*Proceedings of the Edinburgh Mathematical Society* (2011) **54**, 373–379
DOI:10.1017/S0013091509001412

# COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES OF A HALF-PLANE

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(Received 2 October 2009)

*Abstract* We use induction and interpolation techniques to prove that a composition operator induced by a map $\phi$ is bounded on the weighted Bergman space $\mathcal{A}^2_\alpha(\mathbb{H})$ of the right half-plane if and only if $\phi$ fixes the point at $\infty$ non-tangentially and if it has a finite angular derivative $\lambda$ there. We further prove that in this case the norm, the essential norm and the spectral radius of the operator are all equal and are given by $\lambda^{(2+\alpha)/2}$.

*Keywords:* composition operator; weighted Bergman space; interpolation

2010 *Mathematics subject classification:* Primary 47B33

## 1. Introduction

Analytic composition operators have been studied in a number of contexts, primarily on spaces of functions in the unit disc of the complex plane. It has long been known, as a consequence of the Littlewood subordination principle, that all such operators are bounded on all the Hardy spaces, as well as on a large class of other spaces of functions.

On the half-plane, however, things are somewhat more complicated. It is well known that there are unbounded composition operators on the half-plane. Indeed, in [**9**], Matache proved that a composition operator is bounded on the Hardy space $H^2$ of the half-plane if and only if the inducing map fixes the point at infinity and if it has a finite angular derivative $\lambda$ there. Later, in [**5**], Elliott and Jury sharpened this result and showed that in the case when such a composition operator is bounded, the norm, the essential norm and the spectral radius of the operator are all equal to $\sqrt{\lambda}$. In particular, Elliott and Jury's calculation strengthened a result on non-compactness of composition operators produced by Matache in [**8**].

Noting that a Hardy space is effectively a Bergman space with weight $\alpha = -1$, we will take the known situation as a base case and use induction and interpolation techniques to extend the results to all weighted Bergman spaces. In particular, we provide a formula

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for the norm that agrees with the known results for the Hardy space case. For a thorough discussion of Bergman spaces and their composition operators, see [**3**] or [**7**].

## 2. Preliminaries

Let $\mathbb{H}$ denote the right half-plane $\{\operatorname{Re} z > 0\}$. For $\alpha > -1$, the weighted Bergman space $\mathcal{A}^2_\alpha(\mathbb{H})$ contains those analytic functions $F\colon \mathbb{H} \to \mathbb{C}$ for which

$$\|F\|^2_{\mathcal{A}^2_\alpha(\mathbb{H})} := \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty} x^\alpha |F(x+\mathrm{i}y)|^2 \,\mathrm{d}x\,\mathrm{d}y < \infty.$$

For each $\alpha > -1$, the functions $\{k^\alpha_\omega;\ \omega \in \mathbb{H}\}$ defined by

$$k^\alpha_\omega(z) := \frac{2^\alpha(1+\alpha)}{(\bar{\omega}+z)^{2+\alpha}}, \quad z \in \mathbb{H}, \tag{2.1}$$

are the reproducing kernels for $\mathcal{A}^2_\alpha(\mathbb{H})$. As such, they have the property that

$$\langle f, k_\omega\rangle_{\mathcal{A}^2_\alpha(\mathbb{H})} = f(\omega), \quad f \in \mathcal{A}^2_\alpha(\mathbb{H}),\ \omega \in \mathbb{H}. \tag{2.2}$$

In order to prove our result, we will show that a certain kernel is positive. We say that a kernel $K(z,w)$ on $\mathbb{H}\times\mathbb{H}$ is positive if

$$\sum_{i,j=1}^{n} c_i\bar{c}_j K(x_i,x_j) \geqslant 0$$

for all $n \geqslant 1$, and for all scalars $c_1,\dots,c_n \in \mathbb{C}$ and points $x_1,\dots,x_n \in \mathbb{H}$.

**Proposition 2.1 (Nevanlinna).** *A holomorphic function $\psi$ in $\mathbb{H}$ has positive real part if and only if the kernel*

$$\frac{\psi(z)+\overline{\psi(w)}}{z+\bar{w}}$$

*is positive.*

A sequence of points $z_n = x_n + \mathrm{i}y_n$ in $\mathbb{H}$ is said to tend non-tangentially to $\infty$ if

(1) $x_n \to \infty$ and

(2) the ratio $|y_n|/x_n$ is uniformly bounded.

We then say that a map $\phi\colon \mathbb{H}\to\mathbb{H}$ fixes infinity non-tangentially and we write $\phi(\infty)=\infty$ if $\phi(z_n)\to\infty$ whenever $z_n\to\infty$ non-tangentially. If $\phi(\infty)=\infty$, we say that $\phi$ has a finite angular derivative at $\infty$ if the non-tangential limit

$$\lim_{z\to\infty}\frac{z}{\phi(z)} \tag{2.3}$$

exists and is finite; under these circumstances, we write $\phi'(\infty)$ for this quantity.

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If we let $\psi$ be the self-map of $\mathbb{D}$ equivalent to $\phi$ via the standard Möbius identification of the disc with the half-plane given by $\tau(\zeta) = (1+\zeta)/(1-\zeta)$, that is, $\psi = \tau^{-1} \circ \phi \circ \tau$, then (2.3) is equal, by the Julia–Carathéodory Theorem, to the non-tangential limit of $\psi'(\zeta)$ as $\zeta \to 1$, which is where the terminology comes from. Indeed, we have the following half-plane version of the Julia–Carathéodory Theorem, which was proved in [**5**].

**Lemma 2.2 (half-plane Julia–Carathéodory Theorem).** *Let $\phi\colon \mathbb{H} \to \mathbb{H}$ be holomorphic. The following are equivalent:*

(1) $\phi(\infty) = \infty$ *and* $\phi'(\infty)$ *exists;*

(2) $\sup_{z\in\mathbb{H}}(\operatorname{Re} z/\operatorname{Re}\phi(z)) < \infty$*;*

(3) $\limsup_{z\to\infty}(\operatorname{Re} z/\operatorname{Re}\phi(z)) < \infty$.

*Moreover, the quantities in (2) and (3) are both equal to the angular derivative $\phi'(\infty)$.*

## 3. Main results

For a natural number $n \geqslant 1$ and a holomorphic function $\phi\colon \mathbb{H} \to \mathbb{H}$ with finite angular derivative $\lambda$ at infinity, we define the kernel $K^n(\omega, z)$ on $\mathbb{H} \times \mathbb{H}$ by

$$K^n(\omega, z) := \frac{(\phi(z) + \overline{\phi(\omega)})^n - \lambda^{-n}(z+\bar{\omega})^n}{(z+\bar{\omega})^n}, \quad \omega, z \in \mathbb{H}.$$

**Lemma 3.1.** *Suppose that $\phi\colon \mathbb{H} \to \mathbb{H}$ has finite angular derivative $0 < \lambda < \infty$ at infinity. Then, for every natural number $n \geqslant 0$, the kernel $K^{2^n}$ is positive.*

**Proof.** It is shown in [**5**] that $K^1$ is positive. Now suppose that $K^{2^n}$ is positive for some natural number $n \geqslant 1$. Then, using the fact that the numerator of $K^{2^{n+1}}$ is the difference of two squares,

$$\begin{aligned}
K^{2^{n+1}}(\omega, z) &= \frac{((\phi(z) + \overline{\phi(\omega)})^{2^n})^2 - (\lambda^{-2^n}(z+\bar{\omega})^{2^n})^2}{(z+\bar{\omega})^{2^{n+1}}} \\
&= \frac{(\phi(z) + \overline{\phi(\omega)})^{2^n} - \lambda^{-2^n}(z+\bar{\omega})^{2^n}}{(z+\bar{\omega})^{2^n}} \, \frac{(\phi(z) + \overline{\phi(\omega)})^{2^n} + \lambda^{-2^n}(z+\bar{\omega})^{2^n}}{(z+\bar{\omega})^{2^n}} \\
&= K^{2^n}(\omega, z)\biggl(\frac{(\phi(z) + \overline{\phi(\omega)})^{2^n}}{(z+\bar{\omega})^{2^n}} + \lambda^{-2^n}\biggr) \\
&= K^{2^n}(\omega, z)(K^{2^n}(\omega, z) + 2\lambda^{-2^n}).
\end{aligned}$$

By the assumption that $K^{2^n}$ is positive, and since adding a positive constant to a kernel will certainly keep it positive, this is the product of two positive kernels and $K^{2^{n+1}}$ is therefore positive by the Schur Product Theorem [**1**]. The result now follows by induction. □

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As a result of Lemma 3.1, it is possible to provide conditions for boundedness of composition operators on weighted Bergman spaces, for certain integer weights.

**Proposition 3.2.** *Let* $\phi\colon \mathbb{H}\to\mathbb{H}$ *be holomorphic and let* $n \geqslant 1$ *be a natural number. The composition operator* $C_\phi\colon \mathcal{A}^2_{2^n-2}(\mathbb{H}) \to \mathcal{A}^2_{2^n-2}(\mathbb{H})$ *is bounded if and only if* $\phi$ *has finite angular derivative* $0<\lambda<\infty$ *at infinity, in which case* $\|C_\phi\| = \lambda^{2^{n-1}}$.

**Proof.** Let $n \geqslant 1$ be a natural number and define $\alpha := 2^n - 2$. To prove that $C_\phi\colon \mathcal{A}^2_\alpha(\mathbb{H}) \to \mathcal{A}^2_\alpha(\mathbb{H})$ is bounded with $\|C_\phi\| \leqslant \lambda^{2^{n-1}}$, it suffices to show that

$$\lambda^{2^n}\langle k^\alpha_\omega, k^\alpha_z\rangle_{\mathcal{A}^2_\alpha(\mathbb{H})} - \langle C^*_\phi k^\alpha_\omega, C^*_\phi k^\alpha_z\rangle_{\mathcal{A}^2_\alpha(\mathbb{H})} \tag{3.1}$$

is a positive kernel. Using the fact that $C^*_\phi k^\alpha_\omega = k^\alpha_{\phi(\omega)}$ and (2.2), it follows that (3.1) is equal to

$$2^\alpha(1+\alpha)\biggl(\frac{\lambda^{2^n}}{(z+\bar\omega)^{2^n}} - \frac{1}{(\phi(z)+\overline{\phi(\omega)})^{2^n}}\biggr).$$

This can easily be seen to factorize as

$$\lambda^{2^n}\frac{2^\alpha(1+\alpha)}{(\phi(z)+\overline{\phi(\omega)})^{2^n}}\frac{(\phi(z)+\overline{\phi(\omega)})^{2^n} - \lambda^{-2^n}(z+\bar\omega)^{2^n}}{(z+\bar\omega)^{2^n}},$$

which is just

$$\lambda^{2^n}\langle k^\alpha_{\phi(\omega)}, k^\alpha_{\phi(z)}\rangle_{\mathcal{A}^2_\alpha(\mathbb{H})}K^{2^n}(\omega,z).$$

This is positive, being the product of positive kernels and positive scalars.

For the converse, the calculation is similar to the Hardy space case. If the composition operator $C_\phi\colon \mathcal{A}^2_\alpha(\mathbb{H}) \to \mathcal{A}^2_\alpha(\mathbb{H})$ is bounded and if $\|C_\phi\| \leqslant M$, then

$$\begin{aligned}\frac{2^\alpha(1+\alpha)}{2^{2+\alpha}(\operatorname{Re}\phi(z))^{2+\alpha}} &= \|k^\alpha_{\phi(z)}\|^2_{\mathcal{A}^2_\alpha(\mathbb{H})}\\ &= \|C^*_\phi k^\alpha_z\|^2_{\mathcal{A}^2_\alpha(\mathbb{H})}\\ &\leqslant M^2\|k^\alpha_z\|^2_{\mathcal{A}^2_\alpha(\mathbb{H})}\\ &= M^2\frac{2^\alpha(1+\alpha)}{2^{2+\alpha}(\operatorname{Re} z)^{2+\alpha}}.\end{aligned}$$

As such,

$$\frac{\operatorname{Re}(z)}{\operatorname{Re}(\phi(z))} \leqslant M^{2/(2+\alpha)};$$

hence, by Lemma 2.2, $\phi$ has finite angular derivative

$$\phi'(\infty) = \lambda \leqslant \|C_\phi\|^{2/(2+\alpha)} = \|C_\phi\|^{2^{-(n-1)}}.$$

By the first part of the proof the norm of $C_\phi$ must be at most $\lambda^{2^{n-1}}$, and by the second part it must be at least that large. It follows that indeed

$$\|C_\phi\| = \lambda^{2^{n-1}}.$$

□

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Proposition 3.2 tells us that the result holds for particular integral values of $\alpha$ of arbitrarily large size. We proceed by interpolating for the spaces $\mathcal{A}^2_\alpha(\mathbb{H})$, where $2^n < \alpha < 2^{n+1}$. The following weighted version of the Paley–Wiener Theorem (see [**4**] or [**6**]) will be useful.

**Lemma 3.3.** *The Bergman space $\mathcal{A}^2_\alpha(\mathbb{H})$ is isometrically isomorphic, via the Laplace transform $\mathcal{L}$, to the space $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$. Here,*

$$\mathrm{d}\mu_\alpha = \frac{\Gamma(1+\alpha)}{2^\alpha t^{\alpha+1}}\,\mathrm{d}t,$$

*and $\mathrm{d}t$ is the Lebesgue measure on $\mathbb{R}_+ := (0, \infty)$.*

**Theorem 3.4.** *Let $\phi\colon \mathbb{H} \to \mathbb{H}$ be holomorphic and let $\alpha > -1$. The composition operator $C_\phi\colon \mathcal{A}^2_\alpha(\mathbb{H}) \to \mathcal{A}^2_\alpha(\mathbb{H})$ is bounded if and only if $\phi$ has finite angular derivative $0 < \lambda < \infty$ at infinity, in which case $\|C_\phi\| = \lambda^{(2+\alpha)/2}$.*

**Proof.** Let $\alpha > -1$. By Proposition 3.2, the result holds if $\alpha$ is of the form $\alpha = 2^n - 2$. Hence, it may be assumed without loss of generality that there exists a natural number $n \geqslant 0$ such that $\alpha \in (2^n - 2, 2^{n+1} - 2)$. Write $A := 2^n - 2$, $B := 2^{n+1} - 2$. In the following, for simplicity, write $L^2(\mathrm{d}\mu)$ for $L^2(\mathbb{R}_+, \mathrm{d}\mu)$. Define a linear operator

$$\begin{aligned} T\colon L^2(\mathrm{d}\mu_A) &\to L^2(\mathrm{d}\mu_A), \\ T\colon L^2(\mathrm{d}\mu_B) &\to L^2(\mathrm{d}\mu_B) \end{aligned}$$

by $T := \mathcal{L}^{-1} \circ C_\phi \circ \mathcal{L}$. Since $\mathcal{L}$ is an isometric isomorphism between the respective spaces (Lemma 3.3), Proposition 3.2 implies that

$$\begin{aligned} \|T\|_{L^2(\mathrm{d}\mu_A)\to L^2(\mathrm{d}\mu_A)} &= \|C_\phi\|_{\mathcal{A}^2_A(\mathbb{H})\to\mathcal{A}^2_A(\mathbb{H})} = \lambda^{2^{n-1}} = \lambda^{(2+A)/2}, \\ \|T\|_{L^2(\mathrm{d}\mu_B)\to L^2(\mathrm{d}\mu_B)} &= \|C_\phi\|_{\mathcal{A}^2_B(\mathbb{H})\to\mathcal{A}^2_B(\mathbb{H})} = \lambda^{2^{n}} = \lambda^{(2+B)/2}. \end{aligned}$$

(Note that in the case $n = 0$, $\mathcal{A}^2_A(\mathbb{H})$ should be replaced by the Hardy space $H^2(\mathbb{H})$.) Since $\alpha \in (A, B)$, there exists $\theta \in (0, 1)$ such that $\alpha = A(1-\theta) + B\theta$. By the Stein–Weiss Interpolation Theorem [**2**, Corollary 5.5.4],

$$\|T\|_{L^2(\mathrm{d}w)\to L^2(\mathrm{d}w)} \leqslant \lambda^{(2+A)(1-\theta)/2}\lambda^{(2+B)\theta/2} = \lambda^{(2+\alpha)/2}, \tag{3.2}$$

where

$$\begin{aligned} \mathrm{d}w &= \frac{\Gamma(1+A)^{1-\theta}\Gamma(1+B)^\theta}{2^{A(1-\theta)+B\theta}t^{A(1-\theta)+B\theta+1}}\,\mathrm{d}t \\ &= \frac{\Gamma(1+A)^{1-\theta}\Gamma(1+B)^\theta}{2^\alpha t^{1+\alpha}}\,\mathrm{d}t. \end{aligned}$$

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By Lemma 3.3, for any $g \in \mathcal{A}^2_\alpha(\mathbb{H})$ there exists $f \in L^2(\mathrm{d}\mu_\alpha)$ such that $\mathcal{L}f = g$ and $\|g\|_{\mathcal{A}^2_\alpha(\mathbb{H})} = \|f\|_{L^2(\mathrm{d}\mu_\alpha)}$. Thus,

$$\begin{aligned}
\|C_\phi g\|_{\mathcal{A}^2_\alpha(\mathbb{H})} &= \|C_\phi(\mathcal{L}f)\|_{\mathcal{A}^2_\alpha(\mathbb{H})} \\
&= \|\mathcal{L}(Tf)\|_{\mathcal{A}^2_\alpha(\mathbb{H})} \\
&= \|Tf\|_{L^2(\mathrm{d}\mu_\alpha)} \\
&= \frac{\Gamma(1+\alpha)^{1/2}}{\Gamma(1+A)^{(1-\theta)/2}\Gamma(1+B)^{\theta/2}}\|Tf\|_{L^2(\mathrm{d}w)} \\
&\leqslant \frac{\lambda^{(2+\alpha)/2}\Gamma(1+\alpha)^{1/2}}{\Gamma(1+A)^{(1-\theta)/2}\Gamma(1+B)^{\theta/2}}\|f\|_{L^2(\mathrm{d}w)} \quad \text{(by (3.2))} \\
&= \lambda^{(2+\alpha)/2}\|f\|_{L^2(\mathrm{d}\mu_\alpha)} \\
&= \lambda^{(2+\alpha)/2}\|g\|_{\mathcal{A}^2_\alpha(\mathbb{H})}.
\end{aligned}$$

As such, $C_\phi$ is bounded with $\|C_\phi\| \leqslant \lambda^{(2+\alpha)/2}$.

For the converse assume that $C_\phi$ is bounded. Then, by exactly the same proof as for the second half of Proposition 3.2, it follows that $\phi$ has finite angular derivative $\lambda$ and that $\|C_\phi\| \geqslant \lambda^{(2+\alpha)/2}$. □

The following results, concerning the spectral radius and essential norm of $C_\phi$, can be deduced from Theorem 3.4 by the methods used in [**5**] for the Hardy space $H^2(\mathbb{H})$.

**Theorem 3.5.** *If $C_\phi$ is bounded on $\mathcal{A}^2_\alpha(\mathbb{H})$, then its spectral radius and norm are equal.*

**Theorem 3.6.** *Every bounded composition operator on $\mathcal{A}^2_\alpha(\mathbb{H})$ has essential norm equal to its operator norm. In particular, since the zero operator is not a composition operator, there are no compact composition operators on any of the spaces $\mathcal{A}^2_\alpha(\mathbb{H})$.*

**Acknowledgements.** The authors thank Jonathan Partington and Michael Jury for helpful mathematical conversations that have aided the progress of this work.

The authors also thank the staff at the ICMS (Maxwell Institute) in Edinburgh, and the organizers of the Workshop on Operator Theory and Its Applications, held at the ICMS, where this work was initiated.

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