_\circ} \cap yW_f y^{-1}$.

3.5 From $\widehat{\mathfrak{g}}$-modules to $\widehat{\check{\mathfrak{g}}}$-modules

3.5.1 To relate blocks of Category $\mathcal{O}$ for $\widehat{\mathfrak{g}}_{-\kappa}$ and $\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}$, we would like to use the following result of Fiebig.

Let $\mathfrak{k}$ be an affine Lie algebra with Cartan subalgebra $\mathfrak{h}$. Fix $\alpha \in \mathfrak{h}^*$ such that the Verma module M_α is simple, and write $\mathcal{O}_\alpha$ for the corresponding block of Category $\mathcal{O}$ for $\mathfrak{k}$. Let us write W_α for the integral Weyl group of α and W°_α for its subgroup stabilizing α under the dot action.

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Theorem 3.5.2 [Fie06, Theorem 4.1]. *As an abelian highest-weight category, $\mathcal{O}_\alpha$ is determined up to equivalence by the Coxeter system W_α along with its subgroup W_α°.*

Remark 3.5.3. Theorem 3.5.2, as written in [Fie06], applies to symmetrizable Kac–Moody algebras, and in particular affine Kac–Moody algebras. Recall that the latter consists of a Laurent polynomial version of the affine Lie algebra along with a degree operator $t\partial_t$. One may apply it to the present situation as follows. At any noncritical level, Category $\mathcal{O}$ for the affine Lie algebra canonically embeds as a Serre subcategory of Category $\mathcal{O}$ for the affine Kac–Moody algebra; namely, one sets $t\partial_t$ to act by the semisimple part of $-L_0$, where L_0 is the Segal–Sugawara energy operator.

3.5.4 To apply Theorem 3.5.2 in our situation, we must relate $W_{-\kappa,\mathfrak{g}}$ and $W_{\check{\kappa},\check{\mathfrak{g}}}$.

To do so, fix a level $\kappa_\circ$ for $\mathfrak{g}$. Let us write $\check{Q}$ for the coroot lattice and Q for the root lattice of $\mathfrak{g}$. Associated to $\kappa_\circ$ is a map

$$(\kappa_\circ - \kappa_{\mathfrak{g},c}) : \check{Q} \to Q \otimes_{\mathbb{Z}} \mathbb{C},$$

where $\kappa_{\mathfrak{g},c}$ denotes the critical level for $\mathfrak{g}$. In particular, we may consider the sublattice of $\check{Q}$ given by

$$\check{Q}_{\kappa_\circ} := \{\check{\lambda} \in \check{Q} : (\kappa_\circ - \kappa_{\mathfrak{g},c})(\check{\lambda}) \in Q\}.$$

Suppose that $\kappa_\circ$ is noncritical. Recall that if we write $\check{\kappa}_{\check{\mathfrak{g}},c}$ for the critical level for $\check{\mathfrak{g}}$, then by definition the dual nondegenerate bilinear form of $\kappa_\circ - \kappa_{\mathfrak{g},c}$ is $\check{\kappa}_\circ - \check{\kappa}_{\check{\mathfrak{g}},c}$. It follows that we have a canonical identification

$$-(\kappa_\circ - \kappa_{\mathfrak{g},c}) : \check{Q}_{\kappa_\circ} \simeq Q_{\check{\kappa}_\circ} : -(\check{\kappa}_\circ - \check{\kappa}_{\check{\mathfrak{g}},c}). \tag{3.27}$$

3.5.5 With this, we may canonically identify the integral Weyl groups on the opposite sides of quantum Langlands duality.

Theorem 3.5.6. *For any level $\kappa_\circ$, under the identification $W \simeq W_f \ltimes \check{Q}$ one has*

$$W_{\mathfrak{g},\kappa_\circ} \simeq W_f \ltimes \check{Q}_{\kappa_\circ}. \tag{3.28}$$

Moreover, for a noncritical level $\kappa_\circ$, there is a canonical isomorphism of Coxeter systems

$$W_{\mathfrak{g},\kappa_\circ} \simeq W_{\check{\mathfrak{g}},\check{\kappa}_\circ}. \tag{3.29}$$

Remark 3.5.7. As G and $\check{G}$ in general have different affine Weyl groups, there is, perhaps, something surprising about Theorem 3.5.6.

Proof of Theorem 3.5.6. We begin with (3.28). Recall from the proof of Lemma 3.4.3 that for a finite coroot $\check{\alpha}$ and an integer n, the affine coroot $\check{\alpha}_n$ belongs to $W_{\mathfrak{g},\kappa_\circ}$ if and only if

$$\langle \check{\alpha}_n, 0 \rangle = n\frac{\kappa_\circ(\check{\alpha}, \check{\alpha})}{2}$$

is an integer. This integrality condition may be rewritten as

$$n\kappa_\circ(\check{\alpha}) \in \mathbb{Z}\alpha,$$

which in turn is equivalent to $n\check{\alpha} \in \check{Q}_{\kappa_\circ}$. As the affine reflection in W corresponding to $\check{\alpha}_n$ is explicitly given by $s_{\check{\alpha}} t^{n\check{\alpha}}$, it follows that we have an inclusion

$$W_{\mathfrak{g},\kappa_\circ} \subset W_f \ltimes \check{Q}_{\kappa_\circ} \tag{3.30}$$

and that the left-hand side includes the translations $t^{n\check{\alpha}}$ for $n\check{\alpha}$ as above. To see that (3.30) is an equality, it suffices to show that $\check{Q}_{\kappa_\circ}$ lies in the left-hand side. But if we write an element $\check{\lambda}$

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of $\check{Q}$ as a linear combination

$$\check{\lambda} = \sum_{i \in \mathfrak{I}} n_i \check{\alpha}_i \quad \text{for } i \in \mathfrak{I},$$

we have that $\kappa_\circ(\lambda)$ lies in Q if and only if $\kappa_\circ(n_i\check{\alpha}_i)$ lies in Q for all $i \in \mathfrak{I}$. Considering the affine coroots $(\check{\alpha}_i)_{n_i}$ for $i \in \mathfrak{I}$ and composing the translational parts of their reflections yields the desired equality.

Let us use the equality (3.28) to prove (3.29). Via (3.24), we may assume that $\kappa_\circ$ is negative. Under this assumption, we will show that the composite identification

$$W_{\mathfrak{g},\kappa_\circ} \overset{(3.28)}{\simeq} W_f \ltimes \check{Q}_{\kappa_\circ} \overset{(3.27)}{\simeq} W_f \ltimes Q_{\check{\kappa}_\circ} \overset{(3.28)}{\simeq} W_{\check{\mathfrak{g}},\check{\kappa}_\circ}$$

is an isomorphism of Coxeter systems; that is, we claim that the sets of simple reflections are exchanged under the identification

$$-(\kappa_\circ - \kappa_{\mathfrak{g},c}) : W_f \ltimes \check{Q}_{\kappa_\circ} \simeq W_f \ltimes Q_{\check{\kappa}_\circ}. \tag{3.31}$$

If $\kappa_\circ$ is irrational, this is clear, as both sides are W_f. Otherwise, let us write the level as

$$\kappa_\circ = \left(-h^\vee + \frac{p}{q}\right)\kappa_b, \tag{3.32}$$

where p and q are positive coprime integers. Recall the affine simple reflection $s_{0,\kappa_\circ}$ from Lemma 3.4.3. Applying (3.31) to it and writing θ_s for the short dominant root and θ_l for the long dominant root, we obtain

$$-(\kappa_\circ - \kappa_{\mathfrak{g},c})\, s_{0,\kappa_\circ} = \begin{cases} s_{\theta_l} t^{p\cdot\theta_l} & \text{if } (q,r) = 1, \\ s_{\theta_s} t^{p\cdot\theta_s} & \text{if } (q,r) = r. \end{cases} \tag{3.33}$$

To finish, recall that $\check{\kappa}_\circ$ is given by

$$\check{\kappa}_\circ = \left(-h^\vee_{\check{\mathfrak{g}}} + \frac{q}{pr}\right)\check{\kappa}_{\check{\mathfrak{g}},b},$$

where $h^\vee_{\check{\mathfrak{g}}}$ and $\check{\kappa}_{\check{\mathfrak{g}},b}$ are the dual Coxeter number and basic level for $\check{\mathfrak{g}}$, respectively. Comparing the analogue of (3.25) for $\check{\mathfrak{g}}$ to (3.33) shows the intertwining of the affine simple reflections by (3.31), as desired. □

3.5.8 We may apply these results as follows. Suppose that κ and $\check{\kappa}$ are dual levels and that κ' is an integral level for G_s. Suppose we are given a $y \in \tilde{W}$ of minimal length in $W_{\mathfrak{g},-\kappa+\kappa'}y$ and a simple Verma module

$$M_\mu \in \widehat{\mathfrak{g}}_{-\kappa+\kappa'}\text{–mod}^{I_s, y^{-1}\cdot 0}.$$

Suppose we are further given a simple Verma module $M_{\check{\nu}}$ in $\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–mod}^{\check{I}_s}$, such that the stabilizers of μ and $\check{\nu}$ are identified via

$$y^{-1} W_{\mathfrak{g},-\kappa+\kappa'} y \simeq W_{\mathfrak{g},-\kappa+\kappa'} \overset{(3.24)}{\simeq} W_{\mathfrak{g},\kappa} \overset{(3.29)}{\simeq} W_{\check{\mathfrak{g}},\check{\kappa}}. \tag{3.34}$$

COROLLARY 3.5.9. *In the above situation, there is a t-exact equivalence*

$$\widehat{\mathfrak{g}}_{-\kappa+\kappa'}\text{–mod}^{I_s, y^{-1}\cdot 0}_{\mu} \simeq \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–mod}^{\check{I}_s}_{\check{\nu}}.$$

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Proof. For either category, which we temporarily denote by $\mathcal{C}$, and the finite-length objects in its heart, which we denote by $\mathcal{C}^{\heartsuit,c}$, the canonical map

$$D^b(\mathcal{C}^{\heartsuit,c}) \to \mathcal{C}$$

realizes the latter as the ind-completion of the former. To see this, note that since the blocks contain a simple Verma module, $\mathcal{C}^{\heartsuit,c}$ indeed consists of compact objects, and the fully faithfulness may be checked from Verma to dual Verma modules. Either $\mathcal{C}^{\heartsuit,c}$ is a block of Category $\mathcal{O}$ for the corresponding affine algebra, whence we are done by the assumptions on μ and $\check{\nu}$, the identification of Coxeter systems (3.34), and Theorem 3.5.2. □

3.6 The tamely ramified fundamental local equivalence

Recall that we have assumed G to be simple of adjoint type.

Theorem 3.6.1. *For a good, negative level κ, there is a t-exact equivalence*

$$\mathsf{Whit}^{\mathrm{aff}}_{\kappa} \overset{(3.2)}{\simeq} D_{-\kappa}(I\backslash \mathfrak{L}G/\mathfrak{L}N^-, \psi) \simeq \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{I}}. \tag{3.35}$$

Proof. Recall our identification from § 3.2.4 of the isomorphism classes of simple objects in the left-hand side of (3.35) with the coset space

$$\tilde{W}/W_f.$$

We showed in Lemma 3.2.9 that each orbit of $W_{\mathfrak{g},-\kappa}$ on the latter contains a minimal element y with respect to the Bruhat order. In Corollary 3.4.8 we showed that the corresponding block $\mathsf{Whit}^{\mathrm{aff}}_{\kappa,y}$ of the left-hand side of (3.35) identifies with a block of twisted I_s-integrable modules for $\widehat{\mathfrak{g}}_{-\kappa+\kappa'}$, for an integral level κ'. As discussed in the proof of Corollary 3.4.8, its integral Weyl group identifies as a Coxeter system with $W_{\mathfrak{g},-\kappa}$, and with this identification the stabilizer of the highest weight of a simple Verma module is given by

$$W_{\mathfrak{g},-\kappa} \cap yW_f y^{-1}. \tag{3.36}$$

On the other hand, if we denote by $\check{\Lambda}_G$ the cocharacter lattice of T, i.e. the character lattice of $\check{T}$, recall that $\tilde{W}$ is explicitly

$$\tilde{W} \simeq W_f \ltimes \check{\Lambda}_G,$$

Consider its action on $\check{\Lambda}_G$, where W_f acts by the dot action and an element $\check{\lambda}$ in $\check{\Lambda}_G$ acts as translation by $\check{\lambda}$. Acting on $-\check{\rho}$ yields a $\tilde{W}$-equivariant identification

$$\tilde{W}/W_f \simeq \check{\Lambda}_G. \tag{3.37}$$

With this, recalling that κ is negative, the restriction of (3.37) along the composition

$$W_{\check{\mathfrak{g}},\check{\kappa}} \overset{(3.29)}{\simeq} W_{\mathfrak{g},\kappa} \hookrightarrow \tilde{W}$$

yields the standard level-$\check{\kappa}$ dot action of $W_{\check{\mathfrak{g}},\check{\kappa}}$ on $\check{\Lambda}_G$. Moreover, under the equivalence (3.37), for any reflection s_{α_n} in $W_{\check{\mathfrak{g}},\check{\kappa}}$ associated to a positive coroot α_n of $\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}$ and any element x of $\tilde{W}$, one has that

$$xW_f \leqslant s_\alpha x W_f \quad \text{if and only if} \quad \langle \alpha_n, x \cdot (-\check{\rho}) + \check{\rho}\rangle \leqslant 0, \tag{3.38}$$

where we view α_n as an affine linear functional on $\mathfrak{t}$ as in § 2.9.5. To see this, note that if we write Φ^+ for the positive real coroots of $\check{\mathfrak{g}}$ and Φ_f for the finite coroots of $\check{\mathfrak{g}}$, both sides of (3.38) are equivalent to

$$x^{-1}(\alpha) \in \Phi^+ \cup \Phi_f.$$

Using (3.38), we may describe the block decomposition of

$$\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{I}}$$

as follows. In its usual formulation, by the work of Deodhar, Gabber, and Kac [DGK82], blocks correspond to $W_{\check{\mathfrak{g}},\check{\kappa}}$ dot orbits on $\check{\Lambda}_G$, and each contains a unique simple Verma module. Under the identification of its highest weights with $\tilde{W}/W_f$ via (3.37), each orbit of $W_{\check{\mathfrak{g}},\check{\kappa}}$ contains a minimal element y with respect to the Bruhat order. By (3.38), the corresponding Verma module is antidominant, i.e. simple, and has stabilizer

$$W_{\check{\mathfrak{g}},\check{\kappa}} \cap y W_f y^{-1}. \tag{3.39}$$

Comparing (3.36) and (3.39), we are done by Corollary 3.5.9. □

Remark 3.6.2. Having obtained the tamely ramified fundamental local equivalence for good levels, let us outline a variant of the proof that may be desirable.

Presently, we relate blocks of $\mathsf{Whit}^{\mathrm{aff}}_\kappa$ and $\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{I}}$ by relating the former to $\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}^I$ and applying Fiebig's results. However, it should be possible to adapt the arguments of Bezrukavnikov and Yun on $\mathbb{V}$-functors provided in [BY13, §§ 4 and 5] to $\mathsf{Whit}^{\mathrm{aff}}_\kappa$ and thereby identify each block with a category of (possibly singular) Soergel modules. Comparing this with the similar identification provided by Fiebig, and matching the combinatorics exactly as in the proof of Theorem 3.6.1, should yield the desired equivalence.

This would remove the assumption of goodness on κ, and such a description of the Whittaker category should be equally applicable in other sheaf-theoretic contexts, such as metaplectic Whittaker sheaves over function fields.

3.7 Parahoric fundamental local equivalences

3.7.1 Recall the canonical bijection between the simple roots of $\mathfrak{g}$ and $\check{\mathfrak{g}}$, which were indexed by $\mathcal{I}$. In particular, to a standard parahoric subgroup

$$P \subset \mathfrak{L}G,$$

which corresponds to a subset $\mathcal{J}$ of $\mathcal{I}$, we may associate a dual parahoric

$$\check{P} \subset \mathfrak{L}\check{G}.$$

3.7.2 Let us obtain a parahoric extension of the tamely ramified fundamental local equivalence. In particular, we continue to assume that G is simple of adjoint type.

THEOREM 3.7.3. *For a good, negative level κ, there is a t-exact equivalence*

$$D_\kappa(\mathfrak{L}N, \psi\backslash\mathfrak{L}G/P) \simeq \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{P}}. \tag{3.40}$$

Proof. It is enough to produce an equivalence, which we denote provisionally by a dotted line, fitting into the commutative diagram

$$\begin{array}{ccc} D_{-\kappa}(P\backslash\mathfrak{L}G/\mathfrak{L}N^-, \psi)^{\heartsuit,c} & \overset{\sim}{- - - - - -} & \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{P},\heartsuit,c} \\ {\scriptstyle \pi^{!*}}\downarrow & & \downarrow{\scriptstyle \mathrm{Oblv}} \\ D_{-\kappa}(I\backslash\mathfrak{L}G/\mathfrak{L}N^-, \psi)^{\heartsuit,c} & \overset{(3.25)}{\overline{\qquad\qquad}} & \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{I},\heartsuit,c} \end{array} \tag{3.41}$$

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where we normalize the pull-back $\pi^{!*}$ associated to $\pi : \mathfrak{L}G/I \to \mathfrak{L}G/P$ to be t-exact. Here, as in the proof of Corollary 3.5.9, the superscripts $\heartsuit$ and c denote compact objects in the heart, which in the present situation are equivalent to finite-length objects in the heart. Noting that both vertical arrows in (3.41) are full embeddings of Serre subcategories, it is enough to show that the essential images of the simple objects under are intertwined by (3.35). To see this claim, recall that we denoted the subset of simple roots corresponding to the dual parahorics P and $\check{P}$ by $\mathfrak{J}$. With this, the simple objects on the Whittaker side lying in the essential image are intermediate extensions from the orbits

$$Ix\mathfrak{L}N^- \quad \text{for } x \in \tilde{W},$$

where x satisfies the three conditions

$$s_j x < x \ \ \forall\, j \in \mathfrak{J}, \quad x s_i < x \ \ \forall\, i \in \mathfrak{I}, \quad \text{and} \quad W_{\mathfrak{J}} x s_i < W_{\mathfrak{J}} x \ \ \forall\, i \in \mathfrak{I}. \tag{3.42}$$

The simple objects on the Kac–Moody side lying in the essential image have highest weights $\check{\lambda}$ satisfying

$$s_j \cdot \check{\lambda} < \check{\lambda} \quad \forall\, j \in \mathfrak{J}. \tag{3.43}$$

Write $\check{\lambda} = x \cdot -\check{\rho}$ for $x \in \tilde{W}$ acting as in (3.37). We may assume that x is of maximal length in its left W_f coset, in which case (3.43) is equivalent to x satisfying the three conditions

$$s_j x < x \ \ \forall\, j \in \mathfrak{J}, \quad x s_i < x \ \ \forall\, i \in \mathfrak{I}, \quad \text{and} \quad s_j x W_f < x W_f \ \ \forall\, j \in \mathfrak{J}. \tag{3.44}$$

We finish by noting that (3.42) and (3.44) describe the same subset of $\tilde{W}$, namely the unique elements of maximal length in double cosets $W_{\mathfrak{J}} x W_f$ such that the double coset satisfies

$$x(\check{\Phi}_f) \cap \check{\Phi}_{\mathfrak{J}} = \emptyset,$$

where $\check{\Phi}_f$ denotes the finite coroots and $\check{\Phi}_{\mathfrak{J}}$ the coroots of the Levi associated to $\mathfrak{J}$. □

3.7.4 We finish with two remarks.

Remark 3.7.5. Applying Theorem 3.7.3 in the maximal case, i.e. for the parahorics given by the arc groups $\mathfrak{L}^+G$ and $\mathfrak{L}^+\check{G}$, we obtain the spherical fundamental local equivalence for good levels, namely

$$\mathsf{Whit}_\kappa^{\mathrm{sph}} := D_\kappa(\mathfrak{L}N, \psi \backslash \mathfrak{L}G/\mathfrak{L}^+G) \simeq \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\mathfrak{L}^+\check{G}}. \tag{3.45}$$

One can reasonably ask whether the methods of this paper can be directly applied to obtain this without first proving Theorem 3.6.1. The arguments indeed apply *mutatis mutandis*, where one works throughout with double cosets of $W_{\mathfrak{g},\kappa}$ rather than one-sided cosets, exactly as in the proof of Theorem 3.7.3.

In this approach, to cross Langlands duality one needs the parabolic variant of Fiebig's theorem, Theorem 3.5.2. This is indeed true, and may be deduced from the usual case by an argument similar to that for Theorem 3.7.3.

Remark 3.7.6. We would like to record here the expectation that for dual parahorics P and $\check{P}$ as above, local quantum Langlands duality exchanges the operations of taking P and $\check{P}$ invariants. For the Iwahori and arc subgroups, this has already appeared in the literature [ABC+18, Gai18c]. However, while one has a canonical bijection between the affine simple roots for $\mathfrak{g}$ and $\check{\mathfrak{g}}$, we do *not* expect an interchanging of the corresponding invariants of more general parahoric subgroups. Indeed, already the analogue of Theorem 3.7.3 will typically fail, unless $\check{\kappa}$ is integral and $\mathfrak{g}$ is simply laced.

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4. Proof in the general case

In this final section, we spell out how to deduce the general case of the conjectures from the case where G is of adjoint type and κ is a negative level. While we write the reductions in the tamely ramified case, they apply *mutatis mutandis* in the parahoric cases as well.

4.1 Good levels for general G

Recall the notion of a good level for a simple group; see § 3.4.4 and Proposition 3.4.5. Let us say that a level κ for a reductive group G is good if it is good after restriction to each simple factor of G.

The following reductions show that the general case of Conjectures 1.3.1 and 1.3.2 for reductive G and good κ follow from the case of G being simple of adjoint type and κ being a good, negative level. In particular, via Theorems 3.6.1 and 3.7.3, we obtain the fundamental local equivalences for general G at good levels.

Remark 4.1.1. The reductions from the case of a general connected reductive group at a negative level to an adjoint group at a negative level yield, in combination with the results of § 3, t-exact equivalences.

By contrast, at a positive level one cannot hope for a t-exact equivalence. To see this, note that the Verma modules in $\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{I}}$ are now of infinite length, whereas any compact object in the heart of Whit_κ is of finite length. So the arguments of § 3 cannot be applied directly.

Instead, the equivalences at positive level are deduced from the negative-level cases by duality for DG categories. This makes essential use of the renormalizations of the appearing derived categories, and the equivalences are of infinite cohomological amplitude. For example, if $\check{\kappa}$ is a positive integral level, they send the abelian category of positive energy representations of the loop group $\mathfrak{L}\check{G}$ at level $\check{\kappa}$, i.e.

$$\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\mathfrak{L}\check{G},\heartsuit},$$

into objects of Whit_κ concentrated in cohomological degree $-\infty$.

4.2 Finite isogenies

Suppose we are given a finite isogeny of pinned connected reductive groups

$$\iota : G_1 \to G_2.$$

The morphism i yields a closed embedding of affine flag varieties and hence a fully faithful embedding

$$D_\kappa(\mathfrak{L}N_1, \psi\backslash\mathfrak{L}G_1/I_1) \to D_\kappa(\mathfrak{L}N_2, \psi\backslash\mathfrak{L}G_2/I_2). \tag{4.1}$$

Consider the Langlands dual isogeny of connected reductive groups

$$\check{\iota} : \check{G}_2 \to \check{G}_1.$$

Associated to $\check{\iota}$ is a fully faithful restriction map

$$\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{I}_1} \to \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–}\mathsf{mod}^{\check{I}_2}. \tag{4.2}$$

To see the claimed fully faithfulness, one may use the following lemma.

Lemma 4.2.1. *Suppose one is a given a fibre sequence of quasi-compact affine group schemes*

$$1 \to K \to H \to Q \to 1,$$

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where K is of finite type, the pro-unipotent radical H^u is of finite codimension in H, and pt/K is homologically contractible, i.e.

$$H^*(pt/K, \mathbb{Q}) \simeq \mathbb{Q}.$$

Then for any $D(Q)$-module $\mathcal{C}$, the restriction map $\mathcal{C}^Q \to \mathcal{C}^H$ is fully faithful.

Proof. For $\mathcal{C} \simeq D(Q)$, this is exactly the assumption of homological contractibility. The case of general $\mathcal{C}$ follows from taking its cobar resolution as a $D(Q)$-module and using the commutation of Q and H invariants with colimits. □

We may apply the lemma to $\check{I}_2 \to \check{I}_1$, as the kernel identifies with the kernel of $\check{\iota}$. Combining these assertions, we obtain the following result.

Proposition 4.2.2. *Suppose that one has an equivalence of the form (1.2) for (G_2, κ) and $(\check{G}_2, \check{\kappa})$. Assume that it exchanges the full subcategory generated under shifts and colimits by the Whittaker sheaves*

$$j^{\psi}_{\check{\lambda},!} \quad \text{for } \check{\lambda} \in \check{\Lambda}_{G_1}$$

with the full subcategory generated under shifts and colimits by the Verma modules

$$M_{\check{\lambda}} \quad \text{for } \check{\lambda} \in \check{\Lambda}_{G_1}.$$

Then it induces an equivalence of the form (1.2) for (G_1, κ) and $(\check{G}_1, \check{\kappa})$, fitting into a commutative diagram as follows.

$$\begin{array}{ccc}
D_\kappa(\mathfrak{L}N, \psi\backslash\mathfrak{L}G_1/I_1) & \xrightarrow{\sim} & \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–mod}^{\check{I}_1} \\
{\scriptstyle (4.1)}\downarrow & & \downarrow{\scriptstyle (4.2)} \\
D_\kappa(\mathfrak{L}N, \psi\backslash\mathfrak{L}G_2/I_2) & \xrightarrow{\sim} & \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–mod}^{\check{I}_2}
\end{array}$$

4.3 Products

By the preceding subsection, we may replace our group, after passing to a finite quotient thereof, by the product of a semisimple group of adjoint type and a torus. We next reduce to the case of a single factor.

Suppose that G factors as a product of pinned connected reductive groups $G \simeq G_1 \times G_2$. Associated to this is a tensor product decomposition

$$D_\kappa(\mathfrak{L}N, \psi\backslash\mathfrak{L}G/I) \simeq D_{\kappa_1}(\mathfrak{L}N_1, \psi_1\backslash\mathfrak{L}G_1/I_1) \otimes D_{\kappa_2}(\mathfrak{L}N_2, \psi_2\backslash\mathfrak{L}G_2/I_2).$$

On the Langlands dual side, we obtain a decomposition $\check{G} \simeq \check{G}_1 \times \check{G}_2$ and a similar tensor product decomposition

$$\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–mod}^{\check{I}} \simeq \widehat{\check{\mathfrak{g}}}_{1,\kappa_1}\text{–mod}^{\check{I}_1} \otimes \widehat{\check{\mathfrak{g}}}_{2,\kappa_2}\text{–mod}^{\check{I}_2}.$$

In particular, to provide an equivalence as in (1.2) for $G_1 \times G_2$, it is enough to do so for G_1 and G_2 separately.

4.4 Tori

Let us dispose of the torus factor of G. Given dual tori T and $\check{T}$, it is clear that both sides of (1.2) canonically identify as

$$\bigoplus_{\check{\lambda} \in \check{\Lambda}_T} \mathsf{Vect},$$

corresponding to the components of the affine Grassmannian of T and the Fock modules for $\widehat{\check{\mathfrak{t}}}_{\check{\kappa}}$, respectively.

4.5 Positive level

We have reduced the conjecture to simple G of adjoint type and arbitrary κ. We now reduce to negative κ via cohomological duality.

For a connected reductive group G, there is a canonical $D_\kappa(\mathfrak{L}G)$-equivariant duality

$$D_\kappa(\mathfrak{L}G/I)^\vee \simeq D_{-\kappa}(\mathfrak{L}G/I).$$

This induces a duality of the Whittaker invariants (cf. [Dhi21, § 4]), i.e.

$$D_\kappa(\mathfrak{L}N,\psi\backslash\mathfrak{L}G/I)^\vee \simeq D_{-\kappa}(\mathfrak{L}N,-\psi\backslash\mathfrak{L}G/I). \tag{4.3}$$

On the Kac–Moody side, recall that semi-infinite cohomology (defined with respect to any compact open subalgebra) induces an $D_\kappa(\mathfrak{L}G)$-equivariant duality

$$\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–mod}^\vee \simeq \widehat{\check{\mathfrak{g}}}_{-\check{\kappa}}\text{–mod};$$

see [Ras20, § 9]. Accordingly, passing to $\check{I}$ invariants, we obtain a duality

$$(\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{–mod}^{\check{I}})^\vee \simeq \widehat{\check{\mathfrak{g}}}_{-\check{\kappa}}\text{–mod}^{\check{I}}. \tag{4.4}$$

Accordingly, an equivalence as in (1.2) at level κ follows by duality from such an equivalence at level $-\kappa$.

We now check the compatibility of the above with the assumption of Proposition 4.2.2 concerning essential images. For dual categories $\mathcal{C}$ and $\mathcal{C}^\vee$, let us write $\mathcal{C}^c$ and $(\mathcal{C}^\vee)^c$ for their full subcategories of compact objects and denote their induced identification by

$$\mathbb{D}: \mathcal{C}^{c,\mathrm{op}} \simeq (\mathcal{C}^\vee)^c.$$

Lemma 4.5.1. *Fix any $\check{\lambda}$ in $\check{\Lambda}_G$, and write ρ and $\check{\rho}$ for the half-sums of the positive roots and the coroots of G, respectively. With respect to the duality datum (4.3), we have*

$$\mathbb{D}j_{\check{\lambda},!} \simeq j_{\check{\lambda},*}[-2\langle\check{\rho},\rho\rangle].$$

With respect to the duality datum (4.4), normalized with respect to the Lie algebra of the Iwahori $\check{I}$, we have

$$\mathbb{D}M_{\check{\lambda}} \simeq M_{-\check{\lambda}-2\check{\rho}}.$$

The proof of the lemma is straightforward; cf. Lemma 9.8 of [Dhi21] for the assertion regarding Verma modules.

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