Annihilating class groups
in *p*-elementary abelian extensions

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Abstract

We derive new cases of conjectures of Rubin and of Burns–Kurihara–Sano concerning derivatives of Dirichlet L-series at $s = 0$ in p-elementary abelian extensions of number fields for arbitrary prime numbers p. In naturally arising examples of such extensions one therefore obtains annihilators of class groups from S-truncated Dirichlet L-series for 'large-enough' sets of places S.

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1. *Introduction*

1·1. *Dirichlet L-series at $s = 0$ and annihilation of class groups*

Stark's conjecture predicts a description for the leading term of a general Artin L-series at $s = 0$ up to an unspecified rational factor. Formulating an integral refinement of this conjecture turned out to be a delicate task that Stark himself, in [**54**], only found a solution to in the case that the order of vanishing of the L-series at $s = 0$ is one. Initial generalisations to higher orders of vanishing, for example the 'question' of Stark in [**29**, **55**] or a conjecture of Sands [**50**, Conjecture 2·0], were subsequently shown to not hold in general by Rubin [**48**, Section 4] and Popescu [**47**]. Instead, Rubin proposed in loc. cit. what is now commonly referred to as the 'Rubin–Stark conjecture'.

Going beyond mere integrality, it is expected that this unspecified factor encodes important arithmetic information and, in particular, is linked to the Galois module structure of class groups. The primordial example of this phenomenon is Stickelberger's theorem from the 19th century, which asserts that the ideal class group of a cyclotomic field is annihilated by a certain element valued in the group ring over the relevant Galois group and constructed from values of Dirichlet L-series at $s = 0$. The analogous annihilation statement for class groups of finite abelian CM extensions of totally real fields is known as the 'Brumer–Stark conjecture' and has very recently been settled by Dasgupta and Kakde [**21**] with additional

arguments by Dasgupta, Kakde, Silliman and Wang [**23**]. In certain situations, these results can even be extended to non-abelian CM extensions, see [**16**, **26**, **33**].

However, outside the setting of totally imaginary extensions of totally real fields the values of the associated Dirichlet L-series at $s = 0$ usually vanish (the only exception being the case considered by Nomura in [**45**]) and so naive generalisations of Stickelberger's theorem become trivial. This led Burns to formulate the question whether in such cases one can instead use higher derivatives of Dirichlet L-series to produce annihilators of class groups (see [**37**, Question 1·1]) and similar aspects have also been considered by Buckingham [**7**, **8**].

In this article, we prove new results on the Rubin–Stark conjecture and, moreover, on the annihilation of class groups, in cases of higher orders of vanishing. Indeed, in Theorem 1·1 we extend annihilation results concerning multi-quadratic extensions by Sands [**52**] and the second author [**37**] to general p-elementary abelian extensions K/k of number fields, for arbitrary prime numbers p. This result is conditional on the collection of subextensions L/k of K/k that have degree p validating a conjecture of Burns, Kurihara and Sano (which we recall as Conjecture 1·10). Since we are then also able, in Theorem 1·3, to prove new cases of the Burns–Kurihara–Sano conjecture, we derive a method of systematically producing examples in which the annihilation claim of Theorem 1·1 is valid unconditionally (see Corollary 1·8).

1·2. *Statements of the main results*

To describe our results in more detail, we fix a finite abelian extension of number fields K/k with Galois group $G := \mathrm{Gal}(K/k)$ and, following Rubin [**48**, Hypothesis 2·1], a triple (S, V, T) of finite sets of places of k with the following properties:

(H1) S contains both the set S_∞ of infinite places of k and the places that ramify in K;

(H2) $V \subsetneq S$ is a proper subset comprising places which split completely in K/k;

(H3) T is disjoint from S and such that the group $\mathcal{O}_{K,S,T}^\times := \{a \in K^\times \mid \mathrm{ord}_w(a) = 0 \text{ if } w \notin S_K, \mathrm{ord}_w(a - 1) > 0 \text{ if } w \in T_K\}$ is $\mathbb{Z}$-torsion free. (Here S_K and T_K denote the sets of places of K that lie above those in S and T, respectively, and ord_w is the normalised valuation attached to w.)

We refer to such a triple (S, V, T) as a 'Rubin datum' for K/k. For any Rubin datum and character χ in $\widehat{G} := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^\times)$, the ($S$-truncated, T-modified) Dirichlet L-series

$$L_{k,S,T}(\chi, s) := \prod_{v \in T} (1 - \chi(\mathrm{Frob}_v)\mathrm{N}v^{1-s}) \cdot \prod_{v \notin S} (1 - \chi(\mathrm{Frob}_v)\mathrm{N}v^{-s})^{-1} \quad \text{if } \mathrm{Re}(s) > 1$$

is known to admit a meromorphic continuation to $\mathbb{C}$ that is holomorphic and of order of vanishing at least $|V|$ at $s = 0$ (cf. [**56**, Chapter I, Proposition 3·4]). We may therefore define the ($|V|$-th order) 'Stickelberger element'

$$\theta_{K/k,S,T}^{(|V|)}(0) := \sum_{\chi \in \widehat{G}} \Big(\lim_{s \to 0} s^{-|V|} L_{k,S,T}(\chi^{-1}, s) \Big) \cdot e_\chi,$$

with $e_\chi := |G|^{-1} \sum_{\sigma \in G} \chi(\sigma)^{-1}\sigma$ the usual primitive orthogonal idempotent in $\mathbb{C}[G]$ associated with χ. In addition, we define $X_{K,S} \subseteq Y_{K,S} := \bigoplus_{w \in S_K} \mathbb{Z}w$ to be the $\mathbb{Z}[G]$-submodule of elements whose coefficients sum to zero, and denote the Dirichlet regulator isomorphism by

$$\lambda_{K,S} \colon \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_{K,S}^{\times} \xrightarrow{\simeq} \mathbb{R} \otimes_{\mathbb{Z}} X_{K,S}, \quad x \otimes a \mapsto -x \sum_{w \in S_K} \log |a|_w \cdot w. \tag{1.1}$$

The Rubin–Stark conjecture [**48**, Conjecture B'] now predicts, via the reinterpretation given in Lemma 2·2 below, that for every homomorphism of $\mathbb{Z}[G]$-modules $f \colon \mathcal{O}_{K,S,T}^{\times} \to X_{K,S}$ one has

$$\theta_{K/k,S,T}^{(|V|)}(0) \cdot \det{}_{\mathbb{R}[G]}(f_{\mathbb{R}} \circ \lambda_{K,S}^{-1}) \in \mathbb{Z}[G] \tag{1.2}$$

with $f_{\mathbb{R}}$ the scalar extension $\mathbb{R} \otimes_{\mathbb{Z}} f \colon \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_{K,S}^{\times} = \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_{K,S,T}^{\times} \longrightarrow \mathbb{R} \otimes_{\mathbb{Z}} X_{K,S}$ of f. In addition, it is expected that the element in (1·2) annihilates the S_K-class group of K (cf. [**13**, Conjecture 2·4·1] or [**37**, Question 1·1]). Here we study a refinement of this question that instead considers the S_K-ray class group $\mathrm{Cl}_{K,S,T}$ of K mod T_K (defined as the quotient of the group of fractional ideals of $\mathcal{O}_{K,S}$ coprime to T_K, by the subgroup of principal ideals with a generator congruent to 1 modulo all $w \in T_K$).

To state our first main result in this direction we fix a prime number p, consider a p-elementary abelian extension K/k and write Ω for the set of degree-p subextensions L/k of K/k.

THEOREM 1·1. *Let K/k be a p-elementary abelian extension of number fields of degree p^m and fix a Rubin datum (S, V, T) for K/k that satisfies*

$$|S| \geq \max\{|V| + 2, |V| - s_p + (p-1)(m-1) + 3\},$$

where $s_p := \dim_{\mathbb{F}_p}(\mathrm{Cl}_{k,S,T} \otimes_{\mathbb{Z}} \mathbb{F}_p)$ denotes the p-rank of the S-ray class group mod T of k.

*If for all subextensions L/k in Ω the Burns–Kurihara–Sano conjecture [**17**, Conjecture 7·3] is valid for $(L/k, S, V, T)$, then*

$$\left\{ \theta_{K/k,S,T}^{(|V|)}(0) \cdot \det{}_{\mathbb{R}[G]}(f_{\mathbb{R}} \circ \lambda_{K,S}^{-1}) \,\middle|\, f \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^{\times}, X_{K,S}) \right\} \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_{K,S,T}).$$

In particular, the Rubin–Stark conjecture is valid for $(K/k, S, V, T)$.

Remark 1·2. If $p = 2$, then each subextension in Ω is quadratic and Theorem 1·1 is unconditional (see Remark 1·11 (*d*)) and recovers results of Sands [**51**, Theorem 2·2] on the Rubin–Stark conjecture and of Sands [**52**, Main theorem] and the second author [**37**, Theorem 1·4] on the annihilation of class groups.

To prove Theorem 1·1 (in Section 4) we first deduce in Lemma 4·2 the validity of the Rubin–Stark conjecture for K/k from the assumed validity of conjecture [**17**, Conjecture 7·3] for all subextensions in Ω. The annihilation statement in Theorem 1·1 is then deduced from this by varying the Rubin datum in combination with Cebotarev's density theorem, as in the theory of 'Stark systems' (see, for example, [**18**, Section 4]). Although this latter aspect of the argument is of a general nature, we prefer to focus on the concrete situation of Theorem 1·1 in this article and to discuss the general formalism elsewhere.

As our second main result, we prove new cases of the Burns–Kurihara–Sano conjecture.

THEOREM 1·3. *Let K/k be an extension of number fields of one of the following forms*:

(i) *There exists a prime-power q and a subfield κ of k such that K/κ is a Galois extension with Galois group isomorphic to the group $\mathrm{Aff}(q)$ of affine transformations of the field $\mathbb{F}_q$ with q elements, and $G = \mathrm{Gal}(K/k)$ is the unique subgroup of order q of $\mathrm{Gal}(K/\kappa)$.*

(ii) *K/k is a biquadratic extension.*

Then, given any Rubin datum (S, V, T) for K/k that satisfies $|S| > |V| + 1$, the Burns–Kurihara–Sano conjecture is valid for $(K/k, S, V, T)$. In particular, the Rubin–Stark conjecture is also valid for $(K/k, S, V, T)$.

Remark 1·4. The condition $|S| > |V| + 1$ often already follows from (H1) and (H2). For example, if k has odd class number, then class field theory implies that there can only be finitely many biquadratic extensions K of k that admit a Rubin datum (S, V, T) with $|S| = |V| + 1$.

Example 1·5. Fix a prime number p and let ζ_p be a primitive pth root of unity in an algebraic closure of $\mathbb{Q}$. Let κ be a number field with the property that $\kappa \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$. If we pick any element $a \in \kappa^\times$ that is not a pth power in κ, then it is also not a pth power in $k := \kappa(\zeta_p)$ and $K := k(\sqrt[p]{a})$ is an extension of the form (i) with $q = p$.

Remark 1·6. The Burns–Kurihara–Sano conjecture (for arbitrary Rubin datum) is known to be a consequence of the 'equivariant Tamagawa Number Conjecture' (eTNC) for K/k by [17, Theorem 7·5]. (Note that the eTNC is referred to as the 'Leading Term Conjecture' LTC(K/k) in the cited result, cf. [17, Proposition 3·4 and Remark 3·2]) For the extensions K/k considered in Theorem 1·3 and any prime ℓ not dividing $[K:k]$, the 'ℓ-component' of eTNC(K/k) can easily be seen to follow from the analytic class number formula (via Tate's proof [56, Chapter II, Theorem 6·8] of the 'strong Stark conjecture' in this setting) and Johnston and Nickel have proved in [32, Theorem 4·6] that in certain instances of case (i) one can even deduce the ℓ-component of eTNC(K/κ). Of most interest, therefore, is the component of eTNC(K/k) at the unique prime dividing $[K:k]$.

However, a proof of this component seems to be out of reach at present since even in the case (ii) of biquadratic extensions it amounts to a difficult, yet explicit, question regarding signs (see Remark 3·2 for more details). The perhaps surprising insight behind the proof of Theorem 1·3 is that the information provided by the analytic class number formula is nevertheless sufficient to allow for the deduction of the Burns–Kurihara–Sano conjecture, subject only to the restriction that $|S| > |V| + 1$. In fact, the direct argument given in Section 3·1 is uniform and does not require a distinction between $\ell \nmid [K:k]$ and $\ell \mid [K:k]$.

Remark 1·7. Johnston and Nickel [32, Theorem 7·6] have also proved a conjecture of Burns (from [13]) regarding the annihilation of class groups in extensions K/κ as in case (i) for which $k/\mathbb{Q}$ is abelian.

Results on the Rubin–Stark conjecture in the literature outside the classical cases where at most one archimedean place of k splits in K or the degree $[K:k]$ is at most two are sparse (see Remark 1·11 for a list of known cases). By combining Theorems 1·3 and 1·1 with Example 1·5, we now obtain the following method to systematically produce new examples in which the conjecture is valid.

COROLLARY 1·8. *Let p be a prime number, let ζ_p be a primitive p-th root of unity, and let κ be a number field with the property that $\kappa \cap \mathbb{Q}(\mu_p) = \mathbb{Q}$. Let $a_1, \ldots, a_m$ be elements of κ that are $\mathbb{F}_p$-linearly independent in $\kappa^\times/(\kappa^\times)^p$, and set $k := \kappa(\mu_p)$ and $K := k(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_m})$.*

If (S, V, T) is a Rubin datum for K/k with

$$|S| \geq \max\{|V| + 2, |V| - s_p + (p-1)(m-1) + 3\},$$

then

$$\left\{\theta_{K/k,S,T}^{(|V|)}(0) \cdot \det{}_{\mathbb{R}[G]}(f_\mathbb{R} \circ \lambda_{K,S}^{-1}) \,|\, f \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^\times, X_{K,S})\right\} \ \subseteq \ \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_{K,S,T}).$$

In particular, the Rubin–Stark conjecture is valid for $(K/k, S, V, T)$.

Proof. The kernel of the natural map $\kappa^\times/(\kappa^\times)^p \to k^\times/(k^\times)^p$ identifies with $H^1(\mathrm{Gal}(k/\kappa), \mu_p)$, and hence vanishes. It follows that $a_1, \ldots, a_m$ generate an $\mathbb{F}_p$-subvectorspace of $k^\times/(k^\times)^p$ of dimension m. By Kummer theory, one therefore has that $[K:k] = p^m$ and so, noting that $\mathrm{Gal}(k(\sqrt[p]{a_i})/\kappa) \cong \mathrm{Aff}(p)$ for every $i \in \{1, \ldots, m\}$ because $\kappa \cap \mathbb{Q}(\mu_p) = \mathbb{Q}$, the result follows by combining Theorems 1·3 and 1·1.

1·3. *The conjectures of Rubin–Stark and Burns–Kurihara–Sano*

In this section we state the Rubin–Stark conjecture and the conjecture [**17**, Conjecture 7·3] of Burns, Kurihara and Sano, and we discuss the list of cases in which either conjecture is known to be valid. The formulations given here, in terms of the products of the form (1·2), are equivalent to the original versions of the conjectures by Lemma 2·2 below.

We fix a finite abelian extension of number fields K/k with Galois group $G := \mathrm{Gal}(K/k)$ and Rubin datum (S,V,T).

CONJECTURE 1·9 (Rubin–Stark, [**48**, Conjecture B']). *One has*

$$\left\{\theta_{K/k,S,T}^{(|V|)}(0) \cdot \det{}_{\mathbb{R}[G]}(f_\mathbb{R} \circ \lambda_{K,S}^{-1}) \,|\, f \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^\times, X_{K,S})\right\} \ \subseteq \ \mathbb{Z}[G].$$

We set $K_T^\times := \{a \in K^\times \mid \mathrm{ord}_w(a-1) > 0 \text{ if } w \in T_K\}$. Then the 'integral dual Selmer group' $\mathrm{Sel}_{K,S,T}$ is defined by Burns–Kurihara–Sano [**17**, Definition 2·1] as the cokernel of the map

$$\prod_{w \notin S_K \cup T_K} \mathbb{Z} \to \mathrm{Hom}_\mathbb{Z}(K_T^\times, \mathbb{Z}), \quad (x_w)_w \mapsto \left\{a \mapsto \sum_w x_w \mathrm{ord}_w(a)\right\}.$$

It fits into a canonical exact sequence of G-modules

$$0 \longrightarrow \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}_{K,S,T}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Sel}_{K,S,T} \longrightarrow \mathrm{Hom}_\mathbb{Z}(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}) \longrightarrow 0,$$

with all duals endowed with the contragredient G-action.

In the sequel, for $n \geq 0$, we write $\mathrm{Fitt}_{\mathbb{Z}[G]}^n(M)$ for the n-th Fitting ideal in $\mathbb{Z}[G]$ of a finitely presented $\mathbb{Z}[G]$-module M (see, for example, [**46**, Section 3·1] or [**43**]). Given a subset I of $\mathbb{C}[G]$, we denote by $I^\#$ the image of I under the involution of $\mathbb{C}[G]$ that inverts elements of G.

Burns–Kurihara–Sano use the Selmer group to refine Conjecture 1·9 as follows.

CONJECTURE 1·10 (Burns–Kurihara–Sano [17, Conjecture 7·3]). *One has*

$$\left\{\theta_{K/k,S,T}^{(|V|)}(0) \cdot \det{}_{\mathbb{R}[G]}(f_{\mathbb{R}} \circ \lambda_{K,S}^{-1}) \,|\, f \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^{\times}, X_{K,S})\right\} = \mathrm{Fitt}_{\mathbb{Z}[G]}^{|V|}(\mathrm{Sel}_{K,S,T})^{\#}. \quad (1\cdot3)$$

Remark 1·11. To the best of the authors' knowledge, the following is a complete list of cases in which the Rubin–Stark conjecture is known at present.

(a) One can directly verify the conjecture for the following general classes of extensions K/k:

   (i) if $[K:k] \le 2$, then it follows from the analytic class number formula (see [48, Corollary 3·2 and Theorem 3·5]);

   (ii) if $k = \mathbb{Q}$ and $V = S_{\infty}$ is the singleton comprising the unique infinite place of $\mathbb{Q}$, then it follows by means of a direct computation that shows that the relevant Rubin–Stark element (as defined in section 2·1) can be expressed in terms of a cyclotomic unit (cf. [56, Chapter III, Section 5]);

   (iii) if k is an imaginary quadratic field and $V = S_{\infty}$ is the singleton comprising the unique infinite place of k, then it follows from Kronecker's Second Limit Formula for elliptic units (cf. [56, Chapter IV, Proposition 3·9]);

   (iv) if $V = \varnothing$, then it is a consequence of work of Cassou-Noguès [19] and, independently, Deligne and Ribet [24] (cf. [30, Proposition 3·7]).

(b) In addition, the conjecture has been directly verified in the following particular cases.

   (i) Grant [29] has verified it for $k = \mathbb{Q}(\zeta_5)$ and $K = k(\sqrt[5]{\epsilon})$ with ζ a primitive 5th root of unity and $\epsilon := -\zeta^2 - \zeta^3$.

   (ii) If K/k is multi-quadratic, then Dummit, Sands and Tangedal [25], Sands [51], and the second author [37] have verified it in special cases.

   (iii) McGown, Sands and Vallières [38] have numerically verified it for $V = S_{\infty}$ in the 19197 examples of k a real quadratic field and K a totally real cubic extensions of k of discriminant less than 10^{12} and $V = S_{\infty}$

(c) It holds if $S \setminus V$ contains a place that splits completely in K (cf. [48, Proposition 3·1]).

(d) The examples listed in (a) are by now sufficiently well understood to allow for a proof of eTNC(K/k) [17, Conjecture 3·6]. By [17, Theorem 7·5], for any given Rubin datum for K/k, the conjecture [17, Conjecture 7·3], and hence also the Rubin–Stark conjecture, is a consequence of eTNC(K/k). Using functoriality properties of the eTNC, one thus obtains the validity of both the Burns–Kurihara–Sano and Rubin–Stark conjectures for any K/k (and Rubin datum) such that $F \subseteq k \subseteq K \subseteq H$, with H/F a finite Galois extension for which eTNC(H/F) holds. The same conclusion is true if the 'minus part' eTNC$^-$(H/F) of eTNC(H/F) holds and k is totally real and K is totally imaginary.

In this direction, the following is currently known:

(i) eTNC(H/F) holds if $[H{:}F] = 2$; this case is proved by Kim [**36**, Section 2·4];

(ii) eTNC(H/F) holds if $F = \mathbb{Q}$; this is work of Burns and Greither [**15**] with additional arguments for the 2-component by Flach [**28**];

(iii) eTNC(H/F) holds if F an imaginary quadratic field such that all prime divisors of $[H{:}F]$ split in k or validate Iwasawa's μ-vanishing conjecture; this case is proved by Hofer and the first author [**11**, Theorem B] and extends previous work of Bley [**2**, **3**, **4**].

(iv) eTNC(H/F)$^-$ holds if F is a totally real field and H is CM; this is work of the first author, Burns, Daoud and Seo [**10**] with additional arguments for the 2-component by Dasgupta, Kakde and Silliman [**22**]. Earlier work in this direction includes [**1**, **41**, **42**, **44**]. (The results in [**10**] crucially rely on work of Dasgupta and Kakde [**21**] on the Strong Brumer–Stark conjecture, and we remark that the Rubin–Stark conjecture can alternatively be directly deduced from the Strong Brumer–Stark conjecture, see [**21**, Theorem 1·6]).

Further examples of, not necessarily abelian, extensions H/F for which eTNC(H/F) is known at present include the following:

(i) H is a totally real Galois extension of $F = \mathbb{Q}$ such that either $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$ and H has discriminant less than 10^{20} or $\mathrm{Gal}(H/\mathbb{Q}) \cong D_{12}$ and H has discriminant less than 10^{30}; by Hofmann, Johnston, and Nickel [**34**, Corollary A·3].

(ii) a particular family of Quaternionic extensions H of $F = \mathbb{Q}$; by Burns and Flach [**14**, Theorem 4·1].

(iii) one example of a Galois extension H of $F = \mathbb{Q}$ with $\mathrm{Gal}(H/\mathbb{Q}) \cong A_4$; numerical verification by Navilarekallu [**40**].

(iv) when a number of standard conjectures are known to be valid, further results can be deduced from the examples above, see [**32**, Section 4] and [**34**, Section 10].

2. *Preliminaries*

In this preliminary section we review various constructions that will be useful in the sequel.

2·1. *Rubin–Stark elements*

Let K/k be a finite abelian extension of number fields with Galois group $G := \mathrm{Gal}(K/k)$ and let (S, V, T) be a Rubin datum for K/k. We fix a labelling $S = \{v_0, \ldots, v_{|S|-1}\}$ such that $V = \{v_1, \ldots, v_{|V|}\}$ along with an extension w_i to K of each place v_i in S. The 'Rubin–Stark element' $\varepsilon^V_{K/k,S,T}$ for (S, V, T) is then the unique element of $\mathbb{R} \otimes_{\mathbb{Z}} \bigwedge^{|V|}_{\mathbb{Z}[G]} \mathcal{O}^\times_{K,S}$ with the property that

$$\Big(\bigwedge^{|V|} \lambda_{K,S} \Big)(\varepsilon^V_{K/k,S,T}) = \theta^{(|V|)}_{K/k,S,T}(0) \cdot \bigwedge_{1 \leq i \leq |V|} (w_i - w_0)$$

with $\bigwedge^{|V|} \lambda_{K,S} \colon \mathbb{R} \otimes_{\mathbb{Z}} \bigwedge^{|V|}_{\mathbb{Z}[G]} \mathcal{O}^\times_{K,S} \xrightarrow{\simeq} \mathbb{R} \otimes_{\mathbb{Z}} \bigwedge^{|V|}_{\mathbb{Z}[G]} X_{K,S}$ the isomorphism induced by (1·1).

Definition 2·1 . We define a $\mathbb{Z}[G]$-submodule of $\mathbb{R}[G]$ by setting

$$\operatorname{im}(\varepsilon_{K/k,S,T}^V) := \big\{ F(\varepsilon_{K/k,S,T}^V) \mid F \in \bigwedge\nolimits_{\mathbb{Z}[G]}^{|V|} \operatorname{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \big\},$$

where $F(\varepsilon_{K/k,S,T}^V)$ denotes the image of $(\varepsilon_{K/k,S,T}^V, F)$ under the determinant pairing

$$\big(\mathbb{R} \otimes_{\mathbb{Z}} \bigwedge\nolimits_{\mathbb{Z}[G]}^{|V|} \mathcal{O}_{K,S}^\times\big) \times \big(\mathbb{R} \otimes_{\mathbb{Z}} \bigwedge\nolimits_{\mathbb{Z}[G]}^{|V|} \operatorname{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])\big) \to \mathbb{R}[G],$$

$$(a_1 \wedge \cdots \wedge a_{|V|}, f_1 \wedge \cdots \wedge f_{|V|}) \mapsto \det(f_i(a_j))_{1 \leq i,j \leq |V|}.$$

The following result was used in section 1·3 to reformulate both the Rubin–Stark conjecture and the Burns–Kurihara–Sano conjecture in terms of the more explicit products of the form (1·2).

LEMMA 2·2. *For any Rubin datum (S, V, T) for K/k, one has an equality*

$$\operatorname{im}(\varepsilon_{K/k,S,T}^V) = \{\theta_{K/k,S,T}^{|V|}(0) \cdot \det{}_{\mathbb{R}[G]}(f_{\mathbb{R}} \circ \lambda_{K,S}^{-1}) \mid f \in \operatorname{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{K,S,T}^\times, X_{K,S})\}.$$

Proof. This is an immediate consequence of [**37**, Lemma 2·2].

2·2. *Weil-étale cohomology complexes*

We briefly recall key properties of a useful family of complexes constructed by Burns, Kurihara, and Sano in [**17**]. To do so, we let K/F be an arbitrary finite Galois extension of number fields with Galois group $\Delta_F := \operatorname{Gal}(K/F)$.

We write $D(\mathbb{Z}[\Delta_F])$ for the derived category of $\mathbb{Z}[\Delta_F]$-modules and $D^{\mathrm{p}}(\mathbb{Z}[\Delta_F])$ for its full triangulated subcategory comprising complexes that are 'perfect', that is, isomorphic (in $D(\mathbb{Z}[\Delta_F])$) to a bounded complex of finitely generated projective $\mathbb{Z}[\Delta_F]$-modules.

LEMMA 2·3. *Fix sets S and T of places of F that satisfy the conditions* (H1) *and* (H3) *in section* 1 *with k replaced by F. Then the 'Weil-étale cohomology complex'*

$$C_{K,S,T}^\bullet := \operatorname{RHom}_{\mathbb{Z}}(\mathrm{R}\Gamma_{\mathrm{c},T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})[-2]$$

*constructed in [**17**, Proposition 2·4] is an object of $D^{\mathrm{p}}(\mathbb{Z}[\Delta_F])$ that has the following properties:*

(i) $C_{K,S,T}^\bullet$ *is acyclic outside degrees zero and one, with $H^0(C_{K,S,T}^\bullet) = \mathcal{O}_{K,S,T}^\times$, and the 'transpose Selmer group' $\operatorname{Sel}_{K,S,T}^{\mathrm{tr}} := H^1(C_{K,S,T}^\bullet)$ lies in a short exact sequence of Δ_F-modules*

$$0 \longrightarrow \operatorname{Cl}_{K,S,T} \longrightarrow \operatorname{Sel}_{K,S,T}^{\mathrm{tr}} \longrightarrow X_{K,S} \longrightarrow 0;$$

(ii) $C_{K,S,T}^\bullet$ *is isomorphic in $D(\mathbb{Z}[\Delta_F])$ to a complex $[P_0 \xrightarrow{\phi} P_1]$ in which P_0 is finitely generated projective (and placed in degree 0) while P_1 is free of finite rank;*

(iii) *for any normal subgroup Γ of Δ_F there is, in $D^{\mathrm{p}}(\mathbb{Z}[\Delta_F/\Gamma])$, a canonical isomorphism*

$$\mathbb{Z}[\Delta_F/\Gamma] \otimes_{\mathbb{Z}[\Delta_F]}^{\mathbb{L}} C_{K,S,T}^\bullet \cong C_{K^\Gamma,S,T}^\bullet.$$

Proof. $C^\bullet_{K,S,T}$ is an object of $D^p(\mathbb{Z}[\Delta_F])$ by choice of S and by [**17**, Proposition 2·4 (iv)]. Claim (i) is [**17** Remark 2·7], Claim (ii) is proved in [**17**, Section 5·4]. Claim (iii) follows from the diagram in [**17**, Proposition 2·4 (i)], and the functoriality properties of étale cohomology.

3. *The proof of Theorem* 1·3

3·1. *The proof in case* (*i*)

In this subsection we assume the hypotheses of Theorem 1·3 (i). In particular, $\Delta := \mathrm{Gal}(K/\kappa)$ is isomorphic to $\mathrm{Aff}(q)$, and (S, V, T) is a Rubin datum for K/k with $|S| > |V| + 1$. We recall that $\mathrm{Aff}(q)$ is isomorphic to the semidirect product $\mathbb{F}_q \rtimes \mathbb{F}_q^\times$ with the natural action (see for instance [**32**, Example 2·16]).

Since $G = \mathrm{Gal}(K/k)$ is abelian, the complex $C^\bullet_{K,S,T}$ in $D^p(\mathbb{Z}[G])$ admits a well-defined determinant $\mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T})$ (in the sense of Knudsen–Mumford). We then also use the 'zeta element' $z_{K/k,S,T} \in \mathbb{R} \otimes_\mathbb{Z} \mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T})$, the definition of which can be found in [**17**, Definition 3·5] and will be recalled in the course of the proof of Lemma 3·1 below. For the moment we only note that $z_{K/k,S,T}$ is by construction an $\mathbb{R}[G]$-basis of the free rank-one $\mathbb{R}[G]$-module $\mathbb{R} \otimes_\mathbb{Z} \mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T})$.

LEMMA 3·1. *The following claims are valid.*

(*a*) *The zeta element* $z_{K/k,S,T}$ *belongs to* $\mathbb{Q} \otimes_\mathbb{Z} \mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T})$. *In particular,* $z_{K/k,S,T}$ *is a* $\mathbb{Q}[G]$-*basis of the free rank-one* $\mathbb{Q}[G]$-*module* $\mathbb{Q} \otimes_\mathbb{Z} \mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T})$.

(*b*) *For every prime number* ℓ, *there exists an element* $\mathfrak{z}^{(\ell)}_{K/k,S,T}$ *of* $\mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T})$ *with the following properties:*

(i) *the* $\mathbb{Z}[G]$-*submodule of* $\mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T})$ *generated by* $\mathfrak{z}^{(\ell)}_{K/k,S,T}$ *has prime-to-*ℓ *index;*

(ii) *the unique element* $\lambda^{(\ell)} \in \mathbb{Q}[G]$ *defined by* $z_{K/k,S,T} = \lambda^{(\ell)} \cdot \mathfrak{z}^{(\ell)}_{K/k,S,T}$ *belongs to the image of the map*

$$\rho_{\Delta/G} \colon \zeta(\mathbb{C}[\Delta]) \to \mathbb{C}[G], \quad x \mapsto \sum_{\chi \in \widehat{G}} \Big(\prod_{\psi \in \widehat{\Delta}} \psi(x)^{\langle \psi, \mathrm{Ind}^\Delta_G(\chi) \rangle} \Big) \cdot e_\chi,$$

where $\widehat{\Delta}$ *is the set of irreducible characters of* Δ, $\langle \cdot, \cdot \rangle$ *denotes the inner product of characters,* $\zeta(\mathbb{C}[\Delta]) \cong \prod_{\psi \in \widehat{\Delta}} \mathbb{C}$ *denotes the centre of* $\mathbb{C}[\Delta]$, *and we have written* ψ *for the map* $\zeta(\mathbb{C}[\Delta]) \to \mathbb{C}$ *induced by* ψ.

Proof. Claim (*a*) is equivalent to Stark's conjecture for K/k (cf. [**27**, Theorem 7·1 b)]). Since any non-trivial (irreducible) character of G induces a rational-valued character of Δ (see, for example, [**39**, Theorem 5]), the validity of Stark's conjecture follows from Tate's proof of Stark's conjecture for rational-valued characters in [**56**, Chapter II, Theorem 6·8].

To prove claim (b), we may enlarge S and T since, if S' and T' are respective disjoint finite oversets of S and T, then the exact triangles in [**17**, Proposition 2·4, (ii) and right-hand column of (6) in (i)] induce an isomorphism

$$\mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S',T'}) \xrightarrow{\simeq} \mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T})$$

that maps $z_{K/k,S',T'}$ to $z_{K/k,S,T}$. We therefore may and will assume that S contains all places that are ramified in K/κ and that both S and T are stable under the action of Δ.

Since the complex $C^\bullet_{K,S,T}$ depends only on K, S_K and T_K, we may then regard it also as an object of $D^p(\mathbb{Z}[\Delta])$. We fix a representative of $C^\bullet_{K,S,T}$ in $D(\mathbb{Z}[\Delta])$ as in Lemma 2·3 (ii) (applied to $F = \kappa$). We note that (1·1) combines with the Noether–Deuring Theorem to imply that $\mathbb{Q} \otimes_\mathbb{Z} P_0 \cong \mathbb{Q} \otimes_\mathbb{Z} P_1$. For every prime number ℓ, Roiter's Lemma [20, (31·6)] then gives the existence of an injection $i^{(\ell)} \colon P_1 \hookrightarrow P_0$ with finite cokernel of order prime to ℓ.

We fix a set $\{\sigma_1, \ldots, \sigma_{(\Delta:G)}\}$ of representatives for Δ/G and choose an ordered $\mathbb{Z}[\Delta]$-basis $\mathfrak{B} = \{b_1, \ldots, b_d\}$ of P_1. Then P_1 is also a free $\mathbb{Z}[G]$-module, with (ordered) $\mathbb{Z}[G]$-basis

$$\mathfrak{B}' := \{\sigma_1 b_1, \ldots \sigma_{(\Delta:G)} b_1, \ldots, \sigma_1 b_d, \ldots \sigma_{(\Delta:G)} b_d\}.$$

We also define ordered sets $\mathfrak{C}^{(\ell)} := \{i^{(\ell)}(b) \mid b \in \mathfrak{B}\}$ and $\mathfrak{C}'^{(\ell)} = \{i^{(\ell)}(b) \mid b \in \mathfrak{B}'\}$. Setting $P_1^* := \mathrm{Hom}_{\mathbb{Z}[G]}(P_1, \mathbb{Z}[G])$, we now define

$$\mathfrak{z}^{(\ell)}_{K/k,S,T} := \big(\bigwedge_{c \in \mathfrak{C}'^{(\ell)}} c\big) \otimes \big(\bigwedge_{b \in \mathfrak{B}'} b^*\big) \in \big(\bigwedge\nolimits^{(\Delta:G)d}_{\mathbb{Z}[G]} P_0\big) \otimes_{\mathbb{Z}[G]} \big(\bigwedge\nolimits^{(\Delta:G)d}_{\mathbb{Z}[G]} P_1^*\big)$$

$$= \mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T}),$$

where $b^* \colon P_1 \to \mathbb{Z}[G]$ denotes the $\mathbb{Z}[G]$-linear dual of $b \in P_1$. By construction, the element $\mathfrak{z}^{(\ell)}_{K/k,S,T}$ then has property (i).

To justify claim (ii), we first recall the definition of the zeta element $z_{K/k,S,T}$. Our fixed choice of representative for $C^\bullet_{K,S,T}$ gives rise to exact sequences $0 \to \mathcal{O}^\times_{K,S,T} \to P_0 \to \phi(P_0) \to 0$ and $\phi(P_0) \to P_1 \to \mathrm{Sel}^{\mathrm{tr}}_{K,S,T} \to 0$ of $\mathbb{Z}[\Delta]$-modules for which we may choose $\mathbb{R}[\Delta]$-splittings

$$\iota_1 \colon \mathbb{R} \otimes_\mathbb{Z} P_0 \cong (\mathbb{R} \otimes_\mathbb{Z} \mathcal{O}^\times_{K,S,T}) \oplus (\mathbb{R} \otimes_\mathbb{Z} \phi(P_0)),$$

$$\iota_2 \colon \mathbb{R} \otimes_\mathbb{Z} P_1 \cong (\mathbb{R} \otimes_\mathbb{Z} X_{K,S}) \oplus (\mathbb{R} \otimes_\mathbb{Z} \phi(P_0)).$$

Given this, we define the composite isomorphism of $\mathbb{R}[\Delta]$-modules

$$\alpha := (\iota_2^{-1} \circ (\lambda_{K,S} \oplus \mathrm{id}) \circ \iota_1) \colon P_0 \to P_1,$$

where $\lambda_{K,S}$ denotes the Dirichlet regulator map defined in (1·1). We write $A^{(\ell)}$ for the matrix in $\mathrm{GL}_{(\Delta:G)d}(\mathbb{R}[G])$ that represents α with respect to the bases $\mathfrak{C}'^{(\ell)}$ and $\mathfrak{B}'$.

We consider the 'leading term'

$$\theta^*_{K/\kappa,S,T}(0) := \sum_{\psi \in \widehat{\Delta}} L^*_{\kappa,S,T}(\check{\psi}, 0) e_\psi \in \zeta(\mathbb{R}[\Delta])^\times,$$

where $\check{\psi}$ denotes the contragredient of ψ and $L^*_{\kappa,S,T}(\check{\psi}, 0)$ is the leading term of $L_{\kappa,S,T}(\check{\psi}, s)$ at $s = 0$. Similarly, we set $\theta^*_{K/k,S,T}(0) := \sum_{\chi \in \widehat{G}} L^*_{k,S,T}(\check{\chi}, 0) e_\chi \in \mathbb{R}[G]^\times$. One then has that $z_{K/k,S,T} = \lambda^{(\ell)} \cdot \mathfrak{z}^{(\ell)}_{K/k,S,T}$ with $\lambda^{(\ell)} \in \mathbb{R}[G]^\times$ the unique element such that $\lambda^{(\ell)} \cdot \mathrm{det}_{\mathbb{R}[G]}(A^{(\ell)}) = \theta^*_{K/\kappa,S,T}(0)$. The reduced norm of the matrix $B^{(\ell)} \in \mathrm{GL}_d(\mathbb{R}[\Delta])$ that represents α with respect to the bases $\mathfrak{C}^{(\ell)}$ and $\mathfrak{B}$ belongs to $\zeta(\mathbb{R}[\Delta])^\times$, and we define a scalar $\mu^{(\ell)} \in \zeta(\mathbb{R}[\Delta])^\times$ by

$$\mu^{(\ell)} \cdot \mathrm{Nrd}_{\mathbb{R}[\Delta]}(B^{(\ell)}) = \theta^*_{K/\kappa,S,T}(0).$$

By the functoriality of reduced norms under restriction to subgroups (see, for example, [**5**, bottom of p. 291]) one has $\rho_{\Delta/G}(\mathrm{Nrd}_{\mathbb{R}[\Delta]}(B^{(\ell)})) = \det_{\mathbb{R}[G]}(A^{(\ell)})$ and thus also

$$\rho_{\Delta/G}(\mu^{(\ell)}) \cdot \det_{\mathbb{R}[G]}(A^{(\ell)}) = \rho_{\Delta/G}(\theta^*_{K/\kappa,S,T}(0)) = \theta^*_{K/k,S,T}(0),$$

from which we deduce that $\rho_{\Delta/G}(\mu^{(\ell)}) = \lambda^{(\ell)}$. This concludes the proof of claim (*b*).

We now give the proof of Theorem 1·3 in case (i). Since $\Delta \cong \mathrm{Aff}(q)$, one has that $\widehat{\Delta}$ consists of the linear characters of Δ/G and the unique irreducible character of degree $q-1$ that is obtained as $\psi_{\mathrm{nl}} := \mathrm{Ind}_G^{\Delta}(\chi)$ for any non-trivial (irreducible) character χ of G (see, for example, [**39**, Theorem 5]). As a consequence, one has

$$\langle \psi, \mathrm{Ind}_G^{\Delta}(\chi) \rangle = \begin{cases} 1 & \text{if } \chi \neq \mathbf{1}_G, \psi = \psi_{\mathrm{nl}}, \\ 1 & \text{if } \chi = \mathbf{1}_G, \psi = \mathbf{1}_{\Delta}, \\ 0 & \text{otherwise.} \end{cases}$$

For every prime number ℓ, the element $\lambda^{(\ell)}$ provided by Lemma 3·1 (*b*) (ii) is hence of the form $\lambda^{(\ell)} = ae_{\mathbf{1}} + b(1 - e_{\mathbf{1}})$ for suitable $a, b \in \mathbb{Q}$.

Now, the isomorphism $\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} C^{\bullet}_{K,S,T} \cong C^{\bullet}_{k,S,T}$ in Lemma 2·3 (iii) induces an isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}[G]} (\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Det}_{\mathbb{Z}[G]}(C^{\bullet}_{K,S,T})) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Det}_{\mathbb{Z}}(C^{\bullet}_{k,S,T})$ that sends $1 \otimes z_{K/k,S,T}$ to $z_{k/k,S,T}$. In addition, the analytic class number formula for k asserts that $z_{k/k,S,T}$ is a $\mathbb{Z}$-basis of the free rank-one $\mathbb{Z}$-module $\mathrm{Det}_{\mathbb{Z}}(C^{\bullet}_{k,S,T})$ (cf. [**35**, Section 2·2·2] or [**12**, Exercise 2·6]).

For each prime number ℓ, we write $\mathbb{Z}_{(\ell)}$ for the localisation of $\mathbb{Z}$ at the prime ideal $\ell\mathbb{Z}$. The definition of $\mathfrak{z}^{(\ell)}_{K/k,S,T}$ then implies that both $1 \otimes \mathfrak{z}^{(\ell)}_{K/k,S,T}$ and $a \cdot (1 \otimes \mathfrak{z}^{(\ell)}_{K/k,S,T}) = 1 \otimes (\lambda^{(\ell)} \mathfrak{z}^{(\ell)}_{K/k,S,T}) = 1 \otimes z_{K/k,S,T}$ are $\mathbb{Z}_{(\ell)}$-bases of $\mathbb{Z} \otimes_{\mathbb{Z}[G]} (\mathbb{Z}_{(\ell)} \otimes_{\mathbb{Z}} \mathrm{Det}_{\mathbb{Z}[G]}(C^{\bullet}_{K,S,T}))$. We conclude that a belongs to $\mathbb{Z}^{\times}_{(\ell)}$.

We next write $\mathrm{N} = \mathrm{N}_{\mathbb{Q}[G]/\mathbb{Q}} : \mathbb{Q}[G] \to \mathbb{Q}$ for the ring-theoretic norm map and note that the construction of [**10**, Lemma 3·7 (c)] gives the existence of an N-semilinear map $\mathcal{F} : \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Det}_{\mathbb{Z}[G]}(C^{\bullet}_{K,S,T}) \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Det}_{\mathbb{Z}}(C^{\bullet}_{K,S,T})$ that sends $z_{K/k,S,T}$ to $z_{K/K,S,T}$. Since $z_{K/K,S,T}$ is a $\mathbb{Z}$-basis of $\mathrm{Det}_{\mathbb{Z}}(C^{\bullet}_{K,S,T})$ by the analytic class number formula for K, we see that for each prime ℓ, both $\mathcal{F}(\mathfrak{z}^{(\ell)}_{K/k,S,T})$ and $z_{K/k,S,T} = \mathcal{F}(z_{k/k,S,T}) = \mathcal{F}(\lambda^{(\ell)} \mathfrak{z}^{(\ell)}_{K/k,S,T}) = \mathrm{N}(\lambda^{(\ell)}) \cdot \mathcal{F}(\mathfrak{z}^{(\ell)}_{K/k,S,T})$ are $\mathbb{Z}_{(\ell)}$-bases of $\mathbb{Z}_{(\ell)} \otimes_{\mathbb{Z}} \mathrm{Det}_{\mathbb{Z}}(C^{\bullet}_{K,S,T})$. It follows that $\mathrm{N}(\lambda^{(\ell)}) = ab^{q-1}$ must also belong to $\mathbb{Z}^{\times}_{(\ell)}$. Upon recalling that $a \in \mathbb{Z}^{\times}_{(\ell)}$ by the above discussion, we conclude that $b^{q-1} \in \mathbb{Z}^{\times}_{(\ell)}$. Since b is rational, we deduce that b belongs to $\mathbb{Z}^{\times}_{(\ell)}$.

Define an idempotent $e_{K,S,V}$ of $\mathbb{Q}[G]$ as the sum $\sum_{\chi} e_{\chi}$ of all primitive orthogonal idempotents e_{χ} associated with characters χ of G such that e_{χ} annihilates $\mathbb{C} \otimes_{\mathbb{Z}} X_{K,S \setminus V}$.

We then define a 'projection map' $\Theta^V_{K/k,S}$ as the composite map

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Det}_{\mathbb{Z}[G]}(C^{\bullet}_{K,S}) \longrightarrow \mathrm{Det}_{\mathbb{Q}[G]}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}^{\times}_{K,S}) \otimes_{\mathbb{Q}[G]} \mathrm{Det}_{\mathbb{Q}[G]}(\mathbb{Q} \otimes_{\mathbb{Z}} X_{K,S})^{-1}$$

$$\xrightarrow{\cdot e_{K,S,V}} e_{K,S,V} \cdot \left((\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge\nolimits^{|V|}_{\mathbb{Z}[G]} \mathcal{O}^{\times}_{K,S,T}) \otimes_{\mathbb{Q}[G]} (\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge\nolimits^{|V|}_{\mathbb{Z}[G]} Y_{K,V})^{-1} \right)$$

$$\xrightarrow{\simeq} e_{K,S,V} \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge\nolimits^{|V|}_{\mathbb{Z}[G]} \mathcal{O}^{\times}_{K,S,T}), \tag{3.4}$$

where the first arrow is the natural 'passage-to-cohomology' map, the second map is induced by multiplication by $e_{K,S,V}$, and the last arrow by the trivialisation $\bigwedge^{|V|}_{\mathbb{Z}[G]} Y_{K,V} \cong \mathbb{Z}[G]$ that is afforded by sending $\bigwedge_{1 \leq i \leq |V|} w_i$ to 1.

Note that our hypothesis $|S| > |V| + 1$ combines with the short exact sequence $0 \to X_{K,S \setminus V} \to X_{K,S} \to Y_{K,V} \to 0$ to imply that $e_1 \cdot e_{K,S,V} = 0$. In particular, we have $\lambda^{(\ell)} \cdot e_{K,S,V} = (ae_1 + b(1 - e_1)) \cdot e_{K,S,V} = be_{K,S,V}$. Since it is proved in [17, Theorem 5·14] that one has $\Theta^V_{K/k,S}(z_{K/k,S,T}) = \varepsilon^V_{K/k,S,T}$, we therefore deduce that

$$\varepsilon^V_{K/k,S,T} = \Theta^V_{K/k,S,T}(z_{K/k,S,T}) = \lambda^{(\ell)} \cdot \Theta^V_{K/k,S,T}(\mathfrak{z}^{(\ell)}_{K/k,S,T}) = b \cdot \Theta^V_{K/k,S,T}(\mathfrak{z}^{(\ell)}_{K/k,S,T})$$

for each prime ℓ. Now, the equality

$$\mathbb{Z}_{(\ell)} \otimes_{\mathbb{Z}} \mathrm{im}(\Theta^V_{K/k,S,T}(\mathfrak{z}^{(\ell)}_{K/k,S,T})) = \mathbb{Z}_{(\ell)} \otimes_{\mathbb{Z}} \mathrm{Fitt}^{|V|}_{\mathbb{Z}[G]}(\mathrm{Sel}_{K,S,T})^\#$$

that is established via the argument of [17, Theorem 7·5] combines with the last displayed equation and the fact that b is invertible to imply that

$$\mathbb{Z}_{(\ell)} \otimes_{\mathbb{Z}} \mathrm{im}(\varepsilon^V_{K/k,S,T}) = \mathbb{Z}_{(\ell)} \otimes_{\mathbb{Z}} \left(b \cdot \mathrm{im}(\Theta^V_{K/k,S,T}(\mathfrak{z}^{(\ell)}_{K/k,S,T}))\right) = \mathbb{Z}_{(\ell)} \otimes_{\mathbb{Z}} \mathrm{Fitt}^{|V|}_{\mathbb{Z}[G]}(\mathrm{Sel}_{K,S,T})^\#.$$

The claim in Theorem 1·3 (i) now follows upon recalling that ℓ is an arbitrary prime number.

3·2. *The proof in case* (ii)

To prove Theorem 1·3 in case (ii), we let K/k be a biquadratic extension of number fields and note that, by the known validity of Stark's conjecture for K/k, the zeta element $z_{K/k,S,T}$ is a $\mathbb{Q}[G]$-basis of the free rank-one $\mathbb{Q}[G]$-module $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T})$ (cf. the argument of Lemma 3·1 (a)). We then let ℓ be an arbitrary prime number and choose, using Roiter's Lemma, an element $\mathfrak{z}^{(\ell)}_{K/k,S,T}$ that generates a $\mathbb{Z}[G]$-submodule of $\mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T})$ of finite, prime-to-ℓ index. Label the proper intermediate fields of K/k as $K_1 := k$, K_2, K_3, and K_4, and, using Lemma 2·3 (iii), denote the image of $\mathfrak{z}^{(\ell)}_{K/k,S,T}$ under the natural map

$$\mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T}) \to \mathbb{Z}[\mathrm{Gal}(K_i/k)] \otimes_{\mathbb{Z}[G]} \mathrm{Det}_{\mathbb{Z}[G]}(C^\bullet_{K,S,T}) \cong \mathrm{Det}_{\mathbb{Z}[\mathrm{Gal}(K_i/k)]}(C^\bullet_{K_i,S,T})$$

as $\mathfrak{z}^{(\ell)}_{K_i/k,S,T}$ for every $i \in \{1, \ldots, 4\}$. Write χ_i for the trivial character if $i = 1$ and the non-trivial character of $\mathrm{Gal}(K_i/k)$ otherwise. The discussion above (in case (i)) then shows that we have

$$e_{\chi_i} \cdot \mathfrak{z}^{(\ell)}_{K_i/k,S,T} = a_i \cdot e_{\chi_i} \cdot z_{K_i/k,S,T}$$

for some a_i in $\mathbb{Z}^\times_{(\ell)}$. It follows that

$$\mathfrak{z}^{(\ell)}_{K/k,S,T} = \left(\sum_{i=1}^4 a_i e_{\chi_i}\right) \cdot z_{K/k,S,T}.$$

If $\ell \neq 2$, then it is clear that $\lambda^{(\ell)} := \sum_{i=1}^4 a_i e_{\chi_i}$ belongs to $\mathbb{Z}_{(\ell)}[G]^\times$. For $\ell = 2$, the scalar $\lambda^{(2)}$ belongs to $\mathbb{Z}_{(2)}[G]^\times$ if and only if it belongs to $\mathbb{Z}_{(2)}[G]$ because $\mathrm{N}_{\mathbb{Q}[G]/\mathbb{Q}}(\lambda^{(2)}) = \prod_{i=1}^4 a_i$ is a unit in $\mathbb{Z}_{(2)}$. Now, $\lambda^{(2)}$ is in $\mathbb{Z}_{(2)}[G]$ if and only if, for every $\sigma \in G$ we have that

$$\sum_{i=1}^4 a_i \chi_i(\sigma) \equiv 0 \mod 4.$$

Note that $\chi_i(\sigma) = \pm 1$ and $a_i \equiv \pm 1 \mod 4$ for all $i \in \{1, \ldots, 4\}$. One can then check explicitly that the above congruence holds if and only if $\prod_{i=1}^4 a_i \equiv 1 \mod 4$ (cf. also [9, Lemma 6·3 (v)]). In particular, if we let $b \in \{\pm 1\}$ be defined by $b \equiv \prod_{i=1}^4 a_i \mod 4$, then $\lambda' := ba_1 e_1 + \sum_{i=1}^3 a_i e_{\chi_i}$ belongs to $\mathbb{Z}_{(2)}[G]^\times$.

As in case (i), we define $e_{K,S,V}$ as the sum of all e_χ that annihilate $\mathbb{C} \otimes_{\mathbb{Z}} X_{K,S \setminus V}$. The assumption $|S| > |V| + 1$ then ensures that $e_{\mathbf{1}} \cdot e_{K,S,V} = 0$ and thus that $\lambda^{(2)} e_{K,S,V} = \lambda' e_{K,S,V}$. Using the map $\Theta^V_{K/k,S}$ defined in (3·4), we obtain the equality

$$\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} \mathrm{im}(\varepsilon^V_{K/k,S,T}) = \mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} \lambda' \cdot \mathrm{im}(\Theta^V_{K/k,S,T}(\mathfrak{z}^{(2)}_{K/k,S,T})) = \mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} \mathrm{Fitt}^{|V|}_{\mathbb{Z}[G]}(\mathrm{Sel}_{K,S,T})^{\#},$$

where the final equality follows from the argument of [**17**, Theorem 7·5] as in case (i). Since the corresponding identity also holds for each odd ℓ, this completes the proof of Theorem 1·3.

Remark 3·2. The only instances of (i) and (ii) in Theorem 1·3 that can neither be treated by the argument used to prove Theorem 1·3 nor Remark 1·11 (i) are the cases in which $|S| = |V| + 1$ and the unique place $v \in S \setminus V$ has full decomposition group in K/k. In any such situation and for large enough V, the equality (1·3) is in fact equivalent to eTNC(K/k) and amounts to a subtle question about signs. To make this more explicit in case (ii) of Theorem 1·3, we suppose that K/k is biquadratic, $|S| = |V| + 1$, and V is large enough that $\mathrm{Cl}_{K,S,T}$ vanishes. Then $\mathcal{O}^\times_{K,S,T}$ is a free $\mathbb{Z}[G]$-module of rank $|V|$ and we can choose an ordered $\mathbb{Z}[G]$-basis $\mathfrak{B}$ of $\mathcal{O}^\times_{K,S,T}$. Fix an ordering $G = \{g_1, g_2, g_3, g_4\}$ and consider the ordered $\mathbb{Z}$-basis $\mathfrak{B}' := \{gb \mid g \in G, b \in \mathfrak{B}\}$ of $\mathcal{O}^\times_{K,S,T}$, ordered lexicographically. Similarly, we set $\mathfrak{W} := \{gw_i \mid g \in G, 1 \leq i \leq |V|\}$, ordered lexicographically. Then one can show that (1·3) is equivalent to

$$\det_{\mathbb{R}}(\log |b|_w)_{b \in \mathfrak{B}', w \in \mathfrak{W}} < 0.$$

(Cf. [**9**, Proposition 10·5].) This question does not depend on the ordering on G and, since G is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, also not on the choice of basis $\mathfrak{B}$ (or the ordering on it) because every unit in $\mathbb{Z}[G]$ is of the form $\pm g$ for some $g \in G$, and so has norm 1.

In the setting of case (i) of Theorem 1·3 one can similarly derive an explicit criterion by using [**6**, Lemma 3·5].

4. *The proof of Theorem* 1·1

We now fix a p-elementary extension K/k with Galois group $G \cong (\mathbb{Z}/p\mathbb{Z})^m$. Write Ω^* for the set of subgroups H of G of index at most p. The following algebraic observation plays a key role in the sequel.

LEMMA 4·1. *Set* $\mathrm{N}_H = \sum_{\tau \in H} \tau$ *for every* $H \in \Omega^*$. *In* $\mathbb{Z}[G]$ *we then have the equality*

$$\sum_{H \in \Omega^*} \mathrm{N}_H + \left((p^{m-1} - 1) - \left(\sum_{i=0}^{m-1} p^i\right)\right) \cdot \mathrm{N}_G = p^{m-1}.$$

Proof. Observe that G is an $\mathbb{F}_p$-vector space and the (non-trivial) H are exactly the $(m-1)$-dimensional subspaces of G. Recall that the trace pairing

$$\mathbb{F}_p^m \times \mathbb{F}_p^m \to \mathbb{F}_p, \quad (v, w) \mapsto \sum_{i=1}^m v_i w_i$$

is perfect, hence induces a bijection between $(m-1)$-dimensional and 1-dimensional subspaces. The number of 1-dimensional subspaces is exactly $p^m - 1/p - 1$, hence $|\Omega^* \setminus \{G\}|$ is equal to $p^m - 1/p - 1$. If we fix $v \in \mathbb{F}_p \setminus \{0\}$, then the set of all $(m-1)$-dimensional

subspaces of $\mathbb{F}_p^m$ that contain v is in bijection with all 1-dimensional subspaces of the space $\{w \in \mathbb{F}_p^m \mid \sum_{i=1}^m v_i w_i = 0\}$, the kernel of the $(1 \times m)$-matrix v. This space is therefore of dimension $m - 1$ and contains $p^{m-1} - 1/p - 1$ subspaces of dimension one. That is, there are exactly $p^{m-1} - 1/p - 1$ subgroups $H \in \Omega^* \setminus \{H\}$ that contain a given (non-trivial) element of G. It follows that there are exactly

$$\frac{p^m - 1}{p - 1} - \frac{p^{m-1} - 1}{p - 1} = \frac{(p^m - 1) - (p^{m-1} - 1)}{p - 1} = \frac{p^{m-1}(p - 1)}{(p - 1)} = p^{m-1}$$

such H that do *not* contain a given (non-trivial) element. Thus, each element of G appears in the sum $\left(\sum_{H \in \Omega^* \setminus \{G\}} \mathrm{N}_H \right) + p^{m-1}(\mathrm{N}_G - 1)$ exactly $|\Omega^* \setminus \{G\}|$ many times. From this we obtain

$$\left(\sum_{H \in \Omega^* \setminus \{G\}} \mathrm{N}_H \right) + p^{m-1}(\mathrm{N}_G - 1) = |\Omega^* \setminus \{G\}| \cdot \mathrm{N}_G = \frac{(p^m - 1)}{p - 1} \cdot \mathrm{N}_G = \left(\sum_{i=0}^{m-1} p^i \right) \cdot \mathrm{N}_G.$$

For any integer $r \geq 0$ and $H \in \Omega^*$, we consider the injection

$$\nu_H \colon \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge\nolimits_{\mathbb{Z}[G/H]}^r \mathcal{O}_{K^H,S,T}^\times \to \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge\nolimits_{\mathbb{Z}[G]}^r \mathcal{O}_{K,S,T}^\times, \quad a \mapsto |H|^{\max\{0, 1 - r\}} \cdot a$$

that satisfies

$$\nu_H(\mathrm{N}_H^r a) = \mathrm{N}_H a \quad \text{for any } a \in \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge\nolimits_{\mathbb{Z}[G]}^r \mathcal{O}_{K,S,T}^\times. \tag{4.5}$$

As a straightforward application of Lemma 4·1 we obtain the following consequence that recovers [51, Proposition 4·5] in the case $p = 2$.

PROPOSITION 4·2. *In* $\mathbb{R} \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}[G]}^r \mathcal{O}_{K,S,T}^\times$ *we have the equality*

$$\varepsilon_{K/k,S,T}^V = \frac{1}{p^{m-1}} \cdot \left(\sum_{H \in \Omega^*} \nu_H\left(\varepsilon_{K^H/k,S,T}^V \right) + \left((p^{m-1} - 1) - \left(\sum_{i=0}^{m-1} p^i \right) \right) \cdot \nu_G\left(\varepsilon_{k/k,S,T}^V \right) \right).$$

Proof. Using Lemma 4·1 (*a*), equation (4·5), and the norm relations for Rubin–Stark elements [48, Proposition 6·1] we calculate

$$p^{m-1} \cdot \varepsilon_{K/k,S,T}^V = \left(\sum_{H \in \Omega^*} \mathrm{N}_H + \left((p^{m-1} - 1) - \left(\sum_{i=0}^{m-1} p^i \right) \right) \cdot \mathrm{N}_G \right) \cdot \varepsilon_{K/k,S,T}^V$$

$$= \left(\sum_{H \in \Omega^*} \nu_H\left(\mathrm{N}_H^{|V|} \varepsilon_{K/k,S,T}^V \right) \right) + \left((p^{m-1} - 1) - \left(\sum_{i=0}^{m-1} p^i \right) \right) \cdot \nu_G\left(\mathrm{N}_G^{|V|} \varepsilon_{K/k,S,T}^V \right)$$

$$= \left(\sum_{H \in \Omega^*} \nu_H\left(\varepsilon_{K^H/k,S,T}^V \right) \right) + \left((p^{m-1} - 1) - \left(\sum_{i=0}^{m-1} p^i \right) \right) \cdot \nu_G\left(\varepsilon_{k/k,S,T}^V \right),$$

as required to prove the claim.

To prepare for the proof of Theorem 1·1, we now first give a preliminary result in which we write $I_G := \ker\{\mathbb{Z}[G] \to \mathbb{Z}\}$ for the absolute augmentation ideal of $\mathbb{Z}[G]$ and, given a $\mathbb{Z}[G]$-module M and non-negative integer r, define its r-th exterior bidual' to be

$$\bigcap\nolimits_{\mathbb{Z}[G]}^{r} M := \big\{ a \in \mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge\nolimits_{\mathbb{Z}[G]}^{r} M \mid F(a) \in \mathbb{Z}[G] \text{ for all } F \in \bigwedge\nolimits_{\mathbb{Z}[G]}^{r} \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G]) \big\}.$$

LEMMA 4·3. *Fix a Rubin datum (S, V, T) for K/k and a non-negative integer c that satisfies*

$$|S| \geq \max\{|V| + 2, |V| - s_p + (p-1)(m-1) + 2 + c\},$$

where $s_p := \dim_{\mathbb{F}_p} (\mathrm{Cl}_{k,S,T} \otimes_{\mathbb{Z}} \mathbb{F}_p)$ denotes the p-rank of $\mathrm{Cl}_{k,S,T}$. If the equality (1·3) of Conjecture 1·10 holds for all extensions L/k in Ω, then $\varepsilon_{K/k,S,T}^{V}$ belongs to $I_G^c \cdot \bigcap_{\mathbb{Z}[G]}^{|V|} \mathcal{O}_{K,S,T}^{\times}$ (so, in particular, the Rubin–Stark conjecture is valid for (K/k, S, V, T)).

Proof. At the outset we note that, for any $H \in \Omega^*$, the map ν_H restricts to an injection $\bigcap_{\mathbb{Z}[G/H]}^{|V|} \mathcal{O}_{K^H,S,T}^{\times} \to \bigcap_{\mathbb{Z}[G]}^{|V|} \mathcal{O}_{K,S,T}^{\times}$ (cf. [17, Remark 4·13]). By Proposition 4·2, it is hence sufficient to prove that $\varepsilon_{K^H/k,S,T}^{V}$ belongs to $p^{m-1} I_{G/H}^c \bigcap_{\mathbb{Z}[G/H]}^{|V|} \mathcal{O}_{K^H,S,T}^{\times}$ for every $H \in \Omega^*$. By the assumption $|S| \geq |V| + 2$, we may and will assume $K^H \neq k$ so that $K^H \in \Omega$.

We now first claim that for this purpose it is enough to prove that $\mathrm{im}(\varepsilon_{K^H/k,S,T}^{V})$ is contained in $p^{m-1} I_{G/H}^{1+c}$. To justify this, we apply Lemma 2·3 (ii) to fix a representative $[P_0 \xrightarrow{\phi} P_1]$ of the complex $C_{K^H,S,T}^{\bullet}$ in $D^p(\mathbb{Z}[G/H])$. From [49, Lemma B·6] we then obtain an exact sequence

$$0 \longrightarrow \bigcap\nolimits_{\mathbb{Z}[G/H]}^{|V|} \mathcal{O}_{K^H,S,T}^{\times} \longrightarrow \bigwedge\nolimits_{\mathbb{Z}[G/H]}^{|V|} P_0 \xrightarrow{\phi} P_1 \otimes_{\mathbb{Z}[G/H]} \bigwedge\nolimits_{\mathbb{Z}[G/H]}^{|V|-1} P_0. \qquad (4·6)$$

In particular, we may view $\varepsilon_{K^H/k,S,T}^{V}$ as an element of $\bigwedge_{\mathbb{Z}[G/H]}^{|V|} P_0$. Now, if $\mathrm{im}(\varepsilon_{K^H/k,S,T}^{V})$, which equals $\{F(\varepsilon_{K^H/k,S,T}^{V}) \mid F \in \bigwedge_{\mathbb{Z}[G/H]}^{|V|} \mathrm{Hom}_{\mathbb{Z}[G]}(P_0, \mathbb{Z}[G])\}$, is contained in $p^{m-1} I_{G/H}^{1+c}$, then $\varepsilon_{K^H/k,S,T}^{V}$ belongs to the module $p^{m-1} I_{G/H}^{1+c} \bigwedge_{\mathbb{Z}[G]}^{|V|} P_0$ (cf. [17, Proposition 4·17]). We may therefore write $\varepsilon_{K^H/k,S,T}^{V} = p^{m-1} (\sigma_H - 1)^{1+c} a$ with σ_H a generator of G/H and a an element of $\bigwedge_{\mathbb{Z}[G]}^{|V|} P_0$. From the exact sequence (4·6) we then see that

$$p^{m-1} (\sigma_H - 1)^{1+c} \cdot \phi(a) = \phi(p^{m-1} (\sigma_H - 1)^{1+c} a) = \phi(\varepsilon_{K^H/k,S,T}^{V}) = 0.$$

Since $P' := P_1 \otimes_{\mathbb{Z}[G/H]} \bigwedge_{\mathbb{Z}[G/H]}^{|V|-1} P_0$ is $\mathbb{Z}$-torsion free, this implies that $(\sigma_H - 1)^{1+c} \cdot \phi(a)$ vanishes. As $(\sigma_H - 1)P'$ and $(P')^{G/H} = \ker\{P' \xrightarrow{\cdot(\sigma_H-1)} P'\}$ intersect trivially because P' is G/H-cohomologically trivial, it then follows by induction on c that $(\sigma_H - 1)\phi(a)$ vanishes. Exactness of (4·6) now shows that $(\sigma_H - 1)a$ belongs to $\bigcap_{\mathbb{Z}[G/H]}^{|V|} \mathcal{O}_{K^H,S,T}^{\times}$, as required to prove that $\varepsilon_{K^H,S,T}^{V}$ belongs to $p^{m-1} (\sigma_H - 1)^c \bigcap_{\mathbb{Z}[G/H]}^{|V|} \mathcal{O}_{K^H,S,T}^{\times}$.

It now remains to prove that $\mathrm{im}(\varepsilon_{K^H/k,S,T}^{V})$ is contained in $p^{m-1} I_{G/H}^{1+c}$. We may and will assume that no place in $S \setminus V$ splits completely in K^H/k, since otherwise $\varepsilon_{K^H/k,S,T}^{V}$ vanishes. Thus, every place in $S \setminus V$ has full decomposition group in K^H/k. Since we assume (1·3) to hold for K^H/k it is enough to prove, in this situation, that $\mathrm{Fitt}_{\mathbb{Z}[G/H]}^{|V|}(\mathrm{Sel}_{K^H,S,T})^{\#} \subseteq p^{m-1} I_{G/H}^{1+c}$.

To verify this inclusion, we use the 'transpose' Selmer group defined in Lemma 2·3 (i) and the equality

$$\mathrm{Fitt}_{\mathbb{Z}[G/H]}^{|V|}(\mathrm{Sel}_{K^H,S,T})^{\#} = \mathrm{Fitt}_{\mathbb{Z}[G/H]}^{|V|}(\mathrm{Sel}_{K^H,S,T}^{\mathrm{tr}})$$

of [17, Lemma 2·8]. It then suffices to verify that $\mathrm{Fitt}_{\mathbb{Z}[G/H]}^{|V|}(\mathrm{Sel}_{K^H,S,T}^{\mathrm{tr}}) \subseteq p^{m-1} I_{G/H}^{1+c}$.

For this purpose, we first note that $Y_{K^H,V}$ is a free direct summand of $X_{K^H,S} \cong Y_{K^H,V} \oplus X_{K^H,S\setminus V}$, hence also of $\mathrm{Sel}^{\mathrm{tr}}_{K^H,S,T}$. We may thus find a $\mathbb{Z}[G/H]$-module M such that $\mathrm{Sel}^{\mathrm{tr}}_{K^H,S,T} \cong M \oplus Y_{K^H,V}$ and one has the following modified version of the exact sequence in Lemma 2·3 (i):

$$0 \longrightarrow \mathrm{Cl}_{K^H,S,T} \longrightarrow M \longrightarrow X_{K^H,S\setminus V} \longrightarrow 0. \tag{4·7}$$

Setting $d := |S \setminus V|$, one has $X_{K^H,S\setminus V} \cong \mathbb{Z}^{d-1}$ and fixing again a generator σ_H of G/H,

$$\mathrm{Fitt}^0_{\mathbb{Z}[G/H]}(X_{K^H,S\setminus V}) = I_{G/H}^{d-1} = (\sigma_H - 1)^{d-1}\mathbb{Z}[G/H].$$

In particular, $\mathrm{Fitt}^0_{\mathbb{Z}[G/H]}(X_{K^H,S\setminus V})$ is a principal ideal and so we may apply [**31**, Lemma 2·5 (ii)] to the exact sequence (4·7) to infer that

$$\mathrm{Fitt}^{|V|}_{\mathbb{Z}[G/H]}(\mathrm{Sel}^{\mathrm{tr}}_{K^H,S,T}) = \mathrm{Fitt}^0_{\mathbb{Z}[G/H]}(M) = \mathrm{Fitt}^0_{\mathbb{Z}[G/H]}(\mathrm{Cl}_{K^H,S,T}) \cdot \mathrm{Fitt}^0_{\mathbb{Z}[G/H]}(X_{K^H,S\setminus V})$$
$$= \mathrm{Fitt}^0_{\mathbb{Z}[G/H]}(\mathrm{Cl}_{K^H,S,T}) \cdot I_{G/H}^{d-1}.$$

Fix a place $v \in S \setminus V$ and recall that we may assume that v has full decomposition group in K^H/k. If we write $H_{S,T}(K^H)$ and $H_{S,T}(k)$ for the (S, T)-ray class fields of K^H and k, respectively, then $H_{S,T}(k) \cap K^H = k$ since v splits completely in $H_{S,T}(k)$. Thus, we may identify $\mathrm{Gal}(H_{S,T}(k)/k) \cong \mathrm{Gal}(K^H \cdot H_{S,T}(k)/K^H)$ and hence the restriction map $\mathrm{Gal}(H_{S,T}(K^H)/K^H) \to \mathrm{Gal}(H_{S,T}(k)/k)$ is surjective. By class field theory, the restriction map corresponds to the norm map $\mathrm{Cl}_{K^H,S,T} \to \mathrm{Cl}_{k,S,T}$ and so, in particular, the map $\mathrm{Cl}_{K^H,S,T} \to \mathrm{Cl}_{k,S,T} \otimes_{\mathbb{Z}} \mathbb{F}_p \cong (\mathbb{Z}/p\mathbb{Z})^{s_p}$ is surjective as well. This map is G/H-equivariant, thus we obtain an inclusion

$$\mathrm{Fitt}^0_{\mathbb{Z}[G/H]}(\mathrm{Cl}_{K^H,S,T}) \subseteq \mathrm{Fitt}^0_{\mathbb{Z}[G/H]}\big((\mathbb{Z}/p\mathbb{Z})^{s_p}\big) = \prod_{i=1}^{s_p}(p\mathbb{Z}[G/H] + I_{G/H}) \subseteq \sum_{i=0}^{s_p} p^i I_{G/H}^{s_p-i}.$$

By the previous discussion, we therefore have an inclusion

$$\mathrm{Fitt}^{|V|}_{\mathbb{Z}[G/H]}(\mathrm{Sel}^{\mathrm{tr}}_{K^H,S,T}) \subseteq \Big(\sum_{i=0}^{s_p} p^i I_{G/H}^{s_p-i} \Big) \cdot I_{G/H}^{d-1}$$
$$= \sum_{i=0}^{s_p} p^i I_{G/H}^{s_p-i+d-1}$$
$$\subseteq \Big(\sum_{i=0}^{s_p} p^i I_{G/H}^{s_p-i+d-c-1} \Big) \cdot I_{G/H}^c.$$

Since σ_H is of order p, we have $(\sigma_H - 1)^p \equiv \sigma_H^p - 1 = 0 \mod p$ and so $(\sigma_H - 1)^p$ is divisible by p in $\mathbb{Z}[G/H]$. Noting that the quotient $\mathbb{Z}[G/H]/I_{G/H} \cong \mathbb{Z}$ is torsion-free, we see that $(\sigma_H - 1)^p$ is in fact divisible by $p(\sigma_H - 1)$. From this it follows that $(\sigma_H - 1)^{s_p-i+d-c-1}$ is divisible by $p^{\max\{0,\lfloor(s_p-i+d-c-2)/(p-1)\rfloor\}}(\sigma_H - 1)$. As a consequence,

$$\sum_{i=0}^{s_p} p^i I_{G/H}^{s_p-i+d-c-1} \subseteq \sum_{i=0}^{s_p} p^{i+\lfloor(s_p-i+d-c-2)/(p-1)\rfloor} I_{G/H} \subseteq p^{\lfloor(s_p+d-c-2)/(p-1)\rfloor} I_{G/H},$$

where we have used that

$$i + \lfloor \frac{s_p - i + d - c - 2}{p - 1} \rfloor = \lfloor \frac{(p-1)i + s_p - i + d - c - 2}{p - 1} \rfloor \geq \lfloor \frac{s_p + d - c - 2}{p - 1} \rfloor$$

as a consequence of $p - 1 \geq 1$. Now,

$$\frac{d + s_p - c - 2}{p - 1} = \frac{|S| - |V| + s_p - c - 2}{p - 1} \geq m - 1 \quad \Leftrightarrow$$

$$|S| \geq |V| - s_p + (p-1)(m-1) + 2 + c$$

and so $\mathrm{Fitt}^{|V|}_{\mathbb{Z}[G/H]}(\mathrm{Sel}^{\mathrm{tr}}_{K^H,S,T})$ is contained in $p^{m-1}I^{1+c}_{G/H}$ as soon as $|S| \geq |V| - s_p + (p-1)(m-1) + 2 + c$. This concludes the proof that $\mathrm{im}(\varepsilon^V_{K/k,S,T})$ is contained in $p^{m-1}I^{1+c}_{G/H}$, as required.

We can now give the proof of Theorem 1·1.

Proof (of Theorem 1·1): Write $H_{k,p}$ and H_K for the extensions of k and K that correspond with $\mathrm{Cl}_{k,S,T} \otimes_{\mathbb{Z}} \mathbb{F}_p$ and $\mathrm{Cl}_{K,S,T}$ via class field theory. That is, $H_{k,p}$ is the maximal p-elementary abelian extension of k that is unramified outside T and in which all places in S split completely, and H_K is the maximal abelian extension of K that is unramified outside T_K and in which all places in S_K split completely. Note that H_K is Galois over k. By Cebotarev's density theorem, we may then choose a finite set W of prime ideals of k that has all of the following properties:

(i) W is disjoint from $S \cup T$;

(ii) every place in W splits completely in $K \cdot H_{k,p}$;

(iii) $\{\mathrm{Frob}_{\mathfrak{p}} \mid \mathfrak{p} \in W\}$ is a generating set for $\mathrm{Gal}(H_K/K \cdot H_{k,p})$.

In particular, one has $\mathrm{Cl}_{k,S',T} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathrm{Cl}_{k,S,T} \otimes_{\mathbb{Z}} \mathbb{F}_p$ with $S' := S \cup W$. Class field theory then provides for a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Cl}_{K,S',T} & \xrightarrow{\;\simeq\;} & \mathrm{Gal}(H_{k,p}K/K) \\
\downarrow{\widetilde{\mathrm{N}}_{K/k}} & & \downarrow \\
\mathrm{Cl}_{k,S,T} \otimes_{\mathbb{Z}} \mathbb{F}_p & \xrightarrow{\;\simeq\;} & \mathrm{Gal}(H_{k,p}/k),
\end{array}
$$

where the right-hand vertical arrow is the natural restriction map and $\widetilde{\mathrm{N}}_{K/k}$ is the composite of the 'norm' map $\mathrm{Cl}_{K,S',T} \to \mathrm{Cl}_{k,S',T}$ induced by the norm $\mathrm{N}_{K/k} \colon K^\times \to k^\times$ and the projection $\mathrm{Cl}_{k,S',T} \to \mathrm{Cl}_{k,S,T} \otimes_{\mathbb{Z}} \mathbb{F}_p$. As a consequence, we obtain a G-equivariant isomorphism $\mathrm{Cl}_{K,S',T} \cong \widetilde{\mathrm{N}}_{K/k}(\mathrm{Cl}_{K,S',T})$, and hence an exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow \mathcal{O}^\times_{K,S,T} \longrightarrow \mathcal{O}^\times_{K,S',T} \xrightarrow{\psi} Y_{K,W} \xrightarrow{\delta} \mathrm{Cl}_{K,S,T} \longrightarrow \widetilde{\mathrm{N}}_{K/k}(\mathrm{Cl}_{K,S',T}) \longrightarrow 0 \qquad (4\cdot8)$$

with $\psi \colon \mathcal{O}^\times_{K,S',T} \to Y_{K,W}$ the map $a \mapsto \sum_{w \in W_K} \mathrm{ord}_w(a)w$ and $\delta \colon Y_{K,W} \to \mathrm{Cl}_{K,S,T}$ sends $w \in W_K$ to the class of w in $\mathrm{Cl}_{K,S,T}$.

Fix a labelling $W = \{v_{|S|+1}, \dots, v_{|S'|}\}$ and, for each $j \in \{|S| + 1, \dots, |S'|\}$, an extension w_j of v_j to K. By condition (ii) every place of K above a fixed v_j is of the form σw_j for

some $\sigma \in G$, which allows us to define a map $w_j^* \colon Y_{K,W} \to \mathbb{Z}[G]$ by sending $\sum_{w \in W_K} a_w w$ to $\sum_{\sigma \in G} a_{\sigma w} \sigma$ (so w_j^* is the 'dual' of w_j). Now, if $a \in \mathcal{O}_{K,S_j,T}^\times$ with $S_j := S \cup \{v_j\}$, then

$$\psi(a) = \sum_{l=|S|+1}^{|S'|} (w_l^* \circ \psi)(a)w_l = (w_j^* \circ \psi)(a)w_j$$

belongs to the kernel of δ by exactness of (4·8). This shows that $(w_j^* \circ \psi)(a)$ annihilates the class of w_j in $\mathrm{Cl}_{K,S,T}$. Since $A := \ker\{\mathrm{Cl}_{K,S,T} \to \widetilde{\mathrm{N}}_{K/k}(\mathrm{Cl}_{K,S',T})\}$ is generated over $\mathbb{Z}[G]$ by $\delta(w_{|S|+1}), \ldots, \delta(w_{|S'|})$ by exactness of (4·9), we have thereby proved that

$$\bigcap_{j=|S|+1}^{|S'|} (w_j^* \circ \psi)(\mathcal{O}_{K,S_j,T}^\times) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(A). \tag{4·9}$$

We now claim that $\mathrm{im}(\varepsilon_{K/k,S,T}^V)$ is contained in I_G times the intersection on the left- hand side of (4·9). To do this, we first note that $s_p' := \dim_{\mathbb{F}_p}(\mathrm{Cl}_{k,S',T} \otimes_{\mathbb{Z}} \mathbb{F}_p)$ is equal to s_p because $\mathrm{Cl}_{k,S',T} \otimes_{\mathbb{Z}} \mathbb{F}_p = \mathrm{Cl}_{k,S,T} \otimes_{\mathbb{Z}} \mathbb{F}_p$ by condition (ii). Setting $V' := V \cup W$, it then follows that

$$\begin{aligned}
|S'| = |W| + |S| &\geq |W| + \max\{|V| + 2, |V| - s_p + (p-1)(m-1) + 3\} \\
&\geq \max\{|V'| + 2, |V'| - s_p' + (p-1)(m-1) + 3\}.
\end{aligned}$$

By Lemma 4·3, we therefore have that $\varepsilon_{K/k,S',T}^{V'}$ belongs to $I_G \cdot \bigcap_{\mathbb{Z}[G]}^{|V'|} \mathcal{O}_{K,S',T}^\times$, hence can be written as $\varepsilon_{K/k,S',T}^{V'} = \sum_{i=1}^{t} x_i a_i$ with a natural number t and elements $x_1, \ldots, x_t \in I_G$ and $a_1, \ldots, a_t \in \bigcap_{\mathbb{Z}[G]}^{|V'|} \mathcal{O}_{K,S',T}^\times$.

At this stage, it is convenient to introduce some general notation. For a $\mathbb{Z}[G]$-module M, we denote its $\mathbb{Z}[G]$-linear dual by $M^* := \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$. Given $f \in M^*$ and an integer $r \geq 1$, we then define a map

$$\begin{aligned}
f^{(r)} \colon \bigwedge_{\mathbb{Z}[G]}^{r} M &\to \bigwedge_{\mathbb{Z}[G]}^{r-1} M, \\
c_1 \wedge \cdots \wedge c_r &\mapsto \sum_{i=1}^{r} (-1)^i \cdot f(c_i) \cdot c_1 \wedge \cdots \wedge \widehat{c_i} \wedge \cdots \wedge c_r,
\end{aligned}$$

where the notation $\widehat{c_i}$ means omission of c_i. Iteration then yields a morphism

$$\begin{aligned}
\xi^{r,s} \colon \bigwedge_{\mathbb{Z}[G]}^{s} M &\to \mathrm{Hom}_{\mathbb{Z}[G]}\Big(\bigwedge_{\mathbb{Z}[G]}^{r} M, \bigwedge_{\mathbb{Z}[G]}^{r-s} M\Big), \\
f_1 \wedge \cdots \wedge f_s &\mapsto f_s^{(r-s+1)} \circ \cdots \circ f_1^{(r)}
\end{aligned}$$

for every $s \leq r$. (The special case $r = s$ of this construction was already discussed in Definition 2·1.) By abuse of notation we will simply write $f_1 \wedge \cdots \wedge f_s$ in place of $\xi^{r,s}(f_1 \wedge \cdots \wedge f_s)$.

Returning now to the concrete setting at hand, we set $\psi_l := w_l^* \circ \psi$ and, for every $f \in \bigwedge_{\mathbb{Z}[G]}^{|V|} (\mathcal{O}_{K,S',T}^\times)^*$ and $j \in \{|S| + 1, \ldots, |S'|\}$, define the map

$$\Phi_{j,f} := \Big(\big(\bigwedge_{\substack{|S|+1 \leq l \leq |S'| \\ l \neq j}} \psi_l\big) \wedge f\Big) \colon \mathbb{R} \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}[G]}^{|V'|} \mathcal{O}_{K,S',T}^\times \to \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_{K,S',T}^\times.$$

For every $g \in (\mathcal{O}_{K,S',T}^{\times})^*$ and $i \in \{1, \dots, t\}$, one then has that $(g \circ \Phi_{j,f})(a_i)$ belongs to $\mathbb{Z}[G]$. This shows that

$$\Phi_{j,f}(a_i) \in \left\{ a \in \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_{K,S',T}^{\times} \mid g(a) \in \mathbb{Z}[G] \text{ for all } g \in (\mathcal{O}_{K,S',T}^{\times})^* \right\} = \mathcal{O}_{K,S',T}^{\times}$$

because $\mathcal{O}_{K,S',T}^{\times}$ is $\mathbb{Z}$-torsion free. In addition, $\psi_l \circ \Phi_{j,f} = \Phi_{j,f} \wedge \psi_l = 0$ for all $l \neq j$ so that in fact $\Phi_{j,f}(a_i) \in \bigcap_{l \neq j} \ker \psi_l = \mathcal{O}_{K,S_j,T}^{\times}$. A similar argument also shows that $\left(\bigwedge_{|S|+1 \leq l \leq |S'|} \psi_l \right)(a_i)$ belongs to $\bigcap_{\mathbb{Z}[G]}^{|V|} \mathcal{O}_{K,S,T}^{\times}$.

Note that $\mathcal{O}_{K,S',T}^{\times}/\mathcal{O}_{K,S,T}^{\times}$ is $\mathbb{Z}$-torsion free (by (4·8)), hence that the natural restriction map res: $(\mathcal{O}_{K,S',T}^{\times})^* \to (\mathcal{O}_{K,S,T}^{\times})^*$ is surjective. For any $f \in \bigwedge_{\mathbb{Z}[G]}^{|V|} (\mathcal{O}_{K,S,T}^{\times})^*$ we can therefore find $\widetilde{f} \in \bigwedge_{\mathbb{Z}[G]}^{|V|} (\mathcal{O}_{K,S',T}^{\times})^*$ with $(\bigwedge^{|V|} \text{res})(\widetilde{f}) = f$. For any $j \in \{|S|+1, \dots, |S'|\}$, we then obtain that

$$\left(\left(\bigwedge_{|S|+1 \leq l \leq |S'|} \psi_l \right) \wedge f \right)(a_i) = \pm \psi_j((\Phi_{j,\widetilde{f}})(a_i)) \subseteq \psi_j(\mathcal{O}_{K,S_j,T}^{\times}).$$

Since this inclusion holds for every such j, we infer that in fact

$$\left(\left(\bigwedge_{|S|+1 \leq l \leq |S'|} \psi_l \right) \wedge f \right)(a_i) \subseteq \bigcap_{j=|S|+1}^{|S'|} \psi_j(\mathcal{O}_{K,S_j,T}^{\times}).$$

Now, by [53, Proposition 3·6] (see also [48, Proposition 5·2]) one has

$$\left(\bigwedge_{|S|+1 \leq l \leq |S'|} \psi_l \right)(\varepsilon_{K/k,S',T}^{V'}) = \pm \varepsilon_{K/k,S,T}^{V}$$

and so, for any $f \in \bigwedge_{\mathbb{Z}[G]}^{|V|} (\mathcal{O}_{K,S,T}^{\times})^*$, we deduce that

$$\begin{aligned}
f(\varepsilon_{K/k,S,T}^{V}) &= \pm \left(\left(\bigwedge_{|S|+1 \leq l \leq |S'|} \psi_l \right) \wedge f \right)(\varepsilon_{K/k,S',T}^{V'}) \\
&= \pm \sum_{i=1}^{t} x_i \cdot \left(\left(\bigwedge_{|S|+1 \leq l \leq |S'|} \psi_l \right) \wedge f \right)(a_i) \\
&\subseteq I_G \cdot \bigcap_{j=|S|+1}^{|S'|} \psi_j(\mathcal{O}_{K,S_j,T}^{\times}),
\end{aligned}$$

as claimed. From (4·9), it now follows that $\text{im}(\varepsilon_{K/k,S,T}^{V})$ is contained in $I_G \cdot \text{Ann}_{\mathbb{Z}[G]}(A)$. As $\widetilde{N}_{K/k}(\text{Cl}_{K,S',T})$ (which carries the trivial G-action) is annihilated by I_G, we conclude from the tautological exact sequence

$$0 \longrightarrow A \longrightarrow \text{Cl}_{K,S,T} \longrightarrow \widetilde{N}_{K/k}(\text{Cl}_{K,S',T}) \longrightarrow 0$$

that any element in $\text{im}(\varepsilon_{K/k,S,T}^{V})$ annihilates $\text{Cl}_{K,S,T}$, as required to prove Theorem 1·1.

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